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Sudakov-type minoration for log-concave vectors

by

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Abstract. We formulate and discuss a conjecture concerning lower bounds for norms of log-concave vectors, which generalizes the classical Sudakov minoration principle for Gaussian vectors. We show that the conjecture holds for some special classes of log-concave measures and some weaker forms of it are satisfied in the general case. We also present some applications based on chaining techniques.

1. Introduction and formulation of the problem. In numerous problems arising in high-dimensional probability one needs to estimate $\mathbb{E}\,\|X\|$, where X is a random d-dimensional vector and $\|\ \|$ is a norm on $\mathbb{R}^d$. Obviously $\|x\| = \sup_{\|t\|_*\le 1}\langle t, x\rangle$, so the question reduces to finding bounds for $\mathbb{E}\sup_{t\in T}\langle t, X\rangle$ with $T\subset\mathbb{R}^d$. For symmetric random vectors this quantity is half of $\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle$, but in the case of arbitrary (not necessarily centered) random vectors it is more convienient to work with the latter quantity.

There are numerous powerful methods to estimate suprema of stochastic processes (cf. the monograph [22]); let us however present only a very easy upper bound. Namely for any $p\ge 1$,

$$\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle = \mathbb{E}\sup_{t,s\in T}|\langle t-s, X\rangle| \le \Big(\mathbb{E}\sup_{t,s\in T}|\langle t-s, X\rangle|^p\Big)^{1/p}$$
$$\le \Big(\mathbb{E}\sum_{t,s\in T}|\langle t-s, X\rangle|^p\Big)^{1/p} \le |T|^{2/p}\sup_{t,s\in T}\|\langle t-s, X\rangle\|_p.$$

Here and below, $\|Y\|_p := (\mathbb{E}\,|Y|^p)^{1/p}$ for a real random variable Y and $p>0$. In particular

$$\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle \le e^2\sup_{t,s\in T}\|\langle t-s, X\rangle\|_p \quad \text{if } |T|\le e^p.$$

It is natural to ask when the above estimate can be reversed. Namely, when is it true that if the set $T\subset\mathbb{R}^d$ has large cardinality (say at least e^p) and variables $(\langle t, X\rangle)_{t\in T}$ are A-separated with respect to the L_p-distance

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then $\mathbb{E}\sup_{t,s\in T}\langle t,X\rangle$ is at least of the order of A? The following definition gives a more precise formulation of that property.

DEFINITION 1.1. Let X be a random d-dimensional vector. We say that X *satisfies the L_p-Sudakov minoration principle with a constant* $\kappa>0$ ($\mathrm{SMP}_p(\kappa)$ for short) if for any set $T\subset\mathbb{R}^d$ with $|T|\ge e^p$ such that

$$\|\langle t-s,X\rangle\|_p=\Big(\mathbb{E}\Big(\sum_{i=1}^d(t_i-s_i)X_i\Big)^p\Big)^{1/p}\ge A\quad\text{for all } s,t\in T,\ s\ne t, \tag{1}$$

we have

$$\mathbb{E}\sup_{t,s\in T}\langle t-s,X\rangle=\mathbb{E}\sup_{t,s\in T}\sum_{i=1}^d(t_i-s_i)X_i\ge\kappa A. \tag{2}$$

A random vector X *satisfies the Sudakov minoration principle with a constant* κ ($\mathrm{SMP}(\kappa)$ for short) if it satisfies $\mathrm{SMP}_p(\kappa)$ for any $p\ge1$.

REMARK 1.2. One cannot hope to improve the estimate (2) even if X has a regular product distribution and $|T|$ is very large with respect to p. To see this take X uniformly distributed on the cube $[-1,1]^d$; then for $p\ge1$,

$$\|X_i-X_j\|_p\ge\|X_i-X_j\|_1=2/3\quad\text{for all } 1\le i<j\le d$$

and $\mathbb{E}\sup_{i,j\le n}(X_i-X_j)\le2$.

EXAMPLE 1.3. If X has the canonical d-dimensional Gaussian distribution then $\|\langle t,X\rangle\|_p=\gamma_p|t|$, where $\gamma_p=\|\mathcal{N}(0,1)\|_p\sim\sqrt{p}$ for $p\ge1$. Hence condition (1) is equivalent to $|t-s|\ge A/\gamma_p$ for distinct vectors $t,s\in T$, and the classical Sudakov minoration principle for Gaussian processes (cf. [19] and [16, Theorem 3.18]) then yields

$$\mathbb{E}\sup_{t,s\in T}\langle t-s,X\rangle=2\,\mathbb{E}\sup_{t\in T}\langle t,X\rangle\ge\frac{A}{C\gamma_p}\sqrt{\log|T|}\ge\frac{A}{C'}$$

provided that $|T|\ge e^p$ (C and C' denote universal constants). Therefore X satisfies the Sudakov minoration principle with a universal constant. In fact it is not hard to see that for centered Gaussian vectors the Sudakov minoration principle in the sense of Definition 1.1 is formally equivalent to the minoration property established by Sudakov.

EXAMPLE 1.4. If X_i's are independent symmetric ±1 r.v.'s (equivalently one may consider the vector X uniformly distributed on the cube $[-1,1]^d$) then condition (1) means, by the result of Hitczenko [9], that

$$t-s\notin\frac{A}{C}(B_1^n+\sqrt{p}\,B_2^n),$$

and in this case $\mathrm{SMP}(\kappa)$ with universal κ was proven by Talagrand [20].

EXAMPLE 1.5. In the more general case when coordinates of X are independent and symmetric with log-concave densities (or just log-concave tails) the Sudakov minoration priciple with a universal constant was proven in [21] (for random variables with density $\exp(-|x|^p)$, $p \geq 1$) and [12].

The Sudakov minoration principle for vectors X with independent coordinates is investigated in [14], where it is shown that SMP is essentially equivalent to the regular growth of moments of coordinates of X. In this paper we will concentrate on the class of log-concave vectors.

A measure μ on $\mathbb{R}^n$ is called *logarithmically concave* (or *log-concave* for short) if

$$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^{\lambda}\mu(L)^{1-\lambda}$$

for any nonempty compact sets K, L and $\lambda \in [0,1]$. By the result of Borell [5] a measure on $\mathbb{R}^n$ with full-dimensional support is log-concave if and only if it has a log-concave density, i.e. a density of the form $e^{-h(x)}$, where $h\colon \mathbb{R}^n \to (-\infty, \infty]$ is convex. A random vector is called log-concave if its distribution is logarithmically concave. A typical example of a log-concave vector is a vector uniformly distributed on a convex body.

It is quite easy to reduce the investigation of the Sudakov minoration principle to the case of symmetric vectors (see Lemma 2.1 below). Since SMP is preserved under linear transformations (Lemma 2.2), we may additionally assume that the vector X is *isotropic*, i.e. $\mathrm{Cov}(X_i, X_j) = \delta_{i,j}$ for all i, j. In many aspects isotropic log-concave probability measures behave like product measures (cf. [6]). This motivates the following conjecture.

CONJECTURE 1.6. *Every d-dimensional log-concave random vector satisfies the Sudakov minoration principle with a universal constant.*

The purpose of this paper is to discuss the above conjecture. In Section 2 we gather simple facts concerning log-concave vectors and the Sudakov minoration principle. In particular we show how to reduce the problem to the case of isotropic vectors. In Section 3 we establish several results concerning arbitrary log-concave distributions. We show that (1) implies (2) provided that $|T| \geq e^{e^p}$ or $|T| \geq e^p$, but under the additional assumption that the vectors $(\langle t, X\rangle)_{t\in T}$ are uncorrelated. The proof is based on the concentration properties of isotropic log-concave distributions. As a byproduct we get a comparison of weak and strong moments of ℓ_∞^d-norms of isotropic log-concave vectors. In Section 4 we consider unconditional log-concave vectors. We show that in this case, (1) implies (2) provided that $|T| \geq e^{p^2}$. In Section 5 we show that Conjecture 1.6 holds for a class of invariant log-concave vectors, which includes rotationally invariant log-concave vectors and vectors uniformly distributed on l_p^d-balls. In the last section we use chaining arguments to give some consequences of the Sudakov minoration principle.

In particular, we show that it yields comparison of weak and strong moments up to a logarithmic factor.

It should be mentioned that the Sudakov minoration principle and Conjecture 1.6 were formulated independently and studied by Shahar Mendelson, Emanuel Milman and Grigoris Paouris [17]. Their approach is however quite different: it uses geometrical properties of an index set T, duality of entropy numbers and the idea of dimension reduction, similar in spirit to the Johnson–Lindenstrauss lemma.

NOTATION. By $|\cdot|$ and $\langle\cdot,\cdot\rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^d$. The canonical basis of $\mathbb{R}^d$ is denoted by $e_1,\ldots,e_d$. For $1\le p\le\infty$, $\|\cdot\|_p$ stands for the l_p norm on $\mathbb{R}^d$, and B_p^d is the unit ball in this norm.

For two convex sets K,L in $\mathbb{R}^d$, $N(K,L)$ is the *covering number*, i.e. the minimal number of translates of L that cover K. By $|T|$ we denote the cardinality of a set T and by $N(T,d,\varepsilon)$ the minimal number of balls in metric d of radius ε that cover T.

We use the letter C for universal constants; the value of a constant C may differ at each occurrence. Whenever we want to fix the value of an absolute constant we use $C_1,C_2,\ldots$.

2. Basic facts. We start with a lemma showing how to reduce the problem of proving the Sudakov minoration to the case of symmetric vectors.

LEMMA 2.1. *Let $p\ge 1$, X be a random vector in $\mathbb{R}^d$ with finite pth moment and X' be an independent copy of X. If $X-X'$ satisfies* $\mathrm{SMP}_p(\kappa)$ *then X satisfies* $\mathrm{SMP}_p(\min\{1/2,\kappa/4\})$.

Proof. Let $p\ge 1$ and $T\subset\mathbb{R}^d$ be such that $|T|\ge e^p$ and (1) holds. Jensen's inequality yields

$$\mathbb{E}\sup_{t,s\in T}\langle t-s,X\rangle\ge\sup_{t,s\in T}\langle t-s,\mathbb{E}X\rangle=\sup_{t,s\in T}|\langle t-s,\mathbb{E}X\rangle|.$$

Therefore we may assume that $|\langle t-s,\mathbb{E}X\rangle|\le A/2$ for all $t,s\in T$. But then for $t\ne s$,

$$\|\langle t-s,X-X'\rangle\|_p\ge\|\langle t-s,X-\mathbb{E}X\rangle\|_p\ge\|\langle t-s,X\rangle\|_p-\|\langle t-s,\mathbb{E}X\rangle\|_p\ge\frac{A}{2}.$$

Therefore the L_p-Sudakov minoration for $X-X'$ implies

$$\begin{aligned}\kappa\frac{A}{2}&\le\mathbb{E}\sup_{t,s\in T}\langle t-s,X-X'\rangle\le\mathbb{E}\sup_{t,s\in T}\langle t-s,X\rangle+\mathbb{E}\sup_{t,s\in T}\langle s-t,X'\rangle\\&=2\,\mathbb{E}\sup_{t,s\in T}\langle t-s,X\rangle.\ \blacksquare\end{aligned}$$

The next observation states that the Sudakov minoration principle is preserved under linear transformations.

LEMMA 2.2. *If X is a d-dimensional random vector that satisfies* $\mathrm{SMP}_p(\kappa)$ *and* $U\colon \mathbb{R}^d \to \mathbb{R}^{d'}$ *is linear then UX satisfies* $\mathrm{SMP}_p(\kappa)$.

Proof. It is enough to observe that $\langle t, UX\rangle = \langle U^*t, X\rangle$. ∎

Now we recall that moments of log-concave variables grow in a regular way.

LEMMA 2.3. *Let Y be a symmetric real log-concave r.v. Then*

$$\|Y\|_p \le \frac{\Gamma(p+1)^{1/p}}{\Gamma(q+1)^{1/q}}\|Y\|_q \quad \text{for } p \ge q > 0.$$

In particular $\|Y\|_2 \le \sqrt{2}\,\|Y\|_1$ *and* $\|Y\|_p \le \frac{p}{q}\|Y\|_q$ *for* $p \ge q \ge 2$.

Proof. The main inequality is the result of Barlow, Marshall and Proschan [1] (it can also be extracted from the much earlier work of Berwald [3]). To show the "in particular" part for $p \ge q \ge 2$ one needs to estimate Γ functions as in [15, Proposition 3.8]. ∎

REMARK 2.4. Suppose $|T| \ge 2$ and (1) holds. Then if X is symmetric log-concave, we may choose distinct $t^1, t^2 \in T$ and get, by Lemma 2.3,

$$\begin{aligned}\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle &\ge \mathbb{E}\,|\langle t^2 - t^1, X\rangle| \ge \frac{\sqrt{2}}{\max\{p,2\}}\|\langle t^2 - t^1, X\rangle\|_p \\ &\ge \frac{\sqrt{2}}{\max\{p,2\}}A.\end{aligned}$$

Hence every symmetric log-concave vector satisfies $\mathrm{SMP}_p(\sqrt{2}/\max\{p,2\})$ and every log-concave vector satisfies $\mathrm{SMP}_p(\sqrt{2}/\max\{4p,8\})$.

REMARK 2.5. For $p \ge 1$ let us define the distance on $\mathbb{R}^d$ by

$$d_{X,p}(s,t) := \|\langle s-t, X\rangle\|_p.$$

Suppose that (1) is satisfied, but $|T| = e^q$ with $1 \le q \le p$. We know that $d_{X,q}(s,t) \ge \frac{q}{Cp} d_{X,p}(s,t)$, so the Sudakov minoration principle for a log-concave vector X implies the following formally stronger statement: for any nonempty $T \subset \mathbb{R}^d$ and $A > 0$,

$$\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle \ge \frac{\kappa}{C}\sup_{p\ge 1}\min\left\{\frac{A}{p}\log N(T, d_{X,p}, A), A\right\}.$$

The next result says that it is enough to verify the Sudakov minoration property only for $p \le d$.

LEMMA 2.6. *Let X be a symmetric log-concave random vector in $\mathbb{R}^d$ that satisfies* $\mathrm{SMP}_d(\kappa)$. *Then X satisfies* $\mathrm{SMP}_p(\kappa/8)$ *for* $p \ge d$.

Proof. Fix $p \geq d$ and $T \subset \mathbb{R}^d$ such that $|T| \geq e^p$ and $\|\langle s-t, X\rangle\|_p \geq A$ for any distinct points $s, t \in T$. Let

$$\mathcal{M}_p(X) := \{t \in \mathbb{R}^d : \mathbb{E}\,|\langle t, X\rangle|^p \leq 1\}.$$

If $d \leq p \leq 8d$ then by Lemma 2.3, $\|\langle s-t, X\rangle\|_d \geq \frac{1}{8}A$, hence $\mathrm{SMP}_d(\kappa)$ yields $\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle \geq (\kappa/8)A$.

If $p \geq 8d$ then for $u \geq 1$ we have

$$\begin{aligned} e^p &\leq N\Big(T, \frac{A}{2}\mathcal{M}_p(X)\Big) \leq N\Big(T, u\frac{A}{2}\mathcal{M}_p(X)\Big) N\Big(u\frac{A}{2}\mathcal{M}_p(X), \frac{A}{2}\mathcal{M}_p(X)\Big) \\ &\leq N\Big(T, u\frac{A}{2}\mathcal{M}_p(X)\Big)(2u+1)^d, \end{aligned}$$

where the last inequality follows by the standard volumetric argument. This shows that $N(T, u\frac{A}{2}\mathcal{M}_p(X)) \geq e^d$ if $u \leq e^{p/(4d)}$, therefore we may find $T_1 \subset T$ with $|T_1| \geq e^d$ such that for all distinct $s, t \in T_1$,

$$\|\langle s-t, X\rangle\|_d \geq \frac{d}{p}\|\langle s-t, X\rangle\|_p \geq \frac{d}{p}e^{p/(4d)}\frac{A}{2} \geq \frac{A}{8}.$$

Thus again $\mathrm{SMP}_d(\kappa)$ yields

$$\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle \geq \mathbb{E}\sup_{t,s\in T_1}\langle t-s, X\rangle \geq \kappa\frac{A}{8}. \ \blacksquare$$

Remark 2.7. Lemmas 2.1 and 2.6 together with Remark 2.4 show that every log-concave vector satisfies $\mathrm{SMP}(1/(Cd))$.

The following easy observation shows that Sudakov minoration holds with a universal constant if p is large with respect to the dimension d.

Lemma 2.8. *Every symmetric d-dimensional log-concave vector X satisfies* $\mathrm{SMP}_p\big(\frac{1}{\sqrt{2}p}(e^{p/d}-1)\big)$ *for $p \geq 2$. In particular X satisfies* $\mathrm{SMP}_p(1/3)$ *for $p \geq 2d\log(d+e)$.*

Proof. By Lemma 2.2 we may assume that X is isotropic. Assume that $|T| \geq e^p$, $p \geq 2$ and (1) holds. Then by Lemma 2.3,

$$|t-s| = \|\langle t-s, X\rangle\|_2 \geq \frac{2}{p}\|\langle t-s, X\rangle\|_p \geq \frac{2}{p}A.$$

This implies that the sets $\big(t + \frac{A}{p}B_2^d\big)_{t\in T}$ have disjoint interiors. The standard volumetric argument shows that there exist $t, s \in T$ such that $|t-s| \geq (A/p)(|T|^{1/d}-1)$. We have

$$\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle = \sup_{t,s\in T}\mathbb{E}\,|\langle t-s, X\rangle| \geq \frac{1}{\sqrt{2}}\sup_{t,s\in T}|t-s| \geq \frac{A}{\sqrt{2}\,p}(e^{p/d}-1).$$

Finally if $p = ad\log(d+e)$ with $a \geq 2$ then $p \geq 2$ and $e^{p/d}-1 \geq d(d+e)^{a-1} \geq d(a-1)\log(d+e) \geq \frac{1}{2}p$. $\blacksquare$

3. Estimates for general log-concave measures. We say that a random vector X in $\mathbb{R}^d$ satisfies *exponential concentration* with a constant $\alpha < \infty$ if for any Borel set B in $\mathbb{R}^d$,

$$\mathbb{P}(X \in B) \geq \frac{1}{2} \;\Rightarrow\; \mathbb{P}(X \in B + \alpha u B_2^d) \geq 1 - e^{-u} \quad \text{for } u > 0.$$

It is an important open problem [10] whether isotropic log-concave vectors satisfy exponential concentration with a universal constant. E. Milman [18] showed that this problem has numerous equivalent functional and isoperimetrical formulations. Klartag [11] proved that every isotropic d-dimensional log-concave vector satisfies exponential concentration with a constant $\alpha \leq C d^{1/2-\varepsilon}$ with $\varepsilon \geq 1/30$. This bound was improved by Eldan [7] to $\alpha \leq C d^{1/3} \log^{1/2}(d+1)$.

We start this section by deriving a simple consequence of exponential concentration, which will be used to estimate ℓ_∞-norms of log-concave vectors.

Proposition 3.1. *Suppose that a random vector X satisfies exponential concentration with a constant α. Then for any $V > 0$, $p \geq 2$ and $T \subset \mathbb{R}^d$ we have*

$$\Big(\mathbb{E} \sum_{t \in T} (|\langle t, X\rangle| \wedge V)^p\Big)^{1/p} \leq 2\,\mathbb{E}\Big(\sum_{t \in T} (|\langle t, X\rangle| \wedge V)^p\Big)^{1/p} + 2^{1/p} V^{(p-2)/p} (\alpha \beta(T))^{2/p},$$

where

$$\beta(T) := \sup_{|x|=1} \Big(\sum_{t \in T} |\langle t, x\rangle|^2\Big)^{1/2}.$$

Proof. Let

$$S := \Big(\sum_{t \in T} (|\langle t, X\rangle| \wedge V)^p\Big)^{1/p} \quad \text{and} \quad M := \mathbb{E}\, S.$$

Define also

$$B := \Big\{x \in \mathbb{R}^n : \Big(\sum_{t \in T} (|\langle t, x\rangle| \wedge V)^p\Big)^{1/p} \leq 2M\Big\}.$$

Then $\mathbb{P}(X \in B) \geq 1/2$. Notice that for $x = y + z \in B + uB_2^n$ with $u > 0$ we have

$$\begin{aligned}\Big(\sum_{t \in T} (|\langle t, x\rangle| \wedge V)^p\Big)^{1/p} &\leq \Big(\sum_{t \in T} (|\langle t, y\rangle| \wedge V)^p\Big)^{1/p} + \Big(\sum_{t \in T} (|\langle t, z\rangle| \wedge V)^p\Big)^{1/p} \\ &\leq 2M + V^{(p-2)/p} \Big(\sum_{t \in T} |\langle t, z\rangle|^2\Big)^{1/p} \\ &\leq 2M + V^{(p-2)/p} (u \beta(T))^{2/p}.\end{aligned}$$

Hence exponential concentration yields
$$\mathbb{P}\big(S \geq 2M + V^{(p-2)/p}(\alpha\beta(T)u)^{2/p}\big) \leq e^{-u} \quad \text{for } u > 0.$$
Integrating by parts this gives
$$\begin{aligned}
&(\mathbb{E}\,(S-2M)_+^p)^{1/p} \\
&\leq V^{(p-2)/p}(\alpha\beta(T))^{2/p}\Big(p\int_0^\infty u^{p-1}\,\mathbb{P}\big(S \geq 2M + V^{(p-2)/p}(\alpha\beta(T))^{2/p}u\big)\,du\Big)^{1/p} \\
&\leq V^{(p-2)/p}(\alpha\beta(T))^{2/p}\Big(p\int_0^\infty u^{p-1}e^{-u^{p/2}}\,du\Big)^{1/p} \\
&= 2^{1/p}V^{(p-2)/p}(\alpha\beta(T))^{2/p}. \ \blacksquare
\end{aligned}$$

We also need a simple technical lemma.

LEMMA 3.2. *Suppose that Y is a real symmetric log-concave r.v., $p \geq 2$ and $\|Y\|_p \geq V > 0$. Then $\mathbb{E}\,(|Y| \wedge V)^p \geq (V/12)^p$.*

Proof. By Lemma 2.3, $\|Y\|_{2p} \leq 2\|Y\|_p$, hence the Paley–Zygmund inequality yields
$$\mathbb{P}(|Y| \geq 2^{-1/p}V) \geq \mathbb{P}\bigg(|Y|^p \geq \frac{1}{2}\,\mathbb{E}\,|Y|^p\bigg) \geq \frac{(\mathbb{E}\,|Y|^p)^2}{4\,\mathbb{E}\,|Y|^{2p}} \geq \frac{1}{4\cdot 2^{2p}}$$
and
$$\mathbb{E}\,(|Y| \wedge V)^p \geq \frac{1}{2}V^p\,\mathbb{P}(|Y| \geq 2^{-1/p}V) \geq \frac{1}{8\cdot 4^p}V^p \geq \bigg(\frac{V}{12}\bigg)^p. \ \blacksquare$$

We are now ready to state a lower bound for suprema of coordinates of isotropic log-concave vectors.

PROPOSITION 3.3. *Let X be an isotropic log-concave random vector in $\mathbb{R}^d$. Suppose that $p \geq 2$, $d \geq e^p - 1$ and $\|a_iX_i\|_p \geq V$ for $1 \leq i \leq d$. Then*
$$\mathbb{E}\max\{|a_1X_1|,\ldots,|a_dX_d|\} \geq \frac{1}{C_1}V.$$

Proof. A symmetrization argument as in Lemma 2.1 shows that we may additionally assume that X is symmetric. By Lemma 2.3,
$$\mathbb{E}\max_i|a_iX_i| \geq \max_i\|a_iX_i\|_1 \geq \frac{\sqrt{2}}{p}\max_i\|a_iX_i\|_p \geq \frac{\sqrt{2}}{p}V,$$
so we may assume that p (and therefore also d) is sufficiently large. Since $e^p - 1 \geq e^{p/2}$ and by Lemma 2.3, $\|a_iX_i\|_{\lambda p} \geq \lambda\|a_iX_i\|_p \geq \lambda V$ for $\lambda \in (0,1]$ and $p \geq 2/\lambda$, we may assume (changing p to $p/(8\ln(24))$ and V to $V/(8\ln(24))$) that $d \geq 24^{4p}$.

Since it is only a matter of normalization of the coefficients a_i and the number V we may and will assume that $\max_i|a_i| = 1$. But then by

Lemma 2.3,

$$\mathbb{E}\max\{|a_1X_1|,\dots,|a_dX_d|\}\geq\max_i\mathbb{E}\,|a_iX_i|\geq\min_i\mathbb{E}\,|X_i|\geq\frac{1}{\sqrt{2}},$$

so it is enough to consider the case $V\geq 2$.

By the result of Eldan [7], X satisfies exponential concentration with constant at most $d^{1/2-1/8}$ (recall that we assume that d is sufficiently large). Let $T:=\{a_ie_i : i\leq d\}\subset\mathbb{R}^d$. Then

$$\beta(T)^2=\sup_{|x|=1}\sum_{i=1}^d|a_ix_i|^2=\max_i a_i^2=1,$$

hence Proposition 3.1 yields (recall that $V\geq 2$)

$$\Big(\mathbb{E}\sum_{i=1}^d(|a_iX_i|\wedge V)^p\Big)^{1/p}\leq 2\,\mathbb{E}\Big(\sum_{i=1}^d(|a_iX_i|\wedge V)^p\Big)^{1/p}+Vd^{1/p-1/(4p)}.$$

We have

$$\mathbb{E}\,\Big(\sum_{i=1}^d(|a_iX_i|\wedge V)^p\Big)^{1/p}\leq d^{1/p}\,\mathbb{E}\max_{1\leq i\leq d}|a_iX_i|.$$

By Lemma 3.2 we know that $\mathbb{E}\,(|a_iX_i|\wedge V)^p\geq (V/12)^p$, therefore

$$\Big(\mathbb{E}\sum_{i=1}^d(|a_iX_i|\wedge V)^p\Big)^{1/p}\geq\frac{1}{12}Vd^{1/p}.$$

Thus

$$\frac{1}{12}Vd^{1/p}\leq 2d^{1/p}\,\mathbb{E}\max_{1\leq i\leq d}|a_iX_i|+Vd^{1/p-1/(4p)}.$$

However $d^{1/(4p)}\geq 24$ and we get

$$\mathbb{E}\max_{1\leq i\leq d}|a_iX_i|\geq\frac{1}{48}V. \ \blacksquare$$

As a corollary we show that Conjecture 1.6 holds for sets T such that r.v.'s $(\langle t,X\rangle)_{t\in T}$ are uncorrelated.

Corollary 3.4. *Suppose that X is a d-dimensional log-concave random vector, $p\geq 2$, $T\subset\mathbb{R}^d$ satisfies* (1) *and* $\mathrm{Cov}(\langle t,X\rangle,\langle s,X\rangle)=0$ *for $s,t\in T$ with $s\neq t$. Then* (2) *holds with a universal constant κ provided that $|T|\geq e^p$.*

Proof. Using a symmetrization argument as in the proof of Lemma 2.1 we may assume that X is symmetric.

Since $\|\langle t-s,X\rangle\|_p\leq\|\langle t,X\rangle\|_p+\|\langle s,X\rangle\|_p$, there exist $t_1,\dots,t_n\subset T$ with $n\geq|T|-1\geq e^p-1$ such that $\|\langle t_i,X\rangle\|_p\geq A/2$ for all i. Proposition 3.3 applied with $V=A/2$, $a_i:=\|\langle t_i,X\rangle\|_2$ and n-dimensional isotropic vector

$Y = (\langle t_i, X\rangle / a_i)_{i\le n}$ gives

$$\mathbb{E}\max_{t\in T}|\langle t,X\rangle| \ge \mathbb{E}\max_i |\langle t_i,X\rangle| = \mathbb{E}\max_i |a_iY_i| \ge \frac{1}{2C_1}A.$$

Notice that for any $t_0\in T$ we have

$$\mathbb{E}\max_{t,s\in T}\langle t-s,X\rangle \ge \mathbb{E}\max_{t\in T}|\langle t-t_0,X\rangle| \ge \mathbb{E}\max_{t\in T}|\langle t,X\rangle| - \mathbb{E}|\langle t_0,X\rangle|.$$

If $\mathbb{E}|\langle t_0,X\rangle| \le A/(4C_1)$ then we are done, otherwise we may assume that $T\ni t_1\ne t_0$ and get, by Lemma 2.3,

$$\begin{aligned}\mathbb{E}\max_{t,s\in T}\langle t-s,X\rangle &\ge \mathbb{E}|\langle t_1-t_0,X\rangle| \ge \frac{1}{\sqrt{2}}\|\langle t_1-t_0,X\rangle\|_2 \ge \frac{1}{\sqrt{2}}\|\langle t_0,X\rangle\|_2\\ &\ge \frac{1}{\sqrt{2}}\mathbb{E}|\langle t_0,X\rangle| \ge \frac{1}{4\sqrt{2}\,C_1}A,\end{aligned}$$

where the third inequality follows since $\mathrm{Cov}(\langle t_0,X\rangle,\langle t_1,X\rangle)=0$. ■

Before we formulate the next consequence of Proposition 3.3, we show a simple decomposition lemma.

LEMMA 3.5. *Let $r>0$, $\varepsilon\in[0,1)$ and $T\subset rB_2^d$ satisfy $|T|\ge(2/\varepsilon+1)^n$. Then we can find vectors $t_k,s_k\in T$ and $v_k,u_k\in\mathbb{R}^d$, $k=1,\dots,n$, such that $0\ne t_k-s_k=u_k+v_k$, $|v_k|\le\varepsilon r$ for all k and the vectors $u_1,\dots,u_n$ are orthogonal.*

Proof. We proceed by induction. We choose for t_1,s_1 any two distinct vectors in T and set $u_1:=t_1-s_1$ and $v_1:=0$. Suppose that $1\le l\le n-1$ and vectors t_k,s_k,u_k,v_k have been chosen for $1\le k\le l$. Define $E:=\mathrm{Lin}(u_1,\dots,u_l)$, so $\dim E\le l\le n-1$. Since in the ball in E of radius r there are at most $(2/\varepsilon+1)^{\dim E}<|T|$ points with mutual distances at least εr, there exist distinct vectors t_{l+1},s_{l+1} in T such that $|P_E(t_{l+1}-s_{l+1})|\le\varepsilon r$, where P_E denotes the orthogonal projection onto E. We set

$$v_{l+1}:=P_E(t_{l+1}-s_{l+1}) \quad\text{and}\quad u_{l+1}:=t_{l+1}-s_{l+1}-v_{l+1}. \ ■$$

The next theorem is a weaker form of Conjecture 1.6.

THEOREM 3.6. *Let X be a log-concave vector, $p\ge1$ and $T\subset\mathbb{R}^d$ be such that $|T|\ge e^{e^p}$ and $\|\langle t-s,X\rangle\|_p\ge A$ for all distinct $t,s\in T$. Then*

$$\mathbb{E}\sup_{t,s\in T}\langle t-s,X\rangle \ge \frac{1}{C}A.$$

Proof. Arguments as in the proofs of Lemmas 2.1 and 2.2 show that it is enough to consider only symmetric and isotropic vectors X. Since the statement is translation invariant, we can assume that $0\in T$.

Let C_1 be as in Proposition 3.3; we may obviously assume that $C_1\ge1$. By Remark 2.4 it is enough to consider the case $p\ge16eC_1$. Set $p':=p/(8eC_1)$

and $A' := A/(8eC_1)$. Lemma 2.3 yields $\|\langle t-s, X\rangle\|_{p'} \geq A'$ for any distinct vectors $s, t \in T$. Moreover

$$(1+4eC_1p')^{e^{p'}} \leq (e^{p/2})^{e^{p/2}} \leq e^{e^p} \leq |T|.$$

Since $0 \in T$, we have (using again Lemma 2.3)

$$\mathbb{E} \sup_{t,s\in T} \langle t-s, X\rangle \geq \sup_{t\in T} \mathbb{E}\, |\langle t, X\rangle| \geq \frac{1}{\sqrt{2}} \sup_{t\in T} \|\langle t, X\rangle\|_2 = \frac{1}{\sqrt{2}} \sup_{t\in T} |t|.$$

Thus we may assume that $T \subset A'B_2^d$. Let $n := e^{p'}$ and $\varepsilon := 1/(2eC_1p')$. Then $|T| \geq (2/\varepsilon+1)^n$. Therefore we may apply Lemma 3.5 to the set T with $r = A'$ and ε, n as above and get points t_k, s_k, u_k and v_k. We have

$$\begin{aligned}\mathbb{E} \sup_{t,s\in T} \langle t-s, X\rangle &\geq \mathbb{E} \max_{k\leq n} |\langle t_k - s_k, X\rangle| = \mathbb{E} \max_{k\leq n} |\langle u_k + v_k, X\rangle| \\ &\geq \mathbb{E} \max_{k\leq n} |\langle u_k, X\rangle| - \mathbb{E} \max_{k\leq n} |\langle v_k, X\rangle|.\end{aligned}$$

Notice that

$$\begin{aligned}\mathbb{E} \max_{k\leq n} |\langle v_k, X\rangle| &\leq \Big(\mathbb{E} \sum_{k\leq n} |\langle v_k, X\rangle|^{p'}\Big)^{1/p'} \leq n^{1/p'} \max_k \|\langle v_k, X\rangle\|_{p'} \\ &\leq e\frac{p'}{2} \max_k \|\langle v_k, X\rangle\|_2 = \frac{ep'}{2} \max_{k\leq n} |v_k| \leq \frac{ep'}{2}\varepsilon A' = \frac{1}{4C_1}A',\end{aligned}$$

where the third inequality follows by Lemma 2.3. Moreover for any k,

$$\|\langle u_k, X\rangle\|_{p'} \geq \|\langle t_k - s_k, X\rangle\|_{p'} - \|\langle v_k, X\rangle\|_{p'} \geq A' - \frac{p'}{2}\varepsilon A' \geq \frac{A'}{2}.$$

Since u_k are orthogonal and X is isotropic, Proposition 3.3 (applied with $p = p'$, $V = A'/2$, $a_k := |u_k|$ and the n-dimensional isotropic vector $Y = (\langle u_k, X\rangle/a_k)_{k\leq n}$) yields

$$\mathbb{E} \sup_{k\leq n} |\langle u_k, X\rangle| \geq \frac{1}{2C_1}A'.$$

Hence

$$\mathbb{E} \sup_{t,s\in T} \langle t-s, X\rangle \geq \frac{1}{2C_1}A' - \frac{1}{4C_1}A' = \frac{1}{4C_1}A'. \ \blacksquare$$

REMARK 3.7. As in Remark 2.5, we can reformulate the above result in terms of covering numbers: for any log-concave vector X, any nonempty $T \subset \mathbb{R}^d$ and $A > 0$, we have

$$\mathbb{E} \sup_{t,s\in T} \langle t-s, X\rangle \geq \frac{1}{C} \sup_{p\geq 1} \min\Big\{\frac{A}{p} \log\big(1 + \log N(T, d_{X,p}, A)\big), A\Big\}.$$

It is an open problem (cf. [13]) whether weak and strong moments of log-concave vectors are comparable, i.e. whether for $p \geq 1$, log-concave

d-dimensional vectors X and any norm $\| \ \|$ on $\mathbb{R}^d$,

$$(\mathbb{E}\,\|X\|^p)^{1/p} \le C\Big(\mathbb{E}\,\|X\| + \sup_{\|t\|_* \le 1} (\mathbb{E}\,|\langle t, X\rangle|^p)^{1/p}\Big).$$

The next result shows that this is the case for weighted l_∞^d-norms of isotropic vectors.

Corollary 3.8. *Let X be an isotropic d-dimensional random vector. Then for any numbers $a_1, \dots, a_d$ and any $p \ge 1$,*

$$\Big(\mathbb{E}\max_{i\le d} |a_i X_i|^p\Big)^{1/p} \le C\Big(\mathbb{E}\max_{i\le d}|a_iX_i| + \max_{i\le d}(\mathbb{E}\,|a_iX_i|^p)^{1/p}\Big).$$

Proof. Let $M := \mathbb{E}\max_{i\le d}|a_iX_i|$. Define

$$I_q := \{i \le d\colon \|a_iX_i\|_q > C_1M\}, \qquad q \ge 1,$$

and

$$I_\infty := \bigcup_{q\ge 1} I_q = \{i \le d\colon \|a_iX_i\|_\infty > C_1M\}.$$

Then by Proposition 3.3, $|I_q| \le e^q - 1$ for $2 \le q < \infty$.

We have

$$\Big(\mathbb{E}\max_{i\in I_{2p}} |a_iX_i|^p\Big)^{1/p} \le \Big(\mathbb{E}\sum_{i\in I_{2p}} |a_iX_i|^p\Big)^{1/p} \le |I_{2p}|^{1/p}\max_{i\in I_{2p}}\|a_iX_i\|_p$$
$$\le e^2 \max_{i\le d}\|a_iX_i\|_p.$$

Chebyshev's inequality implies, for $u > 0$,

$$\mathbb{P}(|a_iX_i| \ge u) \le u^{-q}\|a_iX_i\|_q^q \le (C_1M/u)^q \quad \text{for } i \notin I_q.$$

Hence for $u \ge 2$,

$$\begin{aligned}\mathbb{P}\Big(\max_{i\notin I_{2p}}|a_iX_i| \ge ue^2C_1M\Big) &\le \sum_{i\in I_\infty\setminus I_{2p}} \mathbb{P}(|a_iX_i| \ge ue^2C_1M)\\ &\le \sum_{k=1}^\infty \sum_{i\in I_{2^{k+1}p}\setminus I_{2^kp}} \mathbb{P}(|a_iX_i| \ge ue^2C_1M)\\ &\le |I_{2^{k+1}p}|\Big(\frac{C_1M}{ue^2C_1M}\Big)^{2^kp} \le \sum_{k=1}^\infty u^{-2^kp} \le 2u^{-2p}.\end{aligned}$$

Integration by parts yields

$$\Big(\mathbb{E}\max_{i\notin I_{2p}}|a_iX_i|^p\Big)^{1/p} \le CM = C\,\mathbb{E}\max_{i\le d}|a_iX_i|. \ \blacksquare$$

4. Unconditional case. In this section we study the Sudakov minoration principle for unconditional log-concave vectors X. A random d-dimensional vector $X=(X_1,\ldots,X_d)$ is called *unconditional* if the vector $(\eta_1 X_1,\ldots,\eta_d X_d)$ has the same distribution as X for any choice of signs $\eta_1,\ldots,\eta_d\in\{-1,1\}$.

Since this is only a matter of normalization, we will also assume that X is isotropic, which in this case means that $\mathbb{E}\,X_i^2=1$ for $i=1,\ldots,d$.

We will denote by ε_i a *Bernoulli sequence*, i.e. a sequence of i.i.d. symmetric ±1 r.v.'s; we will also assume that the variables $(\varepsilon_i)_i$ are independent of X. Let $(\mathcal{E}_i)$ denote a sequence of independent symmetric exponential r.v.'s with variance 1 (i.e. with density $\frac{1}{\sqrt{2}}\exp(-\sqrt{2}\,|x|)$).

The next lemma shows that the vectors $(\varepsilon_i)_{i\le d}$ and $(\mathcal{E}_i)_{i\le d}$ are in a sense extremal in the class of d-dimensional unconditional isotropic log-concave vectors.

Lemma 4.1. *Let X be an isotropic unconditional log-concave vector.*

(i) *For any $t\in\mathbb{R}^d$ and $p\ge1$,*

$$\frac{1}{\sqrt{2}}\Big\|\sum_{i=1}^d t_i\varepsilon_i\Big\|_p\le\Big\|\sum_{i=1}^d t_iX_i\Big\|_p\le 2\sqrt{6}\,\Big\|\sum_{i=1}^d t_i\mathcal{E}_i\Big\|_p. \tag{3}$$

(ii) *For any nonempty bounded set $T\subset\mathbb{R}^d$ we have*

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^d t_i\varepsilon_i\le\sqrt{2}\,\mathbb{E}\sup_{t\in T}\sum_{i=1}^d t_iX_i. \tag{4}$$

Moreover for any $\emptyset\ne I\subset\{1,\ldots,d\}$,

$$\mathbb{E}\sup_{t\in T}\sum_{i\in I}t_i\mathcal{E}_i\le C_2\log(|I|+1)\,\mathbb{E}\sup_{t\in T}\sum_{i\in I}t_iX_i. \tag{5}$$

Proof. (i) By Lemma 2.3, Jensen's inequality and unconditionality of X,

$$\frac{1}{\sqrt{2}}\Big\|\sum_{i=1}^d t_i\varepsilon_i\Big\|_p\le\Big\|\sum_{i=1}^d t_i\varepsilon_i\,\mathbb{E}\,|X_i|\Big\|_p\le\Big\|\sum_{i=1}^d t_i\varepsilon_i|X_i|\Big\|_p=\Big\|\sum_{i=1}^d t_iX_i\Big\|_p.$$

On the other hand, the result of Bobkov–Nazarov [4] and integration by parts give

$$\Big\|\sum_{i=1}^d t_iX_i\Big\|_{2k}\le\sqrt{6}\,\Big\|\sum_{i=1}^d t_i\mathcal{E}_i\Big\|_{2k}\quad\text{for } k=1,2,\ldots,$$

and the upper bound in (3) follows by Lemma 2.3.

(ii) Inequality (4) can be proven in a similar way to the lower bound in (3). To finish the proof observe that

$$\mathbb{E}\sup_{t\in T}\sum_{i\in I} t_i\mathcal{E}_i = \mathbb{E}\sup_{t\in T}\sum_{i\in I} t_i\varepsilon_i|\mathcal{E}_i| \le \mathbb{E}\max_{i\in I}|\mathcal{E}_i|\,\mathbb{E}\sup_{t\in T}\sum_{i\in I} t_i\varepsilon_i$$
$$\le C\log(|I|+1)\,\mathbb{E}\sup_{t\in T}\sum_{i\in I} t_i\varepsilon_i,$$

thus (5) follows by (4). ∎

The next result easily follows by comparing unconditional vectors with the exponential random vector $\mathcal{E} = (\mathcal{E}_i)_{i\le d}$.

PROPOSITION 4.2. *Suppose that X is a d-dimensional log-concave unconditional vector. Then X satisfies* $\mathrm{SMP}(1/(C\log(d+1)))$.

Proof. Recall that we may assume that X is isotropic. Let $p \ge 1$ and T be a set in $\mathbb{R}^d$ with cardinality at least e^p such that (1) holds. Then by (3), for distinct $t, s \in T$, we have $\|\langle t-s, \mathcal{E}\rangle\|_p \ge A/(2\sqrt{6})$, where $\mathcal{E} = (\mathcal{E}_i)_{i\le d}$. We know (see Example 1.5) that $\mathcal{E}$ satisfies the Sudakov minoration principle with a universal constant, thus by (5) we have

$$\frac{A}{C} \le \mathbb{E}\sup_{t,s\in T}\langle t-s, \mathcal{E}\rangle \le C_2\log(d+1)\,\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle. \ \blacksquare$$

We are now ready to present the main result of this section. Its proof is also based on comparison ideas, but in a less straightforward way.

THEOREM 4.3. *Let X be a log-concave unconditional vector in $\mathbb{R}^d$, $p\ge 1$ and $T\subset\mathbb{R}^d$ be such that $|T|\ge e^{p^2}$ and $\|\langle t-s, X\rangle\|_p \ge A$ for distinct points $t, s\in T$. Then*

$$\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle = 2\,\mathbb{E}\sup_{t\in T}\langle t, X\rangle \ge \frac{1}{C}A.$$

Proof. Again we assume that X is isotropic. By Remark 2.4 we may assume that $p\ge 2$. Observe also that if $0\in T$ then $\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle \ge \mathbb{E}\sup_{t\in T}|\langle t, X\rangle|$, so for such T it is enough to show that

$$\mathbb{E}\sup_{t\in T}|\langle t, X\rangle| \ge \frac{1}{C}A. \tag{6}$$

We divide the proof into three steps. In the first two steps we show that we may add additional assumptions on the set T (slightly decreasing its cardinality and rescaling A by a universal constant).

STEP 1. We may assume that $0\in T$, $|T|\ge e^{p^2-p}$ and

$$\Big\|\sum_{i=1}^d t_i\varepsilon_i\Big\|_p \le \delta A \quad \text{for all } t\in T, \tag{7}$$

where $\delta>0$ is a positive universal constant (to be chosen later).

Let $d_p(t,s)=\|\sum_{i=1}^d(t_i-s_i)\varepsilon_i\|_p$. By the result of Talagrand (see Example 1.4) we know that if $N(T,d_p,\alpha)\geq e^p$ then $\mathbb{E}\sup_{t,s\in T}\sum_{i=1}^d(t_i-s_i)\varepsilon_i\geq(1/C)\alpha$. Thus using (4) we may assume that $N(T,d_p,\delta A/2)\leq e^p$; this means that there exists $t^0\in T$ such that

$$|\{t\in T: d_p(t,t^0)\leq\delta A\}|\geq|T|/e^p.$$

So we may consider the new set $T'=\{t-t^0 : d_p(t,t^0)\leq\delta A\}$.

Step 2. We may assume (changing A to $A/2$) that $0\in T$, $|T|\geq e^{p^2-p}$, (7) holds and

$$|\mathrm{supp}(t)|\leq p \quad \text{for all } t\in T. \tag{8}$$

By Step 1 we may assume that $0\in T$ and (7) holds. The result of Hitczenko [9] gives

$$\Big\|\sum_{i=1}^d t_i\varepsilon_i\Big\|_p\geq\frac{1}{C_3}\Big(\sum_{i\leq p}t_i^*+\sqrt{p}\Big(\sum_{i>p}|t_i^*|^2\Big)^{1/2}\Big), \tag{9}$$

where t_i^* denotes the nonincreasing rearrangement of $(|t_i|)$.

Let us define $\varphi(x):=\mathrm{sgn}(x)(|x|-C_3\delta A/p)_+$ for $x\in\mathbb{R}$ and let $\varphi(t):=(\varphi(t_i))$ for $t\in\mathbb{R}^d$. Then (7) and (9) imply that $|\mathrm{supp}(\varphi(t))|\leq p$ for $t\in T$. The upper bound in (3) and the Gluskin–Kwapień estimate [8] yield

$$\|\langle t,X\rangle\|_p\leq 2\sqrt{6}\,\|\langle t,\mathcal{E}\rangle\|_p\leq C_4(p\|t\|_\infty+\sqrt{p}\,\|t\|_2). \tag{10}$$

Thus if $\delta\leq 1/(12C_3C_4)$ we get, for $t\in T$,

$$\begin{aligned}\|\langle t-\varphi(t),X\rangle\|_p&\leq C_4\big(p\|t-\varphi(t)\|_\infty+\sqrt{p}\,\|t-\varphi(t)\|_2\big)\\&\leq C_4\Big(2p\|t-\varphi(t)\|_\infty+\sqrt{p}\,\Big(\sum_{i>p}|t_i^*|^2\Big)^{1/2}\Big)\\&\leq C_3C_4\Big(2\delta A+\Big\|\sum_{i=1}^d t_i\varepsilon_i\Big\|_p\Big)\leq 3C_3C_4\delta A\leq\frac{A}{4}.\end{aligned}$$

Hence for any $t,s\in T$, $t\neq s$, we have

$$\|\langle\varphi(t)-\varphi(s),X\rangle\|_p\geq\|\langle t-s,X\rangle\|_p-2\frac{A}{4}\geq\frac{A}{2}.$$

Moreover the contraction principle for Rademacher processes (see [16, Theorem 4.12]) and the unconditionality of X yield

$$\mathbb{E}\sup_{t\in T}|\langle t,X\rangle|\geq\frac{1}{2}\,\mathbb{E}\sup_{t\in T}|\langle\varphi(t),X\rangle|,$$

so it is enough to prove estimate (6) for the set $\varphi(T)=(\varphi(t))_{t\in T}$. Note that condition (7) holds for $\varphi(T)$ since it holds for T and $|\varphi(t_i)|\leq|t_i|$ for all i.

STEP 3. We consider a finite set T such that $0 \in T$, $|T| \geq e^{p^2-p} \geq e^{p^2/2}$, $\|\langle t-s, X\rangle\|_p \geq A$ for distinct $t,s \in T$, and conditions (7)–(8) hold. To finish the proof it is enough to show (6).

To this end we construct inductively points $t_1,\ldots,t_N$. For t_1 we take any point in T. Suppose that $t_1,\ldots,t_n$ have been constructed. We set

$$I_n := \bigcup_{k\leq n} \operatorname{supp}(t_k), \qquad J_n := \{1,\ldots,d\} \setminus I_n$$

and

$$T_n := \{t \in T : \|\langle t_{J_n}, X\rangle\|_p \geq A/4\},$$

where for $I \subset \{1,\ldots,d\}$ we write $t_I := (t_i \mathbb{1}_{\{i\in I\}})$. If T_n is nonempty, we pick for t_{n+1} any point in this set, otherwise we finish the construction and set $n = N$, $I = I_N$, $J = J_N$.

We distinguish two possibilities.

CASE I: $N \leq e^p$. Then $|I| \leq \sum_{i=1}^N |\operatorname{supp}(t_k)| \leq Np \leq e^{2p}$. Observe that then for any distinct $t,s \in T$,

$$\|\langle t_I - s_I, X\rangle\|_p \geq \|\langle t-s, X\rangle\|_p - \|\langle t_J, X\rangle\|_p - \|\langle s_J, X\rangle\|_p \geq \frac{A}{2}.$$

Thus by (10), (9) and (7),

$$\begin{aligned}\frac{A}{2} &\leq \|\langle t_I - s_I, X\rangle\|_p \leq C_4\big(p\|t_I - s_I\|_\infty + \sqrt{p}\,\|t_I - s_I\|_2\big)\\ &\leq C_4\Big(2p\|t_I - s_I\|_\infty + C_3\Big\|\sum_{i\in I} t_i\varepsilon_i\Big\|_p\Big) \leq C_3C_4(2p\|t_I - s_I\|_\infty + \delta A),\end{aligned}$$

therefore if $\delta \leq 1/(4C_3C_4)$ then $\|t_I - s_I\|_\infty \geq A/(8C_3C_4p)$. By the Gluskin–Kwapień estimate [8] we get

$$\|\langle t_I - s_I, \mathcal{E}\rangle\|_{p^2/2} \geq \frac{p^2}{C}\|t_I - s_I\|_\infty \geq \frac{pA}{C},$$

where $\mathcal{E} = (\mathcal{E}_i)_{i\leq d}$. Since $\mathcal{E}$ satisfies the Sudakov minoration principle with a uniform constant (see Example 1.5) and $|T| \geq e^{p^2/2}$, we obtain

$$2\,\mathbb{E}\sup_{t\in T} |\langle t_I, \mathcal{E}\rangle| \geq \mathbb{E}\sup_{t\in T} \langle t_I - s_I, \mathcal{E}\rangle \geq \frac{pA}{C}.$$

Thus by (5) we have

$$\frac{pA}{C} \leq \mathbb{E}\sup_{t\in T} |\langle t_I, \mathcal{E}\rangle| \leq C_2 \log(|I|+1)\,\mathbb{E}\sup_{t\in T} |\langle t_I, X\rangle| \leq Cp\,\mathbb{E}\sup_{t\in T} |\langle t, X\rangle|.$$

CASE II: $N \geq e^p$. Let $I_0 = \emptyset$, $\Delta_k := I_k \setminus I_{k-1}$ and $s_k := t_{k,\Delta_k}$ for $k = 1,\ldots,N$. Then by our construction the vectors s_k have disjoint supports and $\|\langle s_k, X\rangle\|_p \geq A/4$ for $k = 1,\ldots,N$. Thus by Proposition 3.3 (applied with $V = A/4$, $a_i = |s_i|$ and isotropic vector $(\langle s_i, X\rangle/|s_i|)_{i\le N})$ we get

$$\mathbb{E}\max_{k\le N}|\langle s_k, X\rangle| \ge \frac{A}{4C_1}.$$

Since the sets Δ_k are disjoint and the vector X is unconditional, we get

$$\mathbb{E}\max_{t\in T}|\langle t, X\rangle| \ge \frac{1}{2}\,\mathbb{E}\max_{t\in T}\max_k|\langle t_{\Delta_k}, X\rangle| \ge \frac{1}{2}\,\mathbb{E}\max_{k\le N}|\langle s_k, X\rangle| \ge \frac{A}{8C_1}. \ \blacksquare$$

REMARK 4.4. Following Remark 2.5 we may restate Theorem 4.3 in terms of covering numbers—for any log-concave unconditional vector X, any nonempty $T \subset \mathbb{R}^d$ and $A > 0$,

$$\mathbb{E}\sup_{t,s\in T}\langle t-s, X\rangle = 2\,\mathbb{E}\sup_{t\in T}\langle t, X\rangle \ge \frac{1}{C}\sup_{p\ge 1}\min\bigg\{\frac{A}{p}\sqrt{\log N(T, d_{X,p}, A)}, A\bigg\}.$$

5. Invariant log-concave vectors. In this section we investigate the class of invariant log-concave vectors. The first result shows that pth moments of norms of such vectors are almost constant for $p \le d$.

PROPOSITION 5.1. *Let K be a symmetric convex body in $\mathbb{R}^d$ and X be a random d-dimensional vector with a density of the form $e^{-\varphi(\|x\|_K)}$, where $\varphi\colon [0,\infty) \to (-\infty,\infty]$ is a nondecreasing convex function. Then*

$$(\mathbb{E}\,\|X\|_K^d)^{1/d} \le C_5\,\mathrm{Med}(\|X\|_K).$$

Proof. Let μ denote the law of X and $m := \mathrm{Med}(\|X\|_K)$. Then

$$\frac{1}{2} = \mu(mK) = \int_{mK} e^{-\varphi(\|x\|_K)}\,dx \le \int_{mK} e^{-\varphi(0)}\,dx = e^{-\varphi(0)}m^d\,\mathrm{vol}(K)$$

and

$$\begin{aligned}1 \ge \mu(2emK) &= \int_{2emK} e^{-\varphi(\|x\|_K)}\,dx \ge \int_{2emK} e^{-\varphi(2em)}\,dx\\ &= e^{-\varphi(2em)}(2em)^d\,\mathrm{vol}(K).\end{aligned}$$

Therefore

$$e^{\varphi(2em)-\varphi(0)} \ge \frac{1}{2}(2e)^d \ge e^d.$$

Convexity of φ implies that

$$\varphi(r) - \varphi(m) \ge \frac{d}{2em}(r-m) \quad \text{for } r \ge 2em. \tag{11}$$

Integrating in polar-type coordinates we get

$$\begin{aligned}\frac{1}{2} = \mu(mK) &= c_K\int_0^m e^{-\varphi(r)}r^{d-1}\,dr \ge c_K\int_0^m e^{-\varphi(m)}r^{d-1}\,dr\\ &= \frac{c_K}{d}m^d e^{-\varphi(m)},\end{aligned}$$

where $c_K := d\,\mathrm{vol}(K)$. Hence for $s \geq 0$,

$$\mathbb{P}(\|X\|_K \geq sm) = \mu(\mathbb{R}^d \setminus smK) = c_K \int_{sm}^{\infty} e^{-\varphi(r)} r^{d-1}\,dr$$
$$\leq \frac{d}{2} m^{-d} \int_{sm}^{\infty} e^{\varphi(m)-\varphi(r)} r^{d-1}\,dr.$$

Using (11) we get

$$\mathbb{P}(\|X\|_K \geq sm) \leq \frac{d}{2} m^{-d} e^{\frac{d}{2e}} \int_{sm}^{\infty} e^{-\frac{dr}{2em}} r^{d-1}\,dr \quad \text{for } s \geq 2e.$$

The function $r \mapsto e^{-\frac{dr}{4em}} r^{d-1}$ is decreasing for $r \geq 4\frac{d-1}{d}em$, thus

$$e^{-\frac{dr}{2em}} r^{d-1} \leq e^{-d}(4em)^{d-1} e^{-\frac{dr}{4em}} \quad \text{for } r \geq 4em.$$

Hence for $s \geq 4e$,

$$\mathbb{P}(\|X\|_K \geq sm) \leq \frac{d}{2}(em)^{-d}(4em)^{d-1} e^{\frac{d}{2e}} \int_{sm}^{\infty} e^{-\frac{dr}{4em}}\,dr = \frac{1}{2} e^{\frac{d}{2e}} 4^d e^{-\frac{sd}{4e}}.$$

Integrating by parts we get

$$\mathbb{E}\,\|X\|_K^d \leq (4em)^d + dm^d \int_{4e}^{\infty} s^{d-1}\,\mathbb{P}(\|X\|_K \geq sm)\,ds$$
$$\leq (4em)^d + \frac{d}{2} e^{\frac{d}{2e}} (4m)^d \int_{0}^{\infty} s^{d-1} e^{-\frac{sd}{4e}}\,ds$$

and it easily follows that $(\mathbb{E}\,\|X\|_K^d)^{1/d} \leq C_5 m$. ∎

It turns out that the Sudakov minoration property holds with almost the same constant for all vectors in the same class of invariant log-concave vectors.

THEOREM 5.2. *Let X^i, $i = 1, 2$, be d-dimensional random vectors with densities of the form $e^{-\varphi_i(\|x\|_K)}$, where K is a symmetric convex body in $\mathbb{R}^d$ and $\varphi_i\colon [0,\infty) \to (-\infty,\infty]$ are nondecreasing convex functions. If X^1 satisfies* $\mathrm{SMP}(\kappa)$ *then X^2 satisfies* $\mathrm{SMP}(\kappa/C_6)$.

Proof. Observe that X^i has the same distribution as $R_i Y$, where Y is uniformly distributed on K and R_i are nonnegative r.v's independent of Y. We have $\|X^i\|_K \leq R_i$, in particular $\mathbb{E}\,R_i \geq \frac{1}{2}\,\mathrm{Med}(R_i) \geq \frac{1}{2}\,\mathrm{Med}(\|X^i\|_K)$. Moreover

$$(\mathbb{E}\,\|X^i\|_K^d)^{1/d} = (\mathbb{E}\,R_i^d)^{1/d} (\mathbb{E}\,\|Y\|_K^d)^{1/d} \geq \frac{1}{2}(\mathbb{E}\,R_i^d)^{1/d}.$$

Therefore Proposition 5.1 implies

$$\mathbb{E}\,R_i \leq \|R_i\|_p \leq \|R_i\|_d \leq 4C_5\,\mathbb{E}\,R_i \quad \text{for } 1 \leq p \leq d.$$

We need to show that X^2 satisfies $\mathrm{SMP}_p(\kappa/C)$ for $p \geq 1$. By Lemma 2.6 it is enough to consider only $p \leq d$. Since it is a matter of scaling, we may assume that $\mathbb{E}\, R_i = 1$ for $i = 1, 2$. For any $T \subset \mathbb{R}^d$ we then have

$$\mathbb{E} \sup_{t,s\in T} \langle t - s, X^i \rangle = \mathbb{E}\, R_i\, \mathbb{E} \sup_{t,s\in T} \langle t - s, Y \rangle = \mathbb{E} \sup_{t,s\in T} \langle t - s, Y \rangle.$$

Moreover for $p \geq 1$,

$$\|\langle u, X^i \rangle\|_p = \|R_i\|_p \|\langle u, Y \rangle\|_p \quad \text{for any } u \in \mathbb{R}^n.$$

Fix $1 \leq p \leq d$ and take $T \subset \mathbb{R}^d$ with $|T| \geq e^p$ such that $\|\langle t-s, X^2\rangle\|_p \geq A$ for all $t, s \in T$, $t \neq s$. Then $\|\langle t-s, X^1\rangle\|_p \geq \frac{1}{4C_5} A$ for distinct points $t, s \in T$, and SMP for X^1 yields

$$\mathbb{E} \sup_{t,s\in T} \langle t - s, X^2 \rangle = \mathbb{E} \sup_{t\in T} \langle t - s, X^1 \rangle \geq \frac{\kappa}{4C_5} A. \ \blacksquare$$

As a corollary we show that a large class of invariant log-concave vectors satisfy SMP with a universal constant.

Corollary 5.3. *All d-dimensional random vectors with densities of the form $\exp(-\varphi(\|x\|_p))$, where $1 \leq p \leq \infty$ and $\varphi\colon [0,\infty) \to (-\infty, \infty]$ is nondecreasing and convex, satisfy the Sudakov minoration principle with a universal constant. In particular all rotation invariant log-concave random vectors satisfy the Sudakov minoration principle with a universal constant.*

Proof. We apply Theorem 5.2 with $X^2 = X$ and X^1 having a density of the form $c_p^d \exp(-\varphi_p(\|x\|_p))$, where $\varphi_p(r) = r^p$ for $p < \infty$ and $\varphi_\infty = \infty \mathbb{1}_{[1,\infty)}$. Note that X^1 is log-concave with a product density so it satisfies SMP with a universal constant (see Example 1.5). $\blacksquare$

6. Applications. In the last section we apply chaining techniques to establish properties of vectors satisfying SMP. Before we formulate our results we state a simple general estimate for moments of suprema of stochastic processes based on chaining.

Proposition 6.1. *Let $(X_t)_{t\in T}$ be a stochastic process and $(T_k)_{0\leq k\leq k_1}$ be a sequence of subsets of T such that $|T_k| \leq e^{2^{k+1}}$ for $0 \leq k \leq k_1$ and $T_{k_1} = T$. Moreover suppose that $\pi_k : T \to T_k$ for $0 \leq k \leq k_1$ and $\pi_{k_1}(t) = t$ for all $t \in T$. Then for any $1 \leq k_0 \leq k_1 - 1$ and $2^{k_0-1} \leq p \leq 2^{k_0}$,*

$$\Big\|\sup_{t\in T_k} |X_t|\Big\|_p \leq 3e^3 \Big(\sup_{t\in T} \sum_{k=k_0+1}^{k_1} \|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\|_{2^k} + \sup_{t\in T_{k_0}} \|X_t\|_p\Big).$$

Proof. Define

$$m(l) := \sup_{t\in T} \sum_{k=l+1}^{k_1} \|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\|_{2^k}.$$

Then for $u \ge 2e^3$,

$$
\begin{aligned}
(12)\qquad & \mathbb{P}\Big(\sup_{t\in T} |X_t - X_{\pi_{k_0}(t)}| \ge u m(k_0)\Big) \\
&\le \mathbb{P}\Big(\sup_{t\in T} \sum_{k=k_0+1}^{k_1} |X_{\pi_k(t)} - X_{\pi_{k-1}(t)}| \ge u m(k_0)\Big) \\
&\le \mathbb{P}\big(\exists_{k_0+1\le k\le k_1}\ \exists_{t\in T}\ |X_{\pi_k(t)} - X_{\pi_{k-1}(t)}| \ge u\|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\|_{2^k}\big) \\
&\le \sum_{k=k_0+1}^{k_1} \sum_{s\in T_k} \sum_{s'\in T_{k-1}} \mathbb{P}(|X_s - X_{s'}| \ge u\|X_s - X_{s'}\|_{2^k}) \\
&\le \sum_{k=k_0+1}^{k_1} |T_k|\,|T_{k-1}| u^{-2^k} \le \sum_{k=k_0+1}^{k_1} \Big(\frac{e^3}{u}\Big)^{2^k} \\
&\le 2\Big(\frac{e^3}{u}\Big)^{2^{k_0+1}} \le 2\Big(\frac{e^3}{u}\Big)^{2p}.
\end{aligned}
$$

Hence integrating by parts we get

$$
\begin{aligned}
\mathbb{E}\sup_{t\in T} |X_t - X_{\pi_{k_0}(t)}|^p &\le (e^3 m(k_0))^p \Big(2^p + p\int_2^\infty u^{p-1} 2u^{-2p}\,du\Big) \\
&= (2e^3)^p(1+2^{1-2p}) m(k_0)^p \le (3e^3 m(k_0))^p.
\end{aligned}
$$

Moreover

$$
\mathbb{E}\sup_{t\in T} |X_{\pi_{k_0}(t)}|^p \le \sum_{t\in T_{k_0}} \mathbb{E}\,|X_t|^p \le |T_{k_0}| \sup_{t\in T_{k_0}} \|X_t\|_p^p \le e^{4p} \sup_{t\in T_{k_0}} \|X_t\|_p^p.
$$

Hence

$$
\begin{aligned}
\Big\|\sup_{t\in T} |X_t|\Big\|_p &\le \Big\|\sup_{t\in T} |X_t - X_{\pi_{k_0}(t)}|\Big\|_p + \Big\|\sup_{t\in T} |X_{\pi_{k_0}(t)}|\Big\|_p \\
&\le 3e^3\Big(m(k_0) + \sup_{t\in T_{k_0}} \|X_t\|_p\Big). \ \blacksquare
\end{aligned}
$$

REMARK 6.2. If $2^{k_0-1} \le p \le 2^{k_0}$ but $k_0 \ge k_1$, then $p \ge 2^{k_1-1}$ and

$$
\Big\|\sup_{t\in T} |X_t|\Big\|_p \le |T|^{1/p} \sup_{t\in T} \|X_t\|_p \le e^4 \sup_{t\in T} \|X_t\|_p.
$$

Now we show that if weak moments of a random vector X with the SMP property dominate weak moments of another random vector Y then strong moments of X dominate strong moments of Y up to a logarithmic factor.

PROPOSITION 6.3. *Suppose a random vector X in $\mathbb{R}^d$ satisfies* SMP(κ). *Let Y be a random d-dimensional vector such that $\|\langle t, Y\rangle\|_p \le \|\langle t, X\rangle\|_p$ for* *all $p \geq 1$ and $t \in \mathbb{R}^d$. Then for any norm $\|\ \|$ on $\mathbb{R}^d$ and $p \geq 1$,*

$$
\begin{aligned}
(13)\qquad (\mathbb{E}\,\|Y\|^p)^{1/p} &\leq C\biggl(\frac{1}{\kappa}\log_+\Bigl(\frac{ed}{p}\Bigr)\,\mathbb{E}\,\|X\| + \sup_{\|t\|_*\leq 1}(\mathbb{E}\,|\langle t, Y\rangle|^p)^{1/p}\biggr)\\
&\leq C\biggl(\frac{1}{\kappa}\log_+\Bigl(\frac{ed}{p}\Bigr) + 1\biggr)(\mathbb{E}\,\|X\|^p)^{1/p}.
\end{aligned}
$$

Proof. Let T be a $1/2$-net in $B_{\|\ \|_*} := \{t \in \mathbb{R}^d : \|t\|_* \leq 1\}$ of cardinality at most 5^d. Then $\|x\| \leq 2\max_{t\in T}\langle t, x\rangle$ for any $x \in \mathbb{R}^d$. For $q \geq 1$ choose a maximal set $S_q \subset T$ such that $\|\langle t-s, X\rangle\|_q > \frac{2}{\kappa}\,\mathbb{E}\,\|X\|$ for all distinct $t, s \in S_q$. Since X satisfies $\mathrm{SMP}(\kappa)$ and

$$
\mathbb{E}\max_{t,s\in S_q}\langle t-s, X\rangle \leq \mathbb{E}\sup_{\|u\|_*\leq 2}\langle u, X\rangle = 2\,\mathbb{E}\,\|X\|,
$$

we get $|S_q| < e^q$.

Let k_1 be the smallest integer such that $2^{k_1+1} \geq d\log 5$. Set $T_{k_1} := T$ and $T_k := S_{2^{k+1}}$ for $0 \leq k \leq k_1$. Let $\tilde{\pi}_k\colon T \to S_{2^{k+1}}$ be such that for any $t \in T$,

$$
\|\langle t - \tilde{\pi}_k(t), X\rangle\|_{2^{k+1}} \leq \frac{2}{\kappa}\,\mathbb{E}\,\|X\|.
$$

Define maps $\pi_k\colon T \to T_k$ by $\pi_{k_1}(t) = t$ and for $0 \leq k < k_1$ by $\pi_k := \tilde{\pi}_k \circ \tilde{\pi}_{k+1} \circ \cdots \circ \tilde{\pi}_{k_1-1}$. Let k_0 be the smallest positive integer such that $2^{k_0} \geq p$.

If $k_0 \leq k_1 - 1$, we may apply Proposition 6.1 to $X_t := \langle t, Y\rangle$. Hence

$$
\begin{aligned}
(\mathbb{E}\,\|Y\|^p)^{1/p} &\leq 2\Bigl\|\max_{t\in T}|\langle t, Y\rangle|\Bigr\|_p\\
&\leq 6e^3\Bigl(\sup_{t\in T}\sum_{k=k_0+1}^{k_1}\|\langle \pi_k(t) - \pi_{k-1}(t), Y\rangle\|_{2^k} + \sup_{t\in T_{k_0}}\|\langle t, Y\rangle\|_p\Bigr).
\end{aligned}
$$

To show the first inequality in (13) it is enough to notice that $k_1 - k_0 \leq C\log(ed/p)$ and for any $1 \leq k \leq k_1$ we have

$$
\begin{aligned}
\sup_{t\in T}\|\langle \pi_k(t) - \pi_{k-1}(t), Y\rangle\|_{2^k} &\leq \sup_{t\in T}\|\langle \pi_k(t) - \pi_{k-1}(t), X\rangle\|_{2^k}\\
&\leq \sup_{t\in T}\|\langle t - \tilde{\pi}_{k-1}(t), X\rangle\|_{2^k} \leq \frac{2}{\kappa}\,\mathbb{E}\,\|X\|.
\end{aligned}
$$

If $k_0 \geq k_1$, we use Remark 6.2 instead of Proposition 6.1.

The second inequality in (13) follows since

$$
\sup_{t\in T_{k_0}}\|\langle t, Y\rangle\|_p \leq \sup_{\|t\|_*\leq 1}\|\langle t, X\rangle\|_p \leq (\mathbb{E}\,\|X\|^p)^{1/p}. \ \blacksquare
$$

The next corollary states that weak and strong moments of random vectors with the SMP property are comparable up to a logarithmic factor.

COROLLARY 6.4. *Suppose that X is a d-dimensional random vector which satisfies* SMP(κ). *Then for any norm $\| \ \|$ on $\mathbb{R}^d$ and any $p \geq 1$,*

$$(\mathbb{E}\,\|X\|^p)^{1/p} \leq C\bigg(\frac{1}{\kappa}\log_+\bigg(\frac{ed}{p}\bigg)\,\mathbb{E}\,\|X\| + \sup_{\|t\|_*\leq 1}(\mathbb{E}\,|\langle t, X\rangle|^p)^{1/p}\bigg).$$

Proof. We apply Proposition 6.3 with $Y = X$. ∎

The last result shows that for a class of invariant vectors we may eliminate logarithmic factors.

PROPOSITION 6.5. *Let X be a d-dimensional random vector with a density of the form $e^{-\varphi(\|x\|_r)}$, where $1 \leq r < \infty$ and $\varphi\colon [0,\infty) \to (-\infty,\infty]$ is nondecreasing and convex. Let Y be a random vector in $\mathbb{R}^d$ such that $\|\langle t, Y\rangle\|_p \leq \|\langle t, X\rangle\|_p$ for $p \geq 1$. Then for any norm $\| \ \|$ on $\mathbb{R}^d$ and $p \geq 1$,*

$$(\mathbb{E}\,\|Y\|^p)^{1/p} \leq C(r)\,\mathbb{E}\,\|X\| + C\sup_{\|t\|_*\leq 1}(\mathbb{E}\,|\langle t, Y\rangle|^p)^{1/p} \leq C'(r)(\mathbb{E}\,\|X\|^p)^{1/p},$$

where $C(r)$ and $C'(r)$ depend only on r. In particular for any norm $\| \ \|$ on $\mathbb{R}^d$ and any $p \geq 1$,

$$(\mathbb{E}\,\|X\|^p)^{1/p} \leq C(r)\,\mathbb{E}\,\|X\| + C\sup_{\|t\|_*\leq 1}(\mathbb{E}\,|\langle t, X\rangle|^p)^{1/p}.$$

Proof. Let $\tilde{X}$ have a density of the form $c_p^d\exp(-\|x\|_p^p)$. Then $\tilde{X}$ has independent coordinates. We have $X = RZ$ and $\tilde{X} = \tilde{R}Z$, where Z is uniformly distributed on B_p^d and $R, \tilde{R}$ are nonnegative random variables independent of Z. Since it is only a matter of normalization, we may assume that $\mathbb{E}\,R = \mathbb{E}\,\tilde{R}$. Then $\mathbb{E}\,\|X\| = \mathbb{E}\,R\,\mathbb{E}\,\|Z\| = \mathbb{E}\,\tilde{R}\,\mathbb{E}\,\|Z\| = \mathbb{E}\,\|\tilde{X}\|$. Moreover, Proposition 5.1 easily implies (see the proof of Theorem 5.2) that for $1 \leq p \leq d$, we have $\|R\|_p \leq \|R\|_d \leq 4C_5\|R\|_1 = 4C_5\|\tilde{R}\|_1 \leq 4C_5\|\tilde{R}\|_p$. Thus for any $t \in \mathbb{R}^d$ and $1 \leq p \leq d$,

$$\|\langle t, X\rangle\|_p = \|R\|_p\|\langle t, Z\rangle\|_p \leq 4C_5\|\tilde{R}\|_p\|\langle t, Z\rangle\|_p = 4C_5\|\langle t, \tilde{X}\rangle\|_p.$$

For $t \in \mathbb{R}^d$ and $d \leq p \leq d\log 5$, by Lemma 2.3 we get

$$\|\langle t, X\rangle\|_p \leq (\log 5)\|\langle t, X\rangle\|_d \leq 2C_5(\log 5)\|\langle t, \tilde{X}\rangle\|_d \leq 2C_5(\log 5)\|\langle t, \tilde{X}\rangle\|_p.$$

Therefore we have

$$\|\langle t, Y\rangle\|_p \leq \|\langle t, X\rangle\|_p \leq 2C_5(\log 5)\|\langle t, \tilde{X}\rangle\|_p \qquad \text{for } 1 \leq p \leq d\log 5,\ t \in \mathbb{R}^d. \tag{14}$$

Let T be a $1/2$-net in $B_{\| \ \|_*}$ of cardinality at most 5^d. Let k_1 be the smallest integer such that $e^{2^{k_1+1}} \geq 5^d$. By the result of Talagrand [21] we may find sets $T_k \subset T$, $0 \leq k \leq k_1$, and maps $\pi_k\colon T \to T_k$ such that $T_{k_1} = T$,

$|T_k| \le e^{2^{k+1}}$, $\pi_{k_1}(t) = t$ for $t \in T$ and

$$\sum_{k=1}^{k_1} \|\langle \pi_k(t) - \pi_{k-1}(t), \tilde{X}\rangle\|_{2^k} \le \frac{1}{2} C(r)\, \mathbb{E} \sup_{t\in T} \langle t, \tilde{X}\rangle$$
$$\le C(r)\, \mathbb{E}\, \|\tilde{X}\| = C(r)\, \mathbb{E}\, \|X\|.$$

We may now proceed as in the proof of Proposition 6.3 observing that by (14) we have

$$\|\langle \pi_k(t) - \pi_{k-1}(t), Y\rangle\|_{2^k} \le 2C_5(\log 5)\|\langle \pi_k(t) - \pi_{k-1}(t), \tilde{X}\rangle\|_{2^k}$$

for $0 \le k \le k_1$. ■

REMARK 6.6. Using the two-sided bound for the expected value of suprema of Bernoulli processes [2] one may show that Proposition 6.5 is also satisfied in the case $r = \infty$.

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