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A contractive fixed point free mapping on a weakly compact convex set

by

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Abstract. We prove the existence of a contractive mapping on a weakly compact convex set in a Banach space that is fixed point free. This answers a long-standing open question.

1. Introduction. In this paper we prove the existence of a contractive and fixed point free mapping on a weakly compact convex subset of the Banach space $L^1[0,1]$ (with its usual norm), which answers a long-standing open question. This work constitutes part of the doctoral dissertation of the third author [Siv].

In 1965 Kirk [K] proved that every nonexpansive mapping U on a weakly compact convex subset C of a Banach space X with *normal structure* has a fixed point, extending the analogous results of Browder [B1, B2] and Göhde [G] for uniformly convex spaces.

For a long time it was unknown if every nonexpansive mapping U on a weakly compact convex subset C of a Banach space X has a fixed point. In 1981 Alspach [A] settled this question by inventing the first example of a nonexpansive mapping T on a weakly compact convex set C in a Banach space X for which T is fixed point free. Alspach's mapping is an isometry, and $X = L^1[0,1]$, with its usual norm. Soon after, Sine [Si] and Schechtman [Sc] invented more of these interesting fixed point free isometries T (again on a weakly compact convex $C \subseteq X =$ an L^1-space, with its usual norm).

It is easy to check that for Alspach's mapping T, $S := (I+T)/2$ is another nonexpansive fixed point free map on C. Moreover, S contracts the distance between some pairs of unequal points and preserves the distance between other such pairs. Further, this fact is true for S when T is Sine's

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map. We thank B. Sims for pointing out to us that this is also true for S when T is any one of Schechtman's mappings.

The question as to whether there exists a *contractive* mapping U (i.e., U contracts the distances between *all* pairs of unequal points) that is fixed point free on a weakly compact convex subset of a Banach space was still open, and remained so until the authors recently resolved it (see Theorems 1.1 and 3.6 below).

We now describe this solution. First, we define the set

$$C_{1/2} = \left\{ f : [0,1] \to [0,1] : \int_0^1 f = \frac{1}{2} \right\}.$$

This set is a weakly compact convex subset of the Lebesgue function space $L^1[0,1]$, with its usual norm $\|\cdot\|_1$. For the rest of this paper, T will stand for Alspach's map as defined in [A]. This map preserves areas in the sense that $\|Tf - Tg\|_1 = \|f - g\|_1$ for all integrable functions $f, g : [0,1] \to [0,1]$. In particular $T : C_{1/2} \to C_{1/2}$. This and other facts about Alspach's mapping were discussed in [A], and also in, for example, Day and Lennard [DL] (where the *minimal invariant sets* of T are characterized).

In this paper we will prove the following theorem.

Theorem 1.1. *The mapping*

$$R : C_{1/2} \to C_{1/2} : f \mapsto \sum_{n=0}^{\infty} \frac{T^n f}{2^{n+1}} = \left(\frac{I}{2} + \frac{T}{4} + \frac{T^2}{8} + \cdots \right)(f)$$

is contractive and fixed point free on $C_{1/2}$.

2. Preliminaries. We denote the set of positive integers and the set of real numbers by $\mathbb{N}$ and $\mathbb{R}$ respectively. Our scalar field is $\mathbb{R}$.

We write "closed bounded convex set" instead of "closed, bounded, convex set". Also, all sets that are the domains of a mapping are assumed to be nonempty.

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space and C be a closed bounded convex subset of X. Let $U : C \to C$ be a mapping.

(1) We say that U is *nonexpansive* if for all $x, y \in C$,

$$\|Ux - Uy\| \le \|x - y\|.$$

(2) We say that U is *contractive* if for all $x, y \in C$ with $x \neq y$,

$$\|Ux - Uy\| < \|x - y\|.$$

We remark in passing that *contractive* mappings U on non-weakly compact, closed bounded convex sets C in a Banach space arise quite often. For example, Maurey [M] showed that every weakly compact convex subset C

in the Banach space c_0 of all scalar sequences that converge to zero, with the usual $\|\cdot\|_\infty$-norm, is such that every nonexpansive map $U : C \to C$ has a fixed point. On the other hand, Dowling, Lennard and Turett [DLT] showed the following converse result: on every non-weakly compact, closed bounded convex set C in $(c_0, \|\cdot\|_\infty)$, there exists a nonexpansive mapping $W : C \to C$ that is fixed point free. Moreover, one may arrange for W to be *contractive*.

Also, recall that Alspach's mapping T is given by: for all integrable functions $f : [0,1] \to [0,1]$,

$$(Tf)(x) = \begin{cases} 2f(2x) \wedge 1, & 0 \le x < 1/2, \\ (2f(2x-1) \vee 1) - 1, & 1/2 \le x < 1. \end{cases}$$

Here, for all $\alpha, \beta \in \mathbb{R}$, $\alpha \wedge \beta := \min\{\alpha, \beta\}$ and $\alpha \vee \beta := \max\{\alpha, \beta\}$.

3. Proof of the main theorem. First, let us confirm that R maps $C_{1/2}$ back into $C_{1/2}$. Fix an arbitrary $f \in C_{1/2}$. For each $n \in \mathbb{N}$, we have $0 \le T^n f \le 1$, and therefore $0 \le Rf \le 1$. Further,

$$\int_0^1 Rf\, dm = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int_0^1 T^n f\, dm = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int_0^1 f\, dm = \frac{1}{2}.$$

We will begin the proof that $R : C_{1/2} \to C_{1/2}$ is contractive and fixed point free by defining for every $f \in C_{1/2}$ the set

$$A_n(f) = \{x \in [0,1] : T^n f(x) \in (0,1)\}.$$

LEMMA 3.1. *For every $f \in C_{1/2}$,*

$$\lim_{n\to\infty} m(A_n(f)) = 0.$$

In particular, $m(A_n(f)) \le 2^{-n}$.

Proof. In what follows, we will ignore certain dyadic numbers in the domain. These constitute a set of measure zero.

Decompose the set

$$A_1(f) = (A_1(f) \cap [0,1/2)) \cup (A_1(f) \cap (1/2,1]).$$

If $x \in A_1(f) \cap [0,1/2)$, then $x \in [0,1/2)$ and $Tf(x) \in (0,1)$. By definition, for $x \in [0,1/2)$, $Tf(x) = 2f(2x) \wedge 1$. So

$$x \in [0,1/2) \text{ and } f(2x) \in (0,1/2) \;\Leftrightarrow\; x \in A_1(f) \cap [0,1/2).$$

Similarly, if $x \in A_1(f) \cap (1/2,1]$, then $x \in (1/2,1]$ and $Tf(x) \in (0,1)$. By definition, for $x \in (1/2,1]$, $Tf(x) = (2f(2x-1)-1) \vee 0$. So

$$x \in (1/2,1] \text{ and } f(2x-1) \in (1/2,1) \;\Leftrightarrow\; x \in A_1(f) \cap (1/2,1].$$

Note that

$$m\{x \in [0,1/2) : f(2x) \in (0,1/2)\} = \tfrac{1}{2} m\{x \in (0,1) : f(x) \in (0,1/2)\},$$
$$m\{x \in (1/2,1] : f(2x-1) \in (1/2,1)\} = \tfrac{1}{2} m\{x \in (0,1) : f(x) \in (1/2,1)\}.$$

Putting this together gives

$$\begin{aligned}\tfrac{1}{2} m(A_0(f)) &\geq \tfrac{1}{2} m[f \in (0,1/2)] + \tfrac{1}{2} m[f \in (1/2,1)] \\ &= m(A_1(f) \cap (1/2,1]) + m(A_1(f) \cap [0,1/2)) = m(A_1(f)).\end{aligned}$$

Generalizing, we have

$$m(A_n(f)) = m(A_1(T^{n-1} f)) \leq \tfrac{1}{2} m(A_0(T^{n-1} f)) = \tfrac{1}{2} m(A_{n-1}(f)),$$

giving

$$m(A_n(f)) \leq \frac{1}{2^n} m(A_0(f)) \leq \frac{1}{2^n} \to 0. \ \blacksquare$$

LEMMA 3.2. *Let $h \in C_{1/2}$, and let y be any nondyadic number in $[0,1]$. Also, let $n \in \mathbb{N}$. If $h(y) = 0$, then for all $j \in \{1,\ldots,2^n\}$,*

$$T^n h\biggl(\frac{y+j-1}{2^n}\biggr) = 0.$$

If $h(y) = 1$, then for all $j \in \{1,\ldots,2^n\}$,

$$T^n h\biggl(\frac{y+j-1}{2^n}\biggr) = 1.$$

Proof. We start with $n = 1$. We need to check $j \in \{1,2\}$. First,

$$Th(y/2) = 2h(y) \wedge 1 \quad \text{(because } y/2 \text{ is between 0 and } 1/2),$$

which is 1 when $h(y) = 1$ and is 0 when $h(y) = 0$. This settles the case $j = 1$. Then for $j = 2$,

$$Th\biggl(\frac{y+1}{2}\biggr) = (2h(y) - 1) \vee 0 \quad \text{(because } (y+1)/2 \text{ is between } 1/2 \text{ and } 1),$$

which agrees with h when h is 1 or 0.

By way of induction, suppose for all $j \in \{1,\ldots,2^m\}$ that when $h(y)$ is 0 or 1,

$$h(y) = T^m h\biggl(\frac{y+j-1}{2^m}\biggr).$$

Applying the base case to $T^m h$ and $k \in \{1,2\}$ for all $j \in \{1,\ldots,2^m\}$ we have

$$T^m h\biggl(\frac{y+j-1}{2^m}\biggr) = T^{m+1} h\biggl(\frac{\frac{y+j-1}{2^m} + k - 1}{2}\biggr).$$

It follows from this fact and the inductive assumption that

$$h(y) = T^{m+1} h\biggl(\frac{\frac{y+j-1}{2^m} + k - 1}{2}\biggr) = T^{m+1} h\biggl(\frac{y + j + 2^m(k-1) - 1}{2^{m+1}}\biggr).$$

When $k = 1$ we have $j + 2^m(k-1) = j$ spanning $\{1, \ldots, 2^m\}$. When $k = 2$ we have $j + 2^m(k-1) = j + 2^m$ spanning $\{2^m + 1, 2^m + 2, \ldots, 2^{m+1}\}$. ∎

LEMMA 3.3. *For every f and g in $C_{1/2}$ with $\|f - g\|_1 > 0$ there is some $N \in \mathbb{N}$ such that*

$$\left\| \frac{I + T^N}{2} f - \frac{I + T^N}{2} g \right\|_1 < \|f - g\|_1.$$

Proof. Fix $f, g \in C_{1/2}$ with $f \neq g$. Note that all sets in the domain can vary up to a set of measure zero without affecting the argument. Define

$$\begin{aligned}
B_n &= \{x \in [0,1] : T^n f(x) \in (0,1) \text{ or } T^n g(x) \in (0,1)\} = A_n(f) \cup A_n(g),\\
C_n &= \{x \in [0,1] : T^n f(x) = 1 \text{ and } T^n g(x) = 0\},\\
D_n &= \{x \in [0,1] : (T^n f(x) = 1 \text{ and } T^n g(x) = 1)\\
&\qquad\qquad\qquad\qquad \text{or } (T^n f(x) = 0 \text{ and } T^n g(x) = 0)\},\\
E_n &= \{x \in [0,1] : T^n f(x) = 0 \text{ and } T^n g(x) = 1\}.
\end{aligned}$$

Note that $[0,1] = B_n \cup C_n \cup D_n \cup E_n$ is a disjoint union.

We will show that for a given measurable set W of positive measure, for n large, the measure of the intersection with the sets C_n and E_n can be bounded from below by a positive constant multiple of the measure of W. This will lead us to an index N for which

$$\|(1/2)(I + T^N)f - (1/2)(I + T^N)g\|_1 < \|f - g\|_1.$$

By Lemma 3.1 we have $m(B_n) \to 0$. Because $\|f - g\|_1 > 0$ and $\int_0^1 f = \int_0^1 g = 1/2$, it follows that $m[f > g] > 0$ and $m[g > f] > 0$.

Now we will check that there is some N_0 so that when $n > N_0$ we have $m(C_n) > 0$ and $m(E_n) > 0$. Note that

$$\begin{aligned}
\|f - g\|_1 = \|T^n f - T^n g\|_1 &= \int_{B_n} |T^n f - T^n g| + \int_{D_n} 0 + \int_{C_n} 1 + \int_{E_n} 1\\
&= \int_{B_n} |T^n f - T^n g| + m(C_n) + m(E_n).
\end{aligned}$$

This gives $m(E_n) + m(C_n) = \|f - g\|_1 - \int_{B_n} |T^n f - T^n g|$. Also, $\int T^n f = \int T^n g = \frac{1}{2}$, which implies

$$\begin{aligned}
\int_{B_n} (T^n f - T^n g) + \int_{C_n} (T^n f - T^n g) + \int_{D_n} (T^n f - T^n g) + \int_{E_n} (T^n f - T^n g) = 0\\
\Rightarrow \int_{B_n} (T^n f - T^n g) + \int_{C_n} 1 + \int_{E_n} (-1) = 0\\
\Rightarrow m(E_n) - m(C_n) = \int_{B_n} (T^n f - T^n g).
\end{aligned}$$

We know that $|T^n f(x) - T^n g(x)| \le 1$, and so we can deduce from these facts that

$$\|f-g\|_1 \ge m(E_n) + m(C_n) \ge \|f-g\|_1 - m(B_n)$$

and

$$|m(E_n) - m(C_n)| \le m(B_n).$$

Now, since $m(B_n) \to 0$ it follows that

$$m(E_n) \to \tfrac{1}{2}\|f-g\|_1 \quad \text{and} \quad m(C_n) \to \tfrac{1}{2}\|f-g\|_1.$$

So, we choose n to be sufficiently large so that $m(E_n)$ and $m(C_n)$ are both greater than $\frac{1}{4}\|f-g\|_1$. By Lemma 3.2 we have, for all $k \in \mathbb{N}$,

$$C_{n+k} \supseteq \bigcup_{j=0}^{2^k-1} \left(\frac{j}{2^k} + \frac{1}{2^k} C_n\right) \quad \text{and} \quad E_{n+k} \supseteq \bigcup_{j=0}^{2^k-1} \left(\frac{j}{2^k} + \frac{1}{2^k} E_n\right).$$

(♠) Claim. *There exists $k \in \mathbb{N}$ such that*

$$S_1 := E_{n+k} \cap [f > g] \quad \textit{and} \quad S_2 := C_{n+k} \cap [f < g]$$

both have positive measure.

Proof of (♠). Let $W := [f > g]$. Fix $\varepsilon > 0$. By, for example, Royden [R, Chapter 3, Proposition 15], there exists a finite sequence of open intervals $(I_l)_{l=1}^{\nu}$ such that $m(W \triangle \Gamma) < \varepsilon$ for $\Gamma := \bigcup_{l=1}^{\nu} I_l$. Without loss of generality, we may assume that the intervals I_l are pairwise disjoint, and that each I_l is a dyadic interval of the form $(j_l/2^k, (j_l+1)/2^k)$ for some $j_l \in \{0, \ldots, 2^k - 1\}$ and some $k \in \mathbb{N}$. We may write

$$\chi_\Gamma = \sum_{j=0}^{2^k-1} \beta_j \chi_{(j/2^k, (j+1)/2^k)},$$

where each β_j is in $\{0,1\}$. Then

$$\begin{aligned}
m(E_{n+k} \cap W) &\ge m\Big(\bigcup_{j=0}^{2^k-1} \left(\frac{j}{2^k} + \frac{1}{2^k} E_n\right) \cap W \cap \Gamma\Big) \\
&\ge m\Big(\bigcup_{j=0}^{2^k-1} \left(\frac{j}{2^k} + \frac{1}{2^k} E_n\right) \cap \Gamma\Big) - m\Big(\bigcup_{j=0}^{2^k-1} \left(\frac{j}{2^k} + \frac{1}{2^k} E_n\right) \cap \Gamma \setminus W\Big) \\
&\ge \int_0^1 \sum_{j=0}^{2^k-1} \chi_{(\frac{j}{2^k} + \frac{1}{2^k} E_n)} \sum_{s=0}^{2^k-1} \beta_s \chi_{(\frac{s}{2^k}, \frac{s+1}{2^k})}\, dm - m(\Gamma \setminus W)
\end{aligned}$$

$$
\begin{aligned}
&> \int_0^1 \sum_{j=0}^{2^k-1} \beta_j \chi_{(\frac{j}{2^k}+\frac{1}{2^k}E_n)}\,dm - \varepsilon = m(E_n)\frac{1}{2^k}\sum_{j=0}^{2^k-1}\beta_j - \varepsilon \\
&= m(E_n)m(\mathit{\Gamma}) - \varepsilon > m(E_n)(m(W)-\varepsilon) - \varepsilon \ge m(E_n)m(W) - 2\varepsilon \\
&\ge \frac{\|f-g\|_1}{4}m(W) - 2\varepsilon > \frac{\|f-g\|_1}{8}m(W) > 0
\end{aligned}
$$

for $\varepsilon \in (0,\infty)$ chosen small enough. Observe that the estimate holds for every $k \ge k_1$, for some $k_1 \in \mathbb{N}$.

Similarly, there exists $k_2 \in \mathbb{N}$ such that we also have

$$
m(C_{n+k_2} \cap [f<g]) > \frac{\|f-g\|_1}{4}m[f<g] - 2\varepsilon > \frac{\|f-g\|_1}{8}m[f<g] > 0
$$

for an even smaller choice of $\varepsilon \in (0,\infty)$. Moreover, from the above we see that we may choose k and k_2 to be equal. ■♠

Finally, letting $N = n + k$ we can compute the cancellation. Define $S_3 = [0,1] \setminus (S_1 \cup S_2)$. Then

$$
\begin{aligned}
\left\|\frac{I+T^N}{2}f - \frac{I+T^N}{2}g\right\|_1 &= \int_0^1 \left|\frac{f+T^Nf}{2} - \frac{g+T^Ng}{2}\right| \\
&= \int_{S_1}\left|\frac{f-g-1}{2}\right| + \int_{S_2}\left|\frac{f+1-g}{2}\right| + \int_{S_3}\left|\frac{f+T^Nf}{2} - \frac{g+T^Ng}{2}\right| \\
&= \int_{S_1}\frac{1+g-f}{2} + \int_{S_2}\frac{1+f-g}{2} + \int_{S_3}\left|\frac{f+T^Nf}{2} - \frac{g+T^Ng}{2}\right| \\
&< \int_{S_1}\frac{1+f-g}{2} + \int_{S_2}\frac{1+g-f}{2} + \int_{S_3}\left|\frac{f+T^Nf}{2} - \frac{g+T^Ng}{2}\right| \\
&= \int_{S_1}\left(\left|\frac{T^Nf-T^Ng}{2}\right| + \left|\frac{f-g}{2}\right|\right) + \int_{S_2}\left(\left|\frac{T^Nf-T^Ng}{2}\right| + \left|\frac{f-g}{2}\right|\right) \\
&\quad + \int_{S_3}\left|\frac{f-g}{2} + \frac{T^Nf-T^Ng}{2}\right| \\
&\le \int_0^1\left(\left|\frac{f-g}{2}\right| + \left|\frac{T^Nf-T^Ng}{2}\right|\right) = \|f-g\|_1. \ ■
\end{aligned}
$$

Corollary 3.4. *The mapping R is contractive. That is, for all f and g in $C_{1/2}$ with $\|f-g\|_1 > 0$ we have*

$$
\|Rf - Rg\|_1 < \|f-g\|_1.
$$

Proof. This follows from Lemma 3.3 and the fact that we can rewrite R in the following way:

$$Rf = \left(\frac{I}{2} + \frac{T}{4} + \frac{T^2}{8} + \frac{T^3}{16} + \cdots\right) f$$
$$= \frac{1}{2}\,\frac{I+T}{2} f + \frac{1}{4}\,\frac{I+T^2}{2} f + \frac{1}{8}\,\frac{I+T^3}{2} f + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}\,\frac{I+T^n}{2} f.$$

Each of the pieces $(I+T^n)/2$ is nonexpansive. By Lemma 3.3, every pair $f \neq g$ is contracted by at least one piece, and therefore it is contracted by R. ∎

Before the final lemma, we need yet one more reformulation of R:

$$\begin{aligned} Rf &= \frac{f}{2} + \frac{Tf}{4} + \frac{T^2 f}{8} + \frac{T^3 f}{16} + \cdots \\ &= \frac{f}{2} + \frac{1}{2}\left(\frac{Tf}{2} + \frac{T(Tf)}{4} + \frac{T^2(Tf)}{8} + \cdots\right) \\ &= \frac{I}{2} f + \frac{1}{2} R(T(f)) = \frac{I+RT}{2} f. \end{aligned}$$

Lemma 3.5. *R is fixed point free on $C_{1/2}$.*

Proof. Because R is contractive and $T : C_{1/2} \to C_{1/2}$ is an isometry, we find that for all $f, g \in C_{1/2}$ with $\|f-g\|_1 > 0$,

$$\|RTf - RTg\|_1 < \|Tf - Tg\|_1 = \|f-g\|_1.$$

But then

$$\begin{aligned} \|Rf - Rg\|_1 &= \left\|\frac{f-g}{2} + \frac{RTf - RTg}{2}\right\|_1 \\ &\geq \left\|\frac{f-g}{2}\right\|_1 - \left\|\frac{RTf - RTg}{2}\right\|_1 > 0. \end{aligned}$$

This shows that R is 1-1 on $C_{1/2}$ as a subset of L^1. Now let f_0 be any fixed point of R in this set. We have

$$f_0 = \frac{f_0}{2} + \frac{Tf_0}{4} + \frac{T^2 f_0}{8} + \frac{T^3 f_0}{16} + \cdots \;\Rightarrow\; \frac{f_0}{2} = \frac{Tf_0}{4} + \frac{T^2 f_0}{8} + \frac{T^3 f_0}{16} + \cdots$$
$$\Rightarrow\; f_0 = \frac{Tf_0}{2} + \frac{T^2 f_0}{4} + \frac{T^3 f_0}{8} + \cdots = R(Tf_0).$$

But then $R(f_0) = R(Tf_0)$, with R 1-1, implies $Tf_0 = f_0$, giving a fixed point of Alspach's map in $C_{1/2}$. This is known to be impossible. ∎

Looking back over this section, we see that we have proven Theorem 1.1. We can immediately state the following result, which answers the open question discussed in the Introduction.

Theorem 3.6. *There exists a fixed point free contractive mapping on a weakly compact convex set in a Banach space.*

Proof. By Corollary 3.4 and Lemma 3.5, R is such a map. ■

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