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On operators which factor through l_p or c_0

by

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Abstract. Let $1 < p < \infty$. Let X be a subspace of a space Z with a shrinking F.D.D. (E_n) which satisfies a block lower-p estimate. Then any bounded linear operator T from X which satisfies an upper-(C,p)-tree estimate factors through a subspace of $(\sum F_n)_{l_p}$, where (F_n) is a blocking of (E_n). In particular, we prove that an operator from L_p $(2 < p < \infty)$ satisfies an upper-(C,p)-tree estimate if and only if it factors through l_p. This gives an answer to a question of W. B. Johnson. We also prove that if X is a Banach space with X^* separable and T is an operator from X which satisfies an upper-(C,∞)-estimate, then T factors through a subspace of c_0.

1. Introduction. In [3], W. B. Johnson answered the following question about the relation between the structure of L_p and l_p.

QUESTION 1.1. Give a Banach space condition so that if X is a subspace of L_p $(1 < p < 2)$ which satisfies the condition, then X embeds isomorphically into l_p.

The equivalent dual question would be:

QUESTION 1.2. Give a Banach space condition so that if X is a quotient of L_p which satisfies the condition, then X is isomorphic to a quotient of l_p.

For $p > 2$, W. B. Johnson and E. Odell had already proved in [5] that if a subspace X of L_p has no subspace isomorphic to l_2, then X embeds into l_p. For $p < 2$, W. B. Johnson proved that if there exists a $K > 0$ such that every normalized weakly null sequence in X has a subsequence which is K-equivalent to the unit vector basis of l_p, then X is isomorphic to a subspace of l_p. Further W. B. Johnson also gave a complete answer to the dual question in [3]; namely, a quotient of L_p $(2 < p < \infty)$ which is of type p-Banach–Saks is a quotient of l_p. Recall that an operator T from a

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Banach space X is of *type p-Banach–Saks* (where $1 < p < \infty$) if there exists a constant λ such that every normalized weakly null sequence in X has a subsequence (x_n) which satisfies for $n = 1, 2, \ldots,$

$$\Big\| \sum_{i=1}^{n} T x_i \Big\| \le \lambda n^{1/p}.$$

X is said to be of *type p-Banach–Saks* when the identity operator on X is.

From the results above, a more general question naturally arises.

QUESTION 1.3. Give a necessary and sufficient condition so that if an operator T from L_p to any Banach space Y satisfies the condition, then T factors through l_p.

It was proved in [2] that a bounded linear operator T into L_p ($2 < p < \infty$) factors through l_p if and only if T is compact when considered as an operator into L_2. This actually answers Question 1.3 for $1 < p < 2$. In [2], W. B. Johnson conjectured that an operator T from L_p ($2 < p < \infty$) factors through l_p if and only if T is of type p-Banach–Saks. As mentioned above, this conjecture was verified in [3] in the case when T has closed range. Later, W. B. Johnson discovered in [4] a counterexample in the general case, which led him to formulate a conjecture with a stronger condition, namely an operator T from L_p ($2 < p < \infty$) factors through l_p if and only if T satisfies the following condition (when X is L_p).

CONDITION 1.4. T is an operator from X so that for every normalized weakly null sequence $(x_n) \subset X$, there is a subsequence (x_{n_k}) such that

$$\Big\| T\Big(\sum a_k x_{n_k}\Big)\Big\| \le C\Big(\sum |a_k|^p\Big)^{1/p}, \qquad \forall (a_k) \subset \mathbb{R}.$$

In Section 2, we use a space constructed by E. Odell and Th. Schlumprecht in [12] to show that for an operator T from L_p ($2 < p < \infty$), Condition 1.4 does not imply that T factors through l_p. E. Odell and Th. Schlumprecht used this space to disprove W. B. Johnson's conjecture that Condition 1.5 below and reflexivity of X imply that X embeds into an l_p sum of finite-dimensional spaces. They also formulated Condition 1.6 below and proved that Condition 1.6 and reflexivity of X do imply that X embeds into an l_p sum of finite-dimensional spaces. The above-mentioned conditions are defined as follows:

CONDITION 1.5. For all $\varepsilon > 0$, every normalized weakly null sequence in X admits a subsequence which is $(1+\varepsilon)$-equivalent to the unit vector basis of l_p.

CONDITION 1.6. There is a $C > 1$ such that every normalized weakly null tree in X admits a branch which is C-equivalent to the unit vector basis of l_p.

Let $[\mathbb{N}]^{<\omega}$ denote all finite subsets of the positive integers. By a *normalized weakly null tree*, we mean a family $(x_A)_{A\in[\mathbb{N}]^{<\omega}} \subset S_X$ with the property that every sequence $(x_{A\cup\{n\}})_{n\in\mathbb{N}}$ is weakly null. Let $A = \{n_1, \dots, n_m\}$ with $n_1 < \dots < n_m$ and $B = \{j_1, \dots, j_r\}$ with $j_1 < \dots < j_r$. Then we say A is an *initial segment* of B if $m \le r$ and $n_i = j_i$ when $1 \le i \le m$. The tree order on $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ is given by $x_A \le x_B$ if A is an initial segment of B. A *branch* of a tree is a maximal linearly ordered subset of the tree under the tree order.

Motivated by Condition 1.6, we formulate a condition stronger than Condition 1.4, which is an operator version of Condition 1.6.

CONDITION 1.7. For every normalized weakly null tree in X, there is a branch (x_k) so that

$$\Big\|T\Big(\sum a_k x_k\Big)\Big\| \le C\Big(\sum |a_k|^p\Big)^{1/p}, \qquad \forall (a_k) \subset \mathbb{R}.$$

This condition turns out to be the right one for answering Question 1.3 when $X = L_p$ $(2 < p < \infty)$.

2. A counterexample. In this section, we construct an operator T from l_2 into $X = (\sum X_n)_p$ (which will be defined below) which satisfies Condition 1.4 but does not factor through l_p for $2 < p < \infty$. Since l_2 is isomorphic to a complemented subspace of L_p, we also get an operator from L_p into $X = (\sum X_n)_p$ which satisfies Condition 1.4 but does not factor through l_p.

Let $2 < q < p < \infty$ and $X = (\sum X_n)_p$ be the space defined in [12], where X_n is the completion of $c_{00}([\mathbb{N}]^{\le n})$ under the norm

$$\|x\|_n = \sup\Big\{\Big(\sum_{i=1}^{m} \|x|_{\beta_i}\|_q^p\Big)^{1/p} : (\beta_i)_{i=1}^m \text{ are disjoint segments in } [\mathbb{N}]^{\le n}\Big\}.$$

Here $[\mathbb{N}]^{\le n}$ denotes all sets of natural numbers with cardinality less than n. By a *segment* in $[\mathbb{N}]^{\le n}$, we mean a sequence $(A_i)_{i=1}^k \in [\mathbb{N}]^{\le n}$ with

$$A_1 = \{n_1, \dots, n_l\},$$
$$A_2 = \{n_1, \dots, n_l, n_{l+1}\}, \ \dots,$$
$$A_k = \{n_1, \dots, n_l, \dots, n_{l+k-1}\},$$

for some $n_1 < \dots < n_{l+k-1}$. A *branch* in $[\mathbb{N}]^{\le n}$ is a maximal segment in $[\mathbb{N}]^{\le n}$.

REMARK 2.1. The node basis $(\tilde{e}_A^n)_{A\in[\mathbb{N}]^{\le n}}$ given by $\tilde{e}_A^n(B) = \delta_{A,B}$ for any $B \in [\mathbb{N}]^{\le n}$ is a 1-unconditional basis for X_n. Moreover, $(\tilde{e}_{A_i}^n)_{i=1}^n$ is 1-equivalent to the unit vector basis of l_q^n if $(A_i)_{i=1}^n$ is a branch in $[\mathbb{N}]^{\le n}$.

If we write $l_2 = (\sum l_2)_2$, $(e_A^n)_{A\in[\mathbb{N}]^{\le n}}$ is the unit vector basis of the nth l_2 and $(\tilde{e}_A^n)_{A\in[\mathbb{N}]^{\le n}}$ is the unit vector basis of X_n, then the operator $T : l_2 \to$

$X = (\sum X_n)_p$ is defined so that

$$T(e^n_A) = \tilde{e}^n_A.$$

Since $2 < q < p$ we can linearly extend T to be an operator of norm one from l_2 into X.

Claim 1. *The operator T satisfies Condition* 1.4.

Let (x_n) be a normalized weakly null sequence in l_2, and $\varepsilon > 0$. Then $(T(x_n))$ is a weakly null sequence in $(\sum X_n)_p$. By the proof of Example 4.2 in [12], we can pick a subsequence (x_{n_k}) such that for all $(a_k) \subset \mathbb{N}$,

$$\Big\|T\Big(\sum a_k x_{n_k}\Big)\Big\| \le 2\Big(\sum \|T(a_k x_{n_k})\|^p\Big)^{1/p} \le 2\Big(\sum |a_k|^p\Big)^{1/p}.$$

So we proved Claim 1. Our second claim is

Claim 2. *T does not factor through l_p.*

In order to prove the claim, we need the following lemma which is an application of a result concerning blockings of F.D.D.'s proved in [7]. This result was reformulated as Proposition 1.g.4. in [10].

Lemma 2.2. *Let $p > 2$. Then any bounded linear operator A from l_2 into l_p factors through $(\sum E_n)_{l_p}$ in such a way that $A = A' \circ J$, where (E_n) is a blocking of the canonical basis of l_2 and J is the formal identity from l_2 into $(\sum E_n)_{l_p}$.*

Proof. By Proposition 1.g.4 in [10], we find a blocking (E_n) of the canonical basis of l_2 such that $A(E_n)$ is essentially contained in $F_{n-1} \oplus F_n$, where (F_n) is a blocking of the canonical basis of l_p. Let J be the formal identity map from l_2 into $(\sum E_n)_{l_p}$. Since $p > 2$, J is always bounded. Let A' be the linear map from $(\sum E_n)_{l_p}$ into l_p such that $A = A' \circ J$. We claim that A' is bounded. Indeed, let $x = \sum x_n$ with $x_n \in E_n$. Then by the construction of (E_n) and (F_n), we have

$$\begin{aligned}\|A'(x)\| &\le \Big\|A'\Big(\sum x_{2n}\Big)\Big\| + \Big\|A'\Big(\sum x_{2n-1}\Big)\Big\| \\ &\le (\|A\| + \varepsilon)\Big(\Big(\sum \|x_{2n}\|^p\Big)^{1/p} + \Big(\sum \|x_{2n-1}\|^p\Big)^{1/p}\Big) \\ &\le 2(\|A\| + \varepsilon)\Big(\sum \|x_n\|^p\Big)^{1/p}.\end{aligned}$$

So A' is bounded. ∎

Now we can prove Claim 2.

Proof of Claim 2. Suppose T factors through l_p. Then by Lemma 2.2, T factors through $(\sum E_n)_{l_p}$ for some blocking of the canonical basis of l_2. Let $T = J_1 \circ J_2$, where J_1 is the formal identity from l_2 into $(\sum E_n)_{l_p}$ and J_2 is a bounded linear operator from $(\sum E_n)_{l_p}$ into $(\sum X_n)_{l_p}$. Since T is the formal identity from l_2 into $(\sum X_n)_{l_p}$, we deduce that J_2 is also a formal identity. By the choice of (E_n) and the definition of X_n, for any $k \in \mathbb{N}$, we can find a finite basic subsequence $(e^k_{A_n})^k_{n=1}$ of l_2 such that $e^k_{A_n}$'s sit in different E_{r_n}'s and $(A_n)^k_{n=1}$ is a branch of $[\mathbb{N}]^{\le k}$. As J_2 is the formal identity, we have $J_2(e^k_{A_n}) = \tilde{e}^k_{A_n}$, hence $\|J_2\| \ge k^{1/q-1/p}$. Since k is arbitrary, this shows that J_2 is not bounded. This is a contradiction. ∎

3. Main result. Now we give a sufficient condition for an operator from L_p $(2 < p < \infty)$ to factor through l_p.

DEFINITION 3.1. Let $1 \le p < \infty$, $C > 0$ and X, Y be Banach spaces. Suppose $T : X \to Y$ is a bounded linear operator. We say that T satisfies an *upper-(C,p)-tree estimate* if for every normalized weakly null tree in X, there exists a branch (x_i) such that

$$\Big\|T\Big(\sum a_i x_i\Big)\Big\| \le C\Big(\sum |a_i|^p\Big)^{1/p}, \quad \forall (a_i) \subset \mathbb{R}.$$

When $p = \infty$, T satisfies an *upper-(C,∞)-tree estimate* if for every normalized weakly null tree in X, there exists a branch (x_i) such that

$$\sup_n \Big\{\Big\|T\Big(\sum_{i=1}^n x_i\Big)\Big\|\Big\} \le C.$$

THEOREM 3.2. *Let $2 < p < \infty$, X be a Banach space, and let $T : L_p \to X$ be a bounded linear operator. Then T satisfies an upper-(C,p)-tree estimate if and only if T factors through l_p.*

As preparation for the proof, we present the following known lemmas (see [3]).

LEMMA 3.3. *Let $2 < p < \infty$, X be a Banach space, and let $T : L_p \to X$ be a bounded linear operator. Then T factors through l_p if and only if there are a blocking (H_n) of the Haar system and a bounded linear operator $S : (\sum (H_n, \|\cdot\|_p))_{l_p} \to X$ such that $T = S \circ J$, where J is the formal identity map from L_p into $(\sum (H_n, \|\cdot\|_p))_{l_p}$.*

REMARK 3.4. Since $2 < p < \infty$, the formal identity map J from L_p into $(\sum (H_n, \|\cdot\|_p))_p$ is always bounded.

Proof of Lemma 3.4. For any blocking (H_n) of the Haar system, since H_n is finite-dimensional and uniformly complemented in L_p, it is uniformly complemented in l_p. So $(\sum (H_n, \|\cdot\|_p))_{l_p}$ is complemented in l_p, hence isomorphic to l_p by [14] (or Theorem 2.a.3 in [10]). On the other hand, by Theorem II.1 in [3] any operator T from L_p into l_p factors through $(\sum (H_n, \|\cdot\|_p))_{l_p}$ for some blocking (H_n) of the Haar system in the way that $T = S \circ J$ where J is the formal identity. ∎

LEMMA 3.5. *Let $2 < p < \infty$, X be a Banach space, $T : L_p \to X$ be a bounded linear operator and (H_n) be a blocking of the Haar system. Then there is a bounded linear operator $S : (\sum(H_n, \|\cdot\|_p))_{l_p} \to X$ such that $T = S \circ J$, where J is the formal identity map from L_p into $(\sum(H_n, \|\cdot\|_p))_{l_p}$, if and only if there exists $C > 0$ such that*

$$\Big\|T\Big(\sum a_k x_k\Big)\Big\| \le C\Big(\sum |a_k|^p\Big)^{1/p}, \quad \forall (a_k) \subset \mathbb{R},\ x_k \in S_{H_k}. \tag{3.1}$$

Proof. Inequality (3.1) is equivalent to saying that the map $Q : J(L_p) \to X$ which satisfies $T = Q \circ J$ is bounded. Considering Remark 2.1 and noticing that $J(L_p)$ is obviously dense in $(\sum(H_n, \|\cdot\|_p))_{l_p}$, we are done. ∎

DEFINITION 3.6. (x_n) is said to be a *block sequence with respect to* (E_n) if there exists a sequence of integers $0 = m_1 < m_2 < \cdots$ such that $x_n \in \bigoplus_{j=m_n}^{m_{n+1}-1} E_j$ for all $n \in \mathbb{N}$. (x_n) is said to be a *skipped-block sequence with respect to* (E_n) if there exists a sequence of integers $0 = m_1 < m_2 < \cdots$ such that $m_n + 1 < m_{n+1}$ and $x_n \in \bigoplus_{j=m_n+1}^{m_{n+1}-1} E_j$ for all $n \in \mathbb{N}$. Two skipped-block sequences (x_n) and (y_n) are said to be *intrusive* if $x_1, y_1, x_2, y_2, \ldots$ or $y_1, x_1, y_2, x_2, \ldots$ is a block sequence.

DEFINITION 3.7. A property $P(C)$ with some parameter $C > 0$ for normalized block sequences in X is said to be *closed under combination* if there is a $C' > 0$ depending only on C such that for any two intrusive normalized block sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ satisfying $P(C)$, the natural combination sequence $x_1, y_1, x_2, y_2, \ldots$ or $y_1, x_1, y_2, x_2, \ldots$ satisfies $P(C')$. For any $C > 0$ and $\varepsilon > 0$, if there exists $(\delta_i) \searrow 0$ so that for any normalized sequence (x_n) that has property $P(C)$ with $x_n \in F_n$ for some blocking (F_n) of (E_n), we have that any sequence (y_n) with $y_n \in F_n$ and $\|y_n - x_n\| < \delta_n$ has property $P(C + \varepsilon)$, then we say P is *stable under small perturbations.*

DEFINITION 3.8. Let $C > 0$. A normalized block sequence (x_n) is said to be C-good if (x_n) has property $P(C)$. Otherwise we say that it is *C-bad.* A branch of a normalized block tree is C-good if it is a C-good sequence. A blocking (F_n) of (E_n) is C-good if all normalized sequences (x_n) with $x_n \in F_n$ have property $P(C)$. A blocking (F_n) of (E_n) is *C-semigood* if all normalized sequences (x_n) with $x_n \in F_{2n}$ have property $P(C)$.

REMARK 3.9. If for every blocking (F_n) of (E_n), (F_n) is C-semigood, then any skipped-block sequence (x_n) with respect to (E_n) is C-good. On the other hand, if any skipped-block sequence with respect to (E_n) is C-good, then all blockings of (E_n) are C-semigood.

DEFINITION 3.10. We say x *sits* in a block of (E_n) if $x = \sum_{i=k_1}^{k_2} x_i$ with $x_i \in E_i$. Let $y = \sum_{i=m_1}^{m_2} y_i$ with $y_i \in E_i$. If $k_2 < m_1$, then we say y *sits farther than* x. A *normalized block tree with respect to* (E_n) is a family $(x_A)_{A\in[\mathbb{N}]^{<\omega}} \subset S_X$ such that

(a) For any $A \in [\mathbb{N}]^{<\omega}$, x_A sits in some block of (E_n).
(b) If A is a proper initial segment of B, then x_B sits farther than x_A.
(c) If $\max A < n < m$, then $x_{A\cup\{m\}}$ sits farther than $x_{A\cup\{n\}}$.

PROPOSITION 3.11. *Let X be a Banach space with an F.D.D. (E_n). Consider the three conditions*:

(i) *There exists a $C > 0$ such that every blocking of (E_n) has a further blocking (F_n) so that all further blockings of (F_n) are C-good.*
(ii) *There exists a $C > 0$ such that every blocking of (E_n) has a further blocking (F_n) so that all further blockings of (F_n) are C-semigood.*
(iii) *There exists a $C > 0$ such that every normalized block tree with respect to (E_n) in X has a C-good branch.*

Then:

(a) (i) *implies* (ii) *and* (ii) *implies* (iii).
(b) *If property P is closed under combination*, *then* (ii) *implies* (i).
(c) *If property P is stable under small perturbations and makes D'_C closed under the pointwise topology on $[\mathbb{N}]^\omega$, for all $C > 0$, then* (iii) *implies* (ii).

Here D'_C is defined as

$$D'_C = \{M \in [\mathbb{N}]^\omega : \textit{the blocking of } (E_n) \textit{ corresponding to } M \textit{ is } C\textit{-semigood}\}.$$

$[\mathbb{N}]^\omega$ denotes the set of all infinite subsets of positive integers. For a blocking (F_n) of (E_n), given by $F_n = \sum_{i=n_{i-1}+1}^{n_i} E_i$ and $n_0 = 0$, we say that (F_n) corresponds to the set $\{n_1, n_2, \ldots\}$.

Proof. Since (a) and (b) trivially follow from the definitions above, we omit the proof.

It remains to prove that (iii) implies (ii) when D'_C is closed under pointwise topology on $[\mathbb{N}]^\omega$. This is essentially contained in Theorem 3.3 of [12]. For the convenience of the reader, we write down a direct argument which includes only the part of the proof of Theorem 3.3 in [12] that is needed. For any $C > 0$, set

$$D_C = \{\text{blockings of } (E_n) \text{ which are } C\text{-semigood}\}.$$

So we can identify D_C with

$$D'_C = \{M \in [\mathbb{N}]^\omega : \text{the blocking corresponding to } M \text{ is } C\text{-semigood}\}.$$

Let (G_n) be any blocking of (E_n). Since property P makes D'_C closed under the pointwise topology on $[\mathbb{N}]^\omega$, by the infinite version of Ramsey's theorem (cf. [11]), there are two cases:

CASE 1: there is a blocking (F_n) of (G_n) all further blockings of which are C-semigood.

CASE 2: there is a blocking (F_n) of (G_n) no further blocking of which is C-semigood.

In the first case, we are done. In the second case, we will construct a block tree which results in a contradiction. Let N' be the infinite subset of positive integers corresponding to the blocking (F_n) no further blocking of which is C-semigood. Then for each $\tilde{M} \in [N']^\omega$ (which corresponds to a further blocking of (F_n)), we can pick a C-bad sequence $(x_i^{\tilde{M}})$ which is a skipped-block sequence relative to the blocking corresponding to $\tilde{M}$. Letting $N' = \{n_1, n_2, n_3, \ldots\}$, we know that for any $\tilde{M} \in [\{n_3, n_4, \ldots\}]^\omega$,

$$x_1^{\{n_1,n_2\}\cup\tilde{M}} \in S_{[E_i]|_{i=n_1+1}^{n_2}}.$$

By Ramsey's theorem and the compactness of $S_{[E_i]|_{i=1}^{n_2}}$, we can find an $x_{\{1\}} \in S_{[E_i]|_{i=n_1+1}^{n_2}}$ and an $\tilde{M}^1 \subset \{n_3, n_4, \ldots\}$ such that for all $\tilde{M} \in [\tilde{M}^1]^\omega$, we have

$$\|x_{\{1\}} - x_1^{\{n_1,n_2\}\cup\tilde{M}}\| < \delta_1.$$

Repeating the procedure again, we can find an $x_{\{2\}} \in S_{[E_i]|_{i=1+n_1'}^{n_2'}}$ and an $\tilde{M}^2 \in [\tilde{M}^1]^\omega$ so that for all $\tilde{M} \in [\tilde{M}^2]^\omega$, we have

$$\|x_{\{2\}} - x_1^{\{n_1',n_2'\}\cup\tilde{M}}\| < \delta_1,$$

where n_1', n_2' are the first two elements of $\tilde{M}^2$. Continuing this procedure, we get x_i for all $i \in \mathbb{N}$. For the second level of the tree, by using the same method as above, we can find for x_1 an $x_{1,2} \in S_{[E_i]|_{i=1+n_1'}^{n_2'}}$ and an $\tilde{M}^{1,2} \in [\tilde{M}^1 - \{n_1', n_2'\}]^\omega$ such that for all $\tilde{M} \in [\tilde{M}^{1,2}]^\omega$, we have

$$\|x_{\{1,2\}} - x_2^{\{n_1,n_2,n_1',n_2'\}\cup\tilde{M}}\| < \delta_2.$$

Let $\tilde{n}_1^2, \tilde{n}_2^2$ be the smallest two elements of $\tilde{M}^{1,2}$; then we can find our desired $x_{1,3}$ and so on. Since P is stable under small perturbations, by continuing this process, we get a normalized block tree with respect to (F_n) no branch of which has property $P(C+\varepsilon)$. Since C is arbitrary, we get a contradiction. ■

Now we can prove our main result.

Proof of Theorem 3.2. Given an operator T, we say that a normalized block sequence (x_n) with respect to the canonical Haar system (h_n) has property $P(C)$ if

$$\Big\|T\Big(\sum a_i x_i\Big)\Big\| \le C\Big(\sum |a_i|^p\Big)^{1/p}, \quad \forall (a_i) \subset \mathbb{R}.$$

Let (x_n) and (y_n) be two intrusive normalized skipped-block sequences with respect to (h_n). If both of them have property $P(C)$, then

$$\begin{aligned}\Big\|T\Big(\sum(a_ix_i+b_iy_i)\Big)\Big\| &\le \Big\|T\Big(\sum a_ix_i\Big)\Big\|+\Big\|T\Big(\sum b_iy_i\Big)\Big\|\\ &\le C\Big(\Big(\sum|a_i|^p\Big)^{1/p}+\Big(\sum|b_i|^p\Big)^{1/p}\Big)\\ &\le 2C\Big(\sum(|a_i|^p+|b_i|^p)\Big)^{1/p}.\end{aligned}$$

So P is *closed under combination*. Let (H_n) be a blocking of (h_n), and (x_n) be a normalized block sequence with $x_n\in H_n$ which has property $P(C)$. Let (y_n) be another normalized block sequence with $y_n\in H_n$ such that $\|x_n-y_n\|<\delta_n$ where $\delta_n<\varepsilon/2^n\|T\|$. Then

$$\begin{aligned}\Big\|T\Big(\sum a_iy_i\Big)\Big\| &\le \Big\|T\Big(\sum a_ix_i\Big)\Big\|+\Big\|T\Big(\sum a_i(x_i-y_i)\Big)\Big\|\\ &\le (C+\varepsilon)\Big(\sum|a_i|^p\Big)^{1/p}.\end{aligned}$$

So P is *stable under small perturbations*. Also notice that the set

$$\Omega(C)=\Big\{(x_k)\in S^{\omega}_{L_p}:\Big\|T\Big(\sum a_kx_k\Big)\Big\|\le C\Big(\sum|a_k|^p\Big)^{1/p}\Big\},\quad \forall(a_k)\subset\mathbb{R},$$

is closed under pointwise limits where $S^{\omega}_{L_p}$ denotes the set of all infinite sequences in the unit sphere of L_p. Then the set

$$D'_C=\{M\in[\mathbb{N}]^{\omega}:\text{the blocking corresponding to } M \text{ is } C\text{-semigood}\}$$

is closed under pointwise limits in $[\mathbb{N}]^{\omega}$. For L_p, since every block tree is a weakly null tree, by hypothesis every block tree has a good branch. So by Proposition 3.11 and our argument above, we know that there is a blocking (H_n) of (h_n) and $D<\infty$ such that all block sequences of $(H_n)_{n>1}$ are in $\Omega(D)$. Then it is easy to see that there is a $C'>0$ so that all block sequences of (H_n) are in $\Omega(C')$. Combining Lemmas 3.3 and 3.5, we conclude that T factors through l_p. ■

REMARK 3.12. If T factors through l_p, say $T=T_1\circ T_2$ where T_2 is an operator from L_p into l_p and T_1 is an operator from l_p into X, then for any normalized weakly null tree (x_A) in L_p, $(T_2(x_A))$ is a weakly null tree in l_p. Hence there is a branch of (x_A) which satisfies an upper-$(2,p)$-tree estimate. So the upper-(C,p)-tree estimate is also a necessary condition.

Actually we have the following generalization of Theorem 3.2.

DEFINITION 3.13. Let $1\le p\le\infty$. Let X be a Banach space with an F.D.D. (E_n). We say (E_n) satisfies a *block lower-p estimate* if there exists a $C>0$ such that for any block basis (x_n) with respect to (E_n),

$$\Big\|\sum x_n\Big\|\ge C\Big(\sum\|x_n\|^p\Big)^{1/p}.$$

THEOREM 3.14. *Let $1 < p \le \infty$ and X be a Banach space with a shrinking F.D.D. (E_n) which satisfies a block lower-p estimate. Let $T : X \to Y$ be a bounded linear operator which satisfies an upper-(C,p)-tree estimate. If $p < \infty$, then T factors through $(\sum F_n)_{l_p}$ and if $p = \infty$, T factors through $(\sum F_n)_{c_0}$ for some blocking (F_n) of (E_n).*

Proof. Let $p < \infty$. Let (F_n) be any blocking of (E_n) and $J_F : (\sum F_n)_X \to (\sum F_n)_{l_p}$ be the formal identity map. Since (E_n) satisfies a block lower-p estimate, J_F is always bounded. If the map $S_F : J_F(X) \to Y$ with $T = S_F \circ J_F|_X$ is bounded, i.e. there exists a $C > 0$ such that for all (x_k) with $x_k \in F_k$ and $j \in \mathbb{N}$,

$$\Big\|T\Big(\sum_{k=1}^{j} a_k x_k\Big)\Big\| \le C\Big(\sum |a_k|^p\Big)^{1/p}, \quad \forall (a_k) \subset \mathbb{R},$$

then T factors through the subspace $J_F[X]$ of $(\sum F_n)_{l_p}$. Since $J_F[X]$ is dense in $(\sum F_n)_{l_p}$, the operator S_F can be extended to the whole space $(\sum F_n)_{l_p}$. Hence T factors through $(\sum F_n)_{l_p}$. For an operator T, we say that a normalized block sequence (x_n) with respect to (E_n) has property $P(C)$ if for all $j \in \mathbb{N}$,

$$\Big\|T\Big(\sum_{i=1}^{j} a_i x_i\Big)\Big\| \le C\Big(\sum |a_i|^p\Big)^{1/p}, \quad \forall (a_i) \subset \mathbb{R}.$$

As in the proof of Theorem 3.2, we can check that property P is closed under combination and stable under small perturbation. Since (E_n) is shrinking, every block tree is weakly null, hence by hypothesis every block tree has a good branch. Now by applying Proposition 3.11, we conclude that there is a blocking (F_n) of (E_n) so that the operator S_F defined above is bounded. The proof above works as well when $p = \infty$. ∎

A further question is what if X is only a subspace of a space with a shrinking F.D.D. In the case when p is finite, we can prove the following generalization of Theorem 3.2 by using the method of the proof of Theorem 4.1 in [12].

THEOREM 3.15. *Let $1 < p < \infty$ and X be a subspace of a space Z with a shrinking F.D.D. (E_n) which satisfies a block lower-p estimate. Let $T : X \to Y$ be a bounded linear operator which satisfies an upper-(C,p)-tree estimate. Then T factors through a subspace of $(\sum F_n)_{l_p}$, where (F_n) is a blocking of (E_n).*

In order to prove the above theorem, we need Lemma 3.16, which is a result of W. B. Johnson restated as Corollary 4.4 in [12].

LEMMA 3.16 (Corollary 4.4 in [12]). *Let X be a subspace of the reflexive space Z and let (F_i) be an F.D.D. for Z. Let $\delta_i \downarrow 0$. There exists a blocking*

(G_i) of (F_i) given by $G_i = \bigoplus_{j=N_{i-1}+1}^{N_i} F_j$ for some $0 = N_0 < N_1 < \cdots$ with the following property. For all $x \in S_X$ there exist $(x_i)_{i=1}^{\infty} \subset X$ and $t_i \in (N_{i-1}, N_i]$ for $i \in \mathbb{N}$ so that:

(a) $x = \sum_{i=1}^{\infty} x_i$.
(b) *For $i \in \mathbb{N}$, either $\|x_i\| < \delta_i$ or $\|P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j}(x_i) - x_i\| < \delta_i \|x_i\|$.*
(c) *For $i \in \mathbb{N}$, $\|P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j}(x) - x_i\| < \delta_i$.*

Proof of Theorem 3.15. Let (F_n) be any blocking of (E_n) and $J_F : (\sum F_n)_Z \to (\sum F_n)_{l_p}$ be the formal identity map. Since (E_n) satisfies a block lower-p estimate, J_F is always bounded. If the map $S_F : J_F(X) \to Y$ with $T = S_F \circ J_F|_X$ is bounded, i.e. there exists a $C > 0$ such that for all $x = \sum a_k x_k \in X$ with $x_k \in S_{F_k}$,

$$\Big\| T\Big(\sum a_k x_k\Big)\Big\| \le C\Big(\sum |a_k|^p\Big)^{1/p},$$

then T factors through a subspace of $(\sum F_n)_{l_p}$. Let $C > 0$ and set

$$\mathcal{A} = \Big\{(x_i) \in S_X^{\omega} : \forall j \in \mathbb{N},\ \Big\|T\Big(\sum_{i=1}^{j} a_i x_i\Big)\Big\| \le C\Big(\sum |a_i|^p\Big)^{1/p},\ \forall (a_i) \subset \mathbb{R}\Big\}.$$

Applying Proposition 2.4 in [13] to the set $\mathcal{A}$, we get a blocking (F_i) of (E_i) such that there exists $\delta = (\delta_i)$ so that if $(x_n) \subset S_X$ is a δ-skipped block with respect to (F_n) (see Definition 2.2 in [13]), then whenever $\sum a_i x_i$ converges, we have $\|T(\sum a_i x_i)\| \le 2C(\sum |a_i|^p)^{1/p}$. Because the F.D.D. (E_i) is shrinking and satisfies a block lower-p estimate, Z is reflexive. Now let (G_i) be the blocking of (F_i) given by Lemma 3.16. Let $x \in S_X$, $x = \sum x_i = \sum \tilde{x}_i$ with $\tilde{x}_i \in G_i$ and x_i as in Lemma 3.16. Let $y_i = P_{\bigoplus_{j=t_{i-1}+1}^{t_i-1} F_j} x$; then there exist C_1, C_2 such that

$$C_1 \max(\|y_i\|, \|y_{i+1}\|) - \delta_i \le \|\tilde{x}_i\| \le C_2 \|y_i\| + \delta_i.$$

So when δ_i's are sufficiently small, we have

$$\begin{aligned}\Big\|T\Big(\sum \tilde{x}_i\Big)\Big\| = \Big\|T\Big(\sum x_i\Big)\Big\| &\le C\Big(\sum \|x_i\|^p\Big)^{1/p} \\ &\le 2C\Big(\sum \|y_i\|^p\Big)^{1/p} \le C'\Big(\sum \|\tilde{x}_i\|^p\Big)^{1/p}.\end{aligned}$$

This is exactly what we want. ∎

In particular, when Z is L_p $(2 < p < \infty)$, we have the corollary below.

COROLLARY 3.17. *Let $2 < p < \infty$ and let X be a subspace of L_p. If $T : X \to Y$ is a bounded linear operator which satisfies an upper-(C,p)-tree estimate, then T factors through a subspace of l_p.*

For the case when $p = \infty$, we have the following result, the proof of which was shown to me by W. B. Johnson.

THEOREM 3.18. *Let X be a Banach space with X^* separable. Let $T : X \to Y$ be a bounded linear operator satisfying an upper-(C, ∞)-tree estimate. Then T factors through a subspace of c_0.*

To prove the theorem, we need the following lemma, which is a corollary of Theorem 3.14.

LEMMA 3.19. *Let X be a Banach space with a shrinking F.D.D. (E_i) and let $T : X \to Y$ be a bounded linear operator satisfying an upper-(C, ∞)-tree estimate. Then T factors through a subspace of c_0.*

Proof. By Theorem 3.14, we know that T factors through $(\sum F_i)_{c_0}$ for some blocking (F_i) of (E_i). Since $(\sum F_i)_{c_0}$ embeds into c_0, T factors through a subspace of c_0. ∎

Proof of Theorem 3.18. For convenience, without loss of generality, we assume Y is l_∞. Since X^* is separable, by Theorem IV.4 in [6] (or see Theorem 1.g.2 in [10]), there is a closed subspace E of X so that both E and X/E have a shrinking F.D.D. Let T_E be the restriction of the operator T to E. By Lemma 3.19, T_E factors through a subspace of c_0. We write $T_E = B \circ A$ where A is an operator from E into c_0 and B is an operator from $A[E]$ into l_∞. Since X is separable and $A[E]$ is in c_0, we can extend A to be defined on X. Let $\tilde{A}$ be the extension. Since $Y = l_\infty$, we can also extend B to be defined on c_0. Let $\tilde{B}$ be the extension. So we get a new operator $\tilde{T} = \tilde{B} \circ \tilde{A}$ which factors through a subspace of c_0 (actually through c_0).

Now we consider the operator $T - \tilde{T}$. It is identically zero on E and also satisfies an upper-(C_1,∞)-tree estimate. So it naturally induces an operator S from X/E into l_∞ $(S(x+E) = (T - \tilde{T})(x))$. If we can prove that S satisfies an upper-(C, ∞)-tree estimate, then by Lemma 3.19, S factors through a subspace of c_0. Hence so does $T - \tilde{T}$. Since $\tilde{T}$ factors through a subspace of c_0, we conclude that so does $T = (T - \tilde{T}) + \tilde{T}$.

So it is enough to show S satisfies an upper-(C, ∞)-tree estimate. Let us first prove that for any normalized weakly null sequence (z_i) in X/E, there is a subsequence (z_{k_i}) whose pull back (under the canonical quotient $Q : X \to X/E$) (x_i) in X is also weakly null and $\max\{\|x_i\|\} < 2$. Pick a sequence (x_i) in X such that $Q(x_i) = z_i$ and $\max\{\|x_i\|\} < 1 + \varepsilon$. Since l_1 does not embed into X, by Rosenthal's l_1 theorem (see [15]) and passing to a subsequence, we can assume (x_i) is weakly Cauchy. Since (z_i) is weakly null, we can find convex combinations $y_i = \sum_{j=N_{i-1}+1}^{N_i} \alpha_j z_j$ such that $\|y_i\| < 1/2^i$. Replacing x_i by $x_i - \sum_{j=N_{i-1}+1}^{N_i} \alpha_j x_j$, we see that (x_i) is weakly null and $\|Q(x_i) - z_i\| < 1/2^i$. By replacing x_i by an element in the ball centered at

x_i with radius $1/2^i$, we get a weakly null sequence (x_i) such that $Q(x_i) = z_i$ and $\|x_i\| < 2$.

For any normalized weakly null tree in X/E, using the result above, it is easy to prove by induction that there is a subtree whose pull back in X is also a weakly null tree and the norms of each element of the tree are uniformly bounded. Since $T - \tilde{T}$ satisfies an upper-(C_1, ∞)-tree estimate, we conclude that S satisfies an upper-(C, ∞)-tree estimate. We are done. ∎

When T is the identity map, in view of Lemma 3.21, we have the following corollary.

COROLLARY 3.20 (Theorem 3.2 in [9]). *Let X be a separable Banach space which does not contain l_1. If for every normalized weakly null tree in X, there is a branch (x_i) so that*

$$\sup_n \Big\{ \Big\| \sum_{i=1}^{n} x_i \Big\|_X \Big\} \le C,$$

then X embeds into c_0.

LEMMA 3.21. *Let $1 < p \le \infty$. When X is a Banach space with an upper-(C, p)-tree estimate, then the condition "X is separable and l_1 does not embed into X" and the condition "X^* is separable" are equivalent.*

Proof of Lemma 3.21.

FACT 1 (see Theorem 4.2 in [1]). *If l_1 does not embed into X, then $\eta(X) = I_w^+(X)$.*

Here $\eta(X)$ is the Szlenk index (see Definition 4.1 in [1]) and $I_w^+(X)$ is the l_1^+-weakly null index (see Definition 3.6 in [1]).

FACT 2. *The upper-(C, p)-tree estimate implies that $I_w^+(X) = \omega$.*

FACT 3 (see (ix) of Theorem 3.14 of [1]). *If l_1 does not embed into X, then $\eta(X) < \omega_1$ is equivalent to X^* being separable.*

From the above facts, we know that if l_1 does not embed into X and X satisfies an upper-(C, p)-tree estimate for some $p > 1$, then X^* is separable. The other direction is trivial. So we are done. ∎

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