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The minimal displacement problem in subspaces of the space of continuous functions of finite codimension

by

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Abstract. We show that every subspace of finite codimension of the space $C[0,1]$ is extremal with respect to the minimal displacement problem.

1. Introduction. In 1930 Schauder [14] proved that convex and compact subsets of Banach spaces have the fixed point property (fpp for short) for continuous mappings. Kakutani [10] was probably the first to give examples of bounded, closed and convex subsets in infinite-dimensional Banach spaces without fpp for continuous (even lipschitzian) mappings. Much stronger results concerning the failure of Schauder's theorem have been obtained by Klee [11], Nowak [13], Benyamini and Sternfeld [3] and Sternfeld and Lin [12].

In 1973 Goebel [8] introduced the notion of minimal displacement. Let C be a bounded, closed and convex subset of an infinite-dimensional Banach space X, and let $T : C \to C$. The *minimal displacement* of T is the number

$$d_T = \inf\{\|x - Tx\| : x \in C\}.$$

Goebel showed that d_T can be positive for lipschitzian mappings and he proved a basic property of the minimal displacement for lipschitzian mappings:

$$d_T \le (1 - 1/k)r(C) \qquad \text{for } k \ge 1,$$

where

$$r(C) = \inf\bigl\{\sup\{\|x - y\| : y \in C\} : x \in C\bigr\}$$

is the Chebyshev radius of C and k is a Lipschitz constant of T. There are some spaces and sets for which $d_T = (1 - 1/k)r(C)$. Goebel also introduced the so called *minimal displacement characteristic* of X. This is a function

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defined for $k \geq 1$ as

$$\psi_X(k) = \sup\{d_T : T : B \to B,\, T \in L(k)\},$$

where B is the closed unit ball in X and $L(k)$ denotes the class of lipschitzian mappings with constant k. It is known that

$$\psi_X(k) \leq 1 - 1/k$$

for any space X. There exist some "extremal" spaces for which $\psi_X(k) = 1 - 1/k$. Examples of such spaces are $C[0,1], c_0$ and some others [9]. We show that the class of extremal spaces contains all subspaces of $C[0,1]$ of finite codimension.

Such spaces are also important because of the retraction problem. It is known that in every infinite-dimensional Banach space there exists a lipschitzian retraction $R : B \to S$, where S denotes the unit sphere (see [3], [13]). But it is still an open problem to find a minimal Lipschitz constant of such a retraction for a given infinite-dimensional Banach space X. Some authors constructed lipschitzian retraction of the unit ball onto the unit sphere with relatively small Lipschitz constant using lipschitzian mappings with positive displacement (see [7], [5], [6] and [1]). Using such methods, the best known upper bound of the retraction constant is in "extremal spaces".

2. Results. We start with two technical lemmas [4].

LEMMA 1. *For every $k > 1$ there exists a mapping $T : B \to S$ of class $L(k)$ with $d_T = 1 - 1/k$ and such that*

$$(Tx)(0) = -1 \quad \text{and} \quad (Tx)(1) = 1$$

for every $x \in B$.

Proof. Define $T_1 : B \to C[0,1]$ by $(T_1x)(t) = x(t) + 4t - 2$ and $T : B \to S$ by $(Tx)(t) = f((T_1x)(t))$, where the function $f : \mathbb{R} \to [-1,1]$ is given for positive k as

$$f(t) = \begin{cases} -1 & \text{if } t \in (-\infty, -1/k), \\ kt & \text{if } t \in [-1/k, 1/k], \\ 1 & \text{if } t \in (1/k, \infty). \end{cases}$$

The mapping T_1 is nonexpansive and the function f is lipschitzian with constant k, which implies that $T \in L(k)$. Observe that $(Tx)(0) = -1$ and $(Tx)(1) = 1$ for every $x \in B$. Moreover, if $(T_1x)(1/2) \geq 0$, then from the condition $(T_1x)(0) \leq -1$ for every $x \in B$ we infer that there exists $t_0 \in (0, 1/2)$ such that $(T_1x)(t_0) = -1/k$. From this equality we obtain

$$x(t_0) > (T_1x)(t_0) = -1/k > -1 = (Tx)(t_0),$$

which implies $\|x - Tx\| > 1 - 1/k$. Analogously if $(T_1x)(1/2) < 0$, the condition $(T_1x)(1) \geq 1$ implies that there exists $t_1 \in (1/2, 1)$ for which

$(T_1x)(t_1) = 1/k$. This yields

$$x(t_1) < (T_1x)(t_1) = 1/k < 1 = (Tx)(t_1),$$

and further $\|x - Tx\| > 1 - 1/k$. This combined with the basic property of the minimal displacement implies that $d_T = 1 - 1/k$, which ends the proof. ∎

REMARK 2. From the above proof we conclude that the infimum in the definition of the minimal displacement is not attained for any $x \in B$.

REMARK 3. For $k = 1$ the map T will be given by

$$(Tx)(t) = \max\bigl\{-1, \min\{1, (T_1x)(t)\}\bigr\}.$$

This map is fixed point free because $(Tx)(t) > x(t)$ for some $t > 1/2$ or $(Tx)(t) < x(t)$ for some $t < 1/2$.

From Lemma 1 and the two remarks above, and from the fact that the spaces $C[\alpha, \beta]$ and $C[0,1]$ are isometric, we deduce the following lemma.

LEMMA 4. *Let $0 \le \alpha < \beta \le 1$. Then for every $k \ge 1$ there exists a mapping $T_{[\alpha,\beta]} : B \to S$ of class $L(k)$ such that for every $x \in B$,*

$$(T_{[\alpha,\beta]}x)(t) = 0 \quad \textit{for every } t \in [0, \alpha] \cup [\beta, 1]$$

and

$$\max_{t\in[\alpha,\beta]} |x(t) - (T_{[\alpha,\beta]}x)(t)| > 1 - 1/k.$$

Proof. If $0 < \alpha < \beta < 1$ we choose $\alpha < \gamma < \delta < \beta$ and apply Lemma 1 to the space $C[\gamma, \delta]$. Then there exists a map $\overline{T} : B_{C[\gamma,\delta]} \to S_{C[\gamma,\delta]}$ of class $L(k)$ such that

$$\max_{t\in[\gamma,\delta]} |x(t) - (\overline{T}x)(t)| > 1 - 1/k, \qquad (\overline{T}x)(\gamma) = -1 \quad \text{and} \quad (\overline{T}x)(\delta) = 1$$

for every $x \in B_{C[\gamma,\delta]}$. Now we can define $T_{[\alpha,\beta]} : B \to S$ by $(T_{[\alpha,\beta]}x)(t) = (\overline{T}x)(t)$ if $t \in [\gamma, \delta]$, $(T_{[\alpha,\beta]}x)(t) = 0$ for any $t \in [0, \alpha] \cup [\beta, 1]$, and on the intervals (α, γ) and (δ, β) we define $T_{[\alpha,\beta]}$ as an affine map so as to obtain a continuous map. It is easy to check that this map satisfies the desired conditions. If $\alpha = 0$ or $\beta = 1$ then the proof is very similar. For instance if $\alpha = 0$ and $\beta < 1$ we define $(T_{[\alpha,\beta]}x)(\alpha) = 0$ and on the intervals (α, γ), $[\gamma, \delta]$, (δ, β) and $[\beta, 1]$ we define $T_{[\alpha,\beta]}$ as in the case $0 < \alpha < \beta < 1$. ∎

After these technical lemmas we consider the minimal displacement problem in subspaces of codimension one of $C[0, 1]$. From the Riesz Theorem it is known that each such subspace can be written as

$$C_\mu[0,1] = \Bigl\{x \in C[0,1] : \int_{[0,1]} x\,d\mu = 0\Bigr\},$$

where μ is bounded, real Borel signed measure on $[0, 1]$. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ.

THEOREM 5. *Let $C_\mu[0,1]$ be a subspace of $C[0,1]$ of codimension one. Then for every $0 \le \alpha < \beta \le 1$ and $k \ge 1$ there exists a mapping $T : B \to S$ of class $L(k)$ such that for every $x \in B$,*

$$(Tx)(t) = 0 \quad \text{for } t \in [0,\alpha] \cup [\beta,1]$$

and

$$\max_{t\in[\alpha,\beta]} |x(t) - (Tx)(t)| > 1 - 1/k.$$

Proof. Let $0 \le \alpha < \beta \le 1$ and let $|\mu|$ denote the absolute value of the measure μ, i.e. $|\mu| = \mu^+ + \mu^-$. We consider two cases.

(i) If there exists an interval $[c,d] \subset [\alpha,\beta]$ such that $|\mu|([c,d]) = 0$, then we define $T = T_{[c,d]}$ (see Lemma 4).

(ii) If (i) does not hold then $|\mu|([a,b]) \ne 0$ for any subinterval $[a,b]$ of $[\alpha,\beta]$. This implies that there exists $[a,b] \subset (\alpha,\beta)$ such that $\mu([a,b]) \ne 0$. Without loss of generality we can assume that $\mu([a,b]) > 0$. Observe that it is possible to find an interval $[c,d] \subset (\alpha,\beta)$ disjoint from $[a,b]$ and such that $|\mu|([c,d]) < \mu([a,b])$. Let $T_{[c,d]}$ satisfy the conditions of Lemma 4. Proceeding to the second part of the construction observe that there exists $\delta > 0$ for which $[a-\delta, b+\delta]$ is contained in (α,β) and disjoint from $[c,d]$ and the following condition holds:

$$\begin{aligned}&|\mu|([a-\delta,a) \cup (b,b+\delta] \cup [c,d])\\ &\qquad = |\mu|([a-\delta,a)) + |\mu|((b,b+\delta]) + |\mu|([c,d]) < \mu([a,b]).\end{aligned}$$

Now for $h \in [-1,1]$ consider a mapping $d_h : [0,1] \to [-1,1]$ defined as: $d_h(t) = h$ if $t \in [a,b]$, and $d_h(t) = 0$ if $t \notin [a-\delta, b+\delta]$. On $[a-\delta,a)$ and $(b,b+\delta]$ we define d_h as an affine map so that the function d_h is continuous. For the function x and $h \in [-1,1]$ let us consider the expression

$$\begin{aligned} I(h) &= \int_0^1 \big((T_{[c,d]}x)(t) + d_h(t)\big)\, d\mu(t) \\ &= \int_{[a-\delta,a)} d_h(t)\, d\mu(t) + h\mu([a,b]) + \int_{(b,b+\delta]} d_h(t)\, d\mu(t) \\ &\quad + \int_{[c,d]} (T_{[c,d]}x)(t)\, d\mu(t). \end{aligned}$$

Observe that for $h = -1$ we have

$$I(-1) \le |\mu|([a-\delta,a)) - \mu([a,b]) + |\mu|((b,b+\delta]) + |\mu|([c,d]) < 0.$$

Analogously for $h = 1$ we get

$$I(1) \ge -|\mu|([a-\delta,a)) + \mu([a,b]) - |\mu|((b,b+\delta]) - |\mu|([c,d]) > 0.$$

Thus there exists a constant $h = h_x$ for which $I(h_x) = 0$, which means that $d_{h_x} + T_{[c,d]}x \in C_\mu[0,1]$. We prove that the notation $h = h_x$ is justified, i.e.

the constant h for which $I(h)=0$ is unique. Suppose that $I(h_1)=I(h_2)=0$ for $h_1\neq h_2$. Then

$$\int\limits_{[0,1]} d_{h_1}(t)\,d\mu(t)=\int\limits_{[0,1]} d_{h_2}(t)\,d\mu(t),$$

which means that

$$\int\limits_{[a-\delta,a)\cup(b,b+\delta]} d_{h_1}(t)\,d\mu(t)+h_1\mu([a,b])=\int\limits_{[a-\delta,a)\cup(b,b+\delta]} d_{h_2}(t)\,d\mu(t)+h_2\mu([a,b]).$$

This leads to a contradiction:

$$\begin{aligned}|h_1-h_2|\mu([a,b])&=\Big|\int\limits_{[a-\delta,a)\cup(b,b+\delta]}(d_{h_1}(t)-d_{h_2}(t))\,d\mu(t)\Big|\\&\le\int\limits_{[a-\delta,a)\cup(b,b+\delta]}|d_{h_1}(t)-d_{h_2}(t)|\,|d\mu(t)|\\&\le|h_1-h_2|\,|\mu|([a-\delta,a)\cup(b,b+\delta])\\&<|h_1-h_2|\mu([a,b]).\end{aligned}$$

Now we can define a map $T:B\to S$ by

$$Tx=d_{h_x}+T_{[c,d]}x.$$

Observe that indeed $\|Tx\|=1$ for any $x\in B$ because $\|T_{[c,d]}x\|=1$ and $\operatorname{supp}d_h\cap\operatorname{supp}T_{[c,d]}=\emptyset$.

We show that $T\in L(k)$. Observe that it is sufficient to show that $Dx:=d_{h_x}\in L(k)$ (because $\operatorname{supp}d_h\cap\operatorname{supp}T_{[c,d]}=\emptyset$). It is easy to see that $\|Dx-Dy\|=|h_x-h_y|$. Suppose that $D\notin L(k)$. Then there exist $x,y\in B$ and a constant $k_1>k$ such that $\|Dx-Dy\|=k_1\|x-y\|$. Recall that for all $x,y\in B$ we have

$$\int\limits_{[0,1]}\big((Dx)(t)+(T_{[c,d]}x)(t)\big)\,d\mu(t)=\int\limits_{[0,1]}\big((Dy)(t)+(T_{[c,d]}y)(t)\big)\,d\mu(t)=0.$$

This can be written equivalently as

$$\begin{aligned}&\int\limits_{[a-\delta,a)\cup(b,b+\delta]}Dx\,d\mu+\int\limits_{[c,d]}T_{[c,d]}x\,d\mu+h_x\mu([a,b])\\&\qquad=\int\limits_{[a-\delta,a)\cup(b,b+\delta]}Dy\,d\mu+\int\limits_{[c,d]}T_{[c,d]}y\,d\mu+h_y\mu([a,b]),\end{aligned}$$

which implies

$$\begin{aligned}&|h_x\mu([a,b])-h_y\mu([a,b])|\\&\qquad=\Big|\int\limits_{[a-\delta,a)\cup(b,b+\delta]}(Dx-Dy)\,d\mu+\int\limits_{[c,d]}(T_{[c,d]}x-T_{[c,d]}y)\,d\mu\Big|.\end{aligned}$$

We obtain

$$\begin{aligned}|h_x\mu([a,b]) - h_y\mu([a,b])| &= |h_x - h_y|\mu([a,b])\\ &= \mu([a,b])\|Dx - Dy\|\\ &= k_1\mu([a,b])\|x-y\|,\end{aligned}$$

and on the other hand

$$\begin{aligned}&\Big|\int\limits_{[a-\delta,a)\cup(b,b+\delta]}(Dx-Dy)\,d\mu + \int\limits_{[c,d]}(T_{[c,d]}x - T_{[c,d]}y)\,d\mu\Big|\\ &\quad\le \int\limits_{[a-\delta,a)\cup(b,b+\delta]}\|Dx-Dy\|\,d|\mu| + \int\limits_{[c,d]}\|T_{[c,d]}x - T_{[c,d]}y\|\,d|\mu|\\ &\quad\le k_1|\mu|([a-\delta,a)\cup(b,b+\delta])\|x-y\| + k|\mu|([c,d])\|x-y\|\\ &\quad< k_1\mu([a,b])\|x-y\|.\end{aligned}$$

This contradiction shows that $D \in L(k)$.

Finally, observe that

$$\max_{t\in[\alpha,\beta]}|x(t) - (Tx)(t)| \ge \max_{t\in[c,d]}|x(t) - (T_{[c,d]}x)(t)| > 1 - 1/k. \ \blacksquare$$

Now we generalize this theorem to all subspaces of $C[0,1]$ of finite codimension. Recall that each such subspace can be written as

$$C_{\mu_1,\dots,\mu_n}[0,1] = \Big\{x \in C[0,1] : \int\limits_{[0,1]} x\,d\mu_i = 0,\ i = 1,\dots,n\Big\},$$

where the μ_i are independent, bounded real Borel measures on $[0,1]$.

THEOREM 6. *Let $C_{\mu_1,\dots,\mu_n}[0,1]$ be a subspace of $C[0,1]$ of finite codimension. Then for $0 \le \alpha < \beta \le 1$ and for any $k \ge 1$ there exists a mapping $T : B \to S$ of class $L(k)$ such that for every $x \in B$,*

$$(Tx)(t) = 0 \quad \textit{for every } t \in [0,\alpha]\cup[\beta,1]$$

and

$$\max_{t\in[\alpha,\beta]}|x(t) - (Tx)(t)| > 1 - 1/k.$$

Proof. We argue by induction on n. If $n = 1$, then the conclusion is Theorem 5. Now let $\mu_1,\dots,\mu_{n+1}$ be independent measures on $[0,1]$. For brevity set, for any $l \in \mathbb{N}$,

$$B_l = B_{C_{\mu_1,\dots,\mu_l}[0,1]}, \qquad S_l = S_{C_{\mu_1,\dots,\mu_l}[0,1]}.$$

Let $0 \le \alpha < \beta \le 1$ and assume that for all γ,δ such that $\alpha < \gamma < \delta < \beta$ there exists a lipschitzian mapping $\overline{T} : B_n \to S_n$ with constant $k \ge 1$ such that for any $x \in B_n$,

$$(\overline{T}x)(t) = 0 \quad \text{for any } t \in [0,\gamma]\cup[\delta,1]$$

and

$$\max_{t\in[\gamma,\delta]} |x(t) - (\overline{T}x)(t)| > 1 - 1/k.$$

Let us consider two cases.

(i) There exists an interval $[a,b] \subset (\alpha,\beta)$ such that $\int_{[a,b]} \overline{T}x\, d\mu_{n+1} = 0$ for every $x \in B_{n+1}$. Then we can define $T : B_{n+1} \to S_{n+1}$ to be $\overline{T}$, where $\gamma = a$, $\delta = b$. Observe that $T \in L(k)$ and moreover

$$\max_{t\in[\alpha,\beta]} |x(t) - (Tx)(t)| \ge \max_{t\in[a,b]} |x(t) - (\overline{T}x)(t)| > 1 - 1/k.$$

(ii) If (i) does not hold then there exists an interval $[a,b] \subset (\alpha,\beta)$ and a function $x_0 \in B_{n+1}$ with $\operatorname{supp} \overline{T}x_0 \subset [a,b]$ (it is enough to take $\gamma = a$, $\delta = b$) such that $\int_{[a,b]} \overline{T}x_0\, d\mu_{n+1} \neq 0$. Without loss of generality we may assume that $\int_{[a,b]} \overline{T}x_0\, d\mu_{n+1} > 0$. Now we can choose an interval $[c,d] \subset (\alpha,\beta)$ disjoint from $[a,b]$ and such that

$$|\mu_{n+1}|([c,d]) < \int\limits_{[a,b]} \overline{T}x_0\, d\mu_{n+1}.$$

From the induction assumption we know that there exists a mapping $\widetilde{T} : B_n \to S_n$ of class $L(k), k \ge 1$, such that for every $x \in B_n$,

$$(\widetilde{T}x)(t) = 0 \quad \text{for every } t \in [0,c] \cup [d,1]$$

and

$$\max_{t\in[c,d]} |x(t) - (\widetilde{T}x)(t)| > 1 - 1/k.$$

For any $x \in B_{n+1}$ and every $h \in [-1,1]$ define $T_h x = h\overline{T}x_0 + \widetilde{T}x$. Observe that for $h = -1$ we have

$$\begin{aligned}\int\limits_{[0,1]} T_{-1}x\, d\mu_{n+1} &= -\int\limits_{[a,b]} \overline{T}x_0\, d\mu_{n+1} + \int\limits_{[c,d]} \widetilde{T}x\, d\mu_{n+1}\\ &\le -\int\limits_{[a,b]} \overline{T}x_0\, d\mu_{n+1} + |\mu_{n+1}|([c,d]) < 0.\end{aligned}$$

On the other hand for $h = 1$ we get

$$\begin{aligned}\int\limits_{[0,1]} T_1 x\, d\mu_{n+1} &= \int\limits_{[a,b]} \overline{T}x_0\, d\mu_{n+1} + \int\limits_{[c,d]} \widetilde{T}x\, d\mu_{n+1}\\ &\ge \int\limits_{[a,b]} \overline{T}x_0\, d\mu_{n+1} - |\mu_{n+1}|([c,d]) > 0.\end{aligned}$$

Those two facts imply that there exists $h \in (-1,1)$ such that

$$\int\limits_{[0,1]} T_h x\, d\mu_{n+1} = 0,$$

so that $T_h x \in C_{\mu_1,\dots,\mu_{n+1}}[0,1]$. We show that for every $x \in B_{n+1}$ there exists a unique constant $h = h_x$ for which the above condition is satisfied. Suppose there are constants $h_1 \neq h_2$ such that

$$\int_{[0,1]} T_{h_1} x \, d\mu_{n+1} = \int_{[0,1]} T_{h_2} x \, d\mu_{n+1} = 0.$$

This can be written as

$$\int_{[a,b]} h_1 \overline{T} x_0 \, d\mu_{n+1} + \int_{[c,d]} \widetilde{T} x \, d\mu_{n+1} = \int_{[a,b]} h_2 \overline{T} x_0 \, d\mu_{n+1} + \int_{[c,d]} \widetilde{T} x \, d\mu_{n+1},$$

which means

$$0 = \int_{[a,b]} (h_1 - h_2) \overline{T} x_0 \, d\mu_{n+1} = (h_1 - h_2) \int_{[a,b]} \overline{T} x_0 \, d\mu_{n+1},$$

a contradiction.

Now define $T : B_{n+1} \to S_{n+1}$ by

$$Tx = h_x \overline{T} x_0 + \widetilde{T} x.$$

Because $\|\widetilde{T} x\| = 1$ for any $x \in B_{n+1}$ and $\operatorname{supp} \overline{T} x_0 \cap \operatorname{supp} \widetilde{T} x = \emptyset$, it follows that $\|Tx\| = 1$.

To show that $T \in L(k)$, we should prove that

$$|h_x - h_y| \le k\|x - y\|$$

for all $x, y \in B_{n+1}$. Let $M = \int_{[a,b]} \overline{T} x_0 \, d\mu_{n+1}$. We have

$$\begin{aligned} 0 = \int_{[0,1]} Tx \, d\mu_{n+1} &= \int_{[a,b]} h_x \overline{T} x_0 \, d\mu_{n+1} + \int_{[c,d]} \widetilde{T} x \, d\mu_{n+1} \\ &= h_x \int_{[a,b]} \overline{T} x_0 \, d\mu_{n+1} + \int_{[c,d]} \widetilde{T} x \, d\mu_{n+1} \\ &= h_x M + \int_{[c,d]} \widetilde{T} x \, d\mu_{n+1}. \end{aligned}$$

From this we get

$$\begin{aligned} |h_x - h_y| &= M^{-1} \Big| \int_{[c,d]} (\widetilde{T} x - \widetilde{T} y) \, d\mu_{n+1} \Big| \\ &\le M^{-1} \int_{[c,d]} k\|x - y\| \, d|\mu_{n+1}| \\ &= k M^{-1} |\mu_{n+1}|([c,d]) \|x - y\| \\ &\le k\|x - y\|. \end{aligned}$$

Finally, observe that

$$\max_{t \in [\alpha,\beta]} |x(t) - (Tx)(t)| \ge \max_{t \in [c,d]} |x(t) - (\widetilde{T} x)(t)| > 1 - 1/k. \ \blacksquare$$

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