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The simplex of tracial quantum symmetric states

by

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Abstract. We show that the space of tracial quantum symmetric states of an arbitrary unital C^*-algebra is a Choquet simplex and is a face of the tracial state space of the universal unital C^*-algebra free product of A with itself infinitely many times. We also show that the extreme points of this simplex are dense, making it the Poulsen simplex when A is separable and nontrivial. In the course of the proof we characterize the centers of certain tracial amalgamated free product C^*-algebras.

1. Introduction and description of results. Quantum exchangeable random variables (that is, random variables whose distributions are invariant for the natural co-actions of S. Wang's quantum permtuation groups [11]) were characterized by Köstler and Speicher [6] to be those sequences of identically distributed random variables that are free with respect to the conditional expectation onto their tail algebra (that is, free with amalgamation over the tail algebra).

In [4], Dykema, Köstler and Williams considered, for any unital C^*-algebra A, the analogous notion of quantum symmetric states on the universal unital free product C^*-algebra $\mathfrak{A} = *_1^\infty A$. The symbol $\mathrm{QSS}(A)$ denotes the compact convex set of all quantum symmetric states on $\mathfrak{A}$. The paper [4] contains a convenient characterization of the extreme points of $\mathrm{QSS}(A)$. Also the compact convex set $\mathrm{TQSS}(A) \subseteq \mathrm{QSS}(A)$ of all tracial quantum symmetric states on $\mathfrak{A}$ was considered, and the extreme points of $\mathrm{TQSS}(A)$ were described. Question 8.8 of [4] asks whether $\mathrm{TQSS}(A)$ is a Choquet simplex (when A has a tracial state, for otherwise $\mathrm{TQSS}(A)$ is empty).

The main result of this note is that $\mathrm{TQSS}(A)$ is a Choquet simplex whose extreme points are dense. Thus, when A is separable and nontrivial, $\mathrm{TQSS}(A)$ is the Poulsen simplex [7], which is the unique metrizable simplex whose extreme points are dense. In showing this, we also see that $\mathrm{TQSS}(A)$

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is a face of the simplex $\mathrm{TS}(\mathfrak{A})$ of all tracial states on $\mathfrak{A}$ and we obtain a better description of the extreme points of $\mathrm{TQSS}(A)$.

Along the way, we prove some technical results that we need and that may be useful in other contexts. In Section 2, we provide a proof (not readily found in the literature) of a well known fact that natural conditions are sufficient for an amalgamated free product to have a trace. In Section 3, we characterize the centers of certain tracial von Neumann algebra free products with amalgamation and we use this to characterize the set of conditional-expectation-preserving traces of von Neumann algebras. Section 4 is short and consists of a technical result about conditional expectations. Finally, in Section 5, we prove the main result.

2. Amalgamated free products and tracial amalgamated free products. Let $\mathcal{D}$ be a von Neumann algebra, let I be a nonempty set and for every $i \in I$ let $\mathcal{B}_i$ be a von Neumann algebra containing $\mathcal{D}$ by a unital inclusion of von Neumann algebras, and suppose $E_i : \mathcal{B}_i \to \mathcal{D}$ is a normal conditional expectation with faithful GNS representation. Let

$$(\mathcal{M}, F) = (*_{\mathcal{D}})_{i\in I}(\mathcal{B}_i, E_i)$$

be the von Neumann algebra amalgamated free product. In the case that the E_i are all faithful, details of this construction were given by Ueda [9], and he showed that then F is faithful (see [9, p. 364]). Alternatively, and also when the conditional expectations E_i fail to be faithful but do have faithful GNS representations, the free product construction may be performed by (a) taking the C^*-algebra free product $(\mathcal{M}_0, F_0)$ of the family of $(\mathcal{B}_i, E_i)$ acting on the free product Hilbert C^*-module V, (b) taking any normal, faithful $*$-representation π of $\mathcal{D}$ on a Hilbert space $\mathcal{H}_\pi$, (c) letting $\mathcal{M}$ be the strong-operator-topology closure of the image of the resulting representation of $\mathcal{M}_0$ on the Hilbert space $V \otimes_\pi \mathcal{H}_\pi$, and (d) letting $F : \mathcal{M} \to \mathcal{D}$ be compression by the projection from $V \otimes_\pi \mathcal{H}_\pi$ onto the Hilbert subspace $\mathcal{D} \otimes_\pi \mathcal{H}_\pi$. The fact that $\mathcal{M}$ is independent of the representation π follows from the fact that any two normal faithful representations of $\mathcal{D}$ are related by dilation and compression by a projection in the commutant.

The following result is well known, but since we rely on it, this seems like a good place to give a brief proof.

Proposition 2.1. *Suppose τ is a normal trace on $\mathcal{D}$ such that for all $i \in I$, $\tau \circ E_i$ is a trace on $\mathcal{B}_i$. Then $\tau \circ F$ is a trace on $\mathcal{M}$ and is faithful if and only if τ is faithful. Furthermore, every normal tracial state on $\mathcal{M}$ that is preserved by F arises in this fashion.*

Proof. Since every tracial state τ on $\mathcal{M}$ that is preserved by F must equal $\tau\restriction_{\mathcal{D}} \circ F$, the last assertion of the proposition is clearly true. Moreover, suppose we know that $\tau \circ F$ is a trace; if we assume also that τ is faithful, then the GNS representation of $\tau \circ F$ will be faithful; since it is a trace, it follows that $\tau \circ F$ is itself faithful. Thus, we need only show that $\tau \circ F$ is a trace.

Let $\mathcal{B}_i^{\mathrm{o}} = \mathcal{B}_i \cap \ker E_i$. Let $m, n \in \mathbb{N}$ and let $b_j \in \mathcal{B}_{i(j)}^{\mathrm{o}}$ for $1 \le j \le m$ and $c_j \in \mathcal{B}_{k(j)}^{\mathrm{o}}$ for all $1 \le j \le n$, with $i(j) \ne i(j+1)$ and $k(j) \ne k(j+1)$. If $d \in \mathcal{D}$, then by freeness, we have

$$F(d(c_1 \cdots c_n)) = 0 = F((c_1 \cdots c_n)d), \tag{2.1}$$

so the composition with τ is also zero. We will show by induction on $\min(m, n)$ that

$$\tau \circ F((b_m \cdots b_2 b_1)(c_1 c_2 \cdots c_n)) = \tau \circ F((c_1 c_2 \cdots c_n)(b_m \cdots b_2 b_1)), \tag{2.2}$$

and furthermore that the above quantity is zero unless $m = n$ and $i(j) = k(j)$ for all j, in which case it equals

$$\begin{aligned} &\tau \circ E_{i(m)}(b_m\, E_{i(m-1)}(b_{m-1} \cdots E_{i(2)}(b_2\, E_{i(1)}(b_1 c_1)\, c_2) \cdots c_{m-1})\, c_m) \\ &\quad = \tau \circ E_{i(1)}(c_1\, E_{i(2)}(c_2 \cdots E_{i(m-1)}(c_{m-1} E_{i(m)}(c_m b_m)\, b_{m-1}) \cdots b_2)\, b_1). \end{aligned} \tag{2.3}$$

This will suffice to prove the proposition, because the span of $\mathcal{D}$ and such elements $b_1 \cdots b_m$ is dense in $\mathcal{M}$.

By freeness, we have

$$\begin{aligned} &F((b_m \cdots b_2 b_1)(c_1 c_2 \cdots c_n)) \\ &\qquad = \delta_{i(1),k(1)}\, F((b_m \cdots b_2) E_{i(1)}(b_1 c_1)(c_2 \cdots c_n)). \end{aligned} \tag{2.4}$$

If $m = n = 1$, then (2.2) and (2.3) follow from traciality of $\tau \circ E_{i(1)} : \mathcal{B}_{i(1)} \to \mathbb{C}$. If $\min(m, n) = 1$ and $\max(m, n) > 1$, then the right-hand side of (2.4) is zero by (2.1), and by symmetry also $F((c_1 c_2 \cdots c_n)(b_m \cdots b_2 b_1)) = 0$, as required.

We may thus suppose $\min(m, n) > 1$ and $i(1) = k(1)$. Then, using the induction hypothesis (and noting that $\mathcal{D} c_2 \subseteq \mathcal{B}_{k(2)}^{\mathrm{o}}$), we have

$$\begin{aligned} &\tau \circ F((b_m \cdots b_2 b_1)(c_1 c_2 \cdots c_n)) \\ &\qquad = \delta_{i(1),k(1)}\, \tau \circ F((b_m \cdots b_2) E_{i(1)}(b_1 c_1)(c_2 \cdots c_n)) \\ &\qquad = \delta_{i(1),k(1)} \delta_{m,n} \delta_{i(2),k(2)} \cdots \delta_{i(m),k(m)} \\ &\qquad\quad \cdot \tau \circ E_{i(m)}(b_m\, E_{i(m-1)}(b_{m-1} \cdots E_{i(2)}(b_2 E_{i(1)}(b_1 c_1)\, c_2) \cdots c_{m-1})\, c_m). \end{aligned}$$

If $m \ne n$ or if $m = n$ but $i(j) \ne k(j)$ for some j, then not only is the above quantity zero but, by symmetry, also $\tau \circ F((c_1 c_2 \cdots c_n)(b_m \cdots b_2 b_1))$ vanishes.

We may thus suppose $m = n > 1$ and $i(j) = k(j)$ for all j. Treating $E_{i(1)}(b_1 c_1) c_2$ as an element of $\mathcal{B}_{k(2)}^{\mathrm{o}}$, by the induction hypothesis of (2.3), we get

$$\begin{aligned}\tau\circ E_{i(m)}&(b_m\,E_{i(m-1)}(b_{m-1}\cdots E_{i(2)}(b_2E_{i(1)}(b_1c_1)\,c_2)\cdots c_{m-1})\,c_m)\\&=\tau\circ E_{i(2)}(E_{i(1)}(b_1c_1)c_2\,E_{i(3)}(c_3\cdots E_{i(m)}(c_mb_m)\cdots b_3)b_2)\\&=\tau(E_{i(1)}(b_1c_1)\,E_{i(2)}(c_2\,E_{i(3)}(c_3\cdots E_{i(m)}(c_mb_m)\cdots b_3)b_2))\\&=\tau\circ E_{i(1)}(b_1c_1\,E_{i(2)}(c_2\,E_{i(3)}(c_3\cdots E_{i(m)}(c_mb_m)\cdots b_3)b_2))\\&=\tau\circ E_{i(1)}(c_1\,E_{i(2)}(c_2\,E_{i(3)}(c_3\cdots E_{i(m)}(c_mb_m)\cdots b_3)b_2)b_1),\end{aligned}$$

where in the last equality we have used the traciality of $\tau\circ E_{i(1)}$. Thus, we have proved the identity (2.3) and that this quantity equals

$$\tau\circ F((b_m\cdots b_2b_1)(c_1c_2\cdots c_n)).$$

By symmetry, it is equal also to $\tau\circ F((c_1c_2\cdots c_n)(b_m\cdots b_2b_1))$. ∎

Of course, the result analogous to Proposition 2.1 for amalgamated free products of C^*-algebras is true by the same proof.

3. Centers of certain amalgamated free products. Let $\mathcal{D}\subseteq\mathcal{B}$ be a unital inclusion of von Neumann algebras with a normal conditional expectation $E:\mathcal{B}\to\mathcal{D}$ whose GNS representation is faithful. Suppose there is a normal, faithful, tracial state $\tau_{\mathcal{D}}$ on $\mathcal{D}$ such that $\tau_{\mathcal{B}}:=\tau_{\mathcal{D}}\circ E$ is a trace on $\mathcal{B}$. The GNS representation of $\tau_{\mathcal{B}}$ is an action of $\mathcal{B}$ on the Hilbert space $L^2(\mathcal{B},\tau_{\mathcal{B}})=L^2(\mathcal{B},E)\otimes_{\mathcal{D}}L^2(\mathcal{D},\tau)$ by multiplication on the left, and thus the GNS representation of $\tau_{\mathcal{B}}$ is faithful. Since $\tau_{\mathcal{B}}$ is a trace, it follows that $\tau_{\mathcal{B}}$ itself is faithful, and hence E must be faithful.

For an element x of a von Neumann algebra, we will let $[x]$ denote the range projection of x. Thus, $[x]$ is the orthogonal projection onto the closure of the range of x, considered as a Hilbert space operator, and it belongs to the von Neumann algebra generated by x. The notation $Z(\mathcal{A})$ means the center of $\mathcal{A}$.

LEMMA 3.1. *With $E:\mathcal{B}\to\mathcal{D}$ and trace $\tau_{\mathcal{B}}$ as above, let*

$$q=q(E)=\bigvee\{[E(b^*b)]\mid b\in\ker E\}.$$

Then $q\in\mathcal{D}\cap Z(\mathcal{B})$ and $(1-q)\mathcal{B}=(1-q)\mathcal{D}$.

Proof. If $b\in\ker E$ and u is a unitary in $\mathcal{D}$ then $bu\in\ker E$, and

$$[E((bu)^*(bu)]=[u^*E(b^*b)u]=u^*[E(b^*b)]u$$

and we get $u^*qu=q$. Thus, $q\in Z(\mathcal{D})$.

If $q\notin Z(\mathcal{B})$, then there would be a partial isometry $v\in\mathcal{B}$ so that $0\neq v^*v\le 1-q$ and $vv^*\le q$. Since $q\in Z(\mathcal{D})$ we get $E(v)=qE(v)(1-q)=0$. But, since E is faithful, $E(v^*v)\neq 0$ and $[E(v^*v)]\le 1-q$, contrary to the definition of q. Thus, we must have $q\in Z(\mathcal{B})$.

If $(1-q)\mathcal{B}\neq(1-q)\mathcal{D}$, then there would be $b\in(1-q)\mathcal{B}\cap\ker E$ with $b\neq 0$. But again, this yields $0\neq E(b^*b)=(1-q)E(b^*b)$, contrary to the choice of q. Thus, we must have $(1-q)\mathcal{B}=(1-q)\mathcal{D}$. ∎

Let

$$(\mathcal{M}, F) = (*_{\mathcal{D}})_1^\infty(\mathcal{B}, E) \tag{3.1}$$

be the von Neumann algebra free product with amalgamation over $\mathcal{D}$ of infinitely many copies of $(\mathcal{B}, E)$. Our main goal in this section is to show that the center of $\mathcal{M}$ is contained in $\mathcal{D}$.

Let $\tau = \tau_{\mathcal{D}} \circ F$. By Proposition 2.1, τ is a faithful trace on $\mathcal{M}$. Let $(\mathcal{B}_i, E_i)$ denote the ith copy of $(\mathcal{B}, E)$ in the construction of $\mathcal{M}$. We now describe some standard notation for $\mathcal{M}$ and related objects. The von Neumann algebra $\mathcal{M}$ is constructed on the Hilbert space $L^2(\mathcal{M}, \tau)$, and we write $\mathcal{M} \ni x \mapsto \hat{x} \in L^2(\mathcal{M}, \tau)$ for the usual mapping with dense range. For convenience, we will take the inner product on $L^2(\mathcal{M}, \tau)$ to be linear in the second variable and conjugate linear in the first variable. Thus, we have, for $x_1, x_2 \in \mathcal{M}$,

$$\langle \hat{x}_1, \hat{x}_2 \rangle = \tau(x_1^* x_2),$$

and, as usual, we write the corresponding norm $\|x\|_2 = \tau(x^*x)^{1/2}$. Then we have $L^2(\mathcal{M}, \tau) = L^2(\mathcal{M}, F) \otimes_{\mathcal{D}} L^2(\mathcal{D}, \tau_{\mathcal{D}})$, and this is isomorphic to

$$L^2(\mathcal{D}, \tau_{\mathcal{D}}) \oplus \bigoplus_{\substack{k \geq 1 \\ i_1, \ldots, i_k \geq 1 \\ i_j \neq i_{j+1}}} \mathcal{H}_{i_1}^{\mathrm{o}} \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} \mathcal{H}_{i_k}^{\mathrm{o}} \otimes_{\mathcal{D}} L^2(\mathcal{D}, \tau_{\mathcal{D}}),$$

where $\mathcal{H}_i^{\mathrm{o}}$ is the Hilbert $\mathcal{D}, \mathcal{D}$-bimodule $L^2(\mathcal{B}_i, E_i) \ominus \mathcal{D}$. We will denote by λ the left action of $\mathcal{M}$ on $L^2(\mathcal{M}, \tau)$ and by ρ the anti-multiplicative right action, $\rho(x) = J\lambda(x^*)J$, where J is the standard conjugate linear isometry of $L^2(\mathcal{M}, \tau)$ defined by $\hat{x} \mapsto (x^*)\hat{}$.

Lemma 3.2. *Let $N \in \mathbb{N}$, let*

$$\eta_1,\, \eta_2 \in L^2(\mathcal{D}, \tau_{\mathcal{D}}) \oplus \bigoplus_{\substack{k \geq 1 \\ 1 \leq i_1, \ldots, i_k \leq N \\ i_j \neq i_{j+1}}} \mathcal{H}_{i_1}^{\mathrm{o}} \otimes_{\mathcal{D}} \cdots \otimes_{\mathcal{D}} \mathcal{H}_{i_k}^{\mathrm{o}} \otimes_{\mathcal{D}} L^2(\mathcal{D}, \tau_{\mathcal{D}})$$

and let $b_1, b_2 \in \mathcal{B}_{N+1}$. Let $c_1, c_2, d_1, d_2 \in \mathcal{D}$ be such that

$$c_1^* c_2 = E_{N+1}(b_1^* b_2), \qquad d_2 d_1^* = E_{N+1}(b_2 b_1^*).$$

Then

$$\langle \lambda(b_1)\eta_1, \lambda(b_2)\eta_2 \rangle = \langle \lambda(c_1)\eta_1, \lambda(c_2)\eta_2 \rangle,$$
$$\langle \rho(b_1)\eta_1, \rho(b_2)\eta_2 \rangle = \langle \rho(d_1)\eta_1, \rho(d_2)\eta_2 \rangle.$$

Proof. We may without loss of generality assume $\eta_j = \hat{x}_j$ for some x_j in $W^*(\bigcup_{j=1}^N \mathcal{B}_j)$. Then

$$\langle \lambda(b_1)\eta_1, \lambda(b_2)\eta_2 \rangle = \tau(x_1^* b_1^* b_2 x_2) = \tau_{\mathcal{D}}(F(x_1^* b_1^* b_2 x_2)).$$

By freeness, we have

$$F(x_1^* b_1^* b_2 x_2) = F(x_1^* F(b_1^* b_2) x_2) = F(x_1^* c_1^* c_2 x_2),$$

from which we get

$$\langle\lambda(b_1)\eta_1,\lambda(b_2)\eta_2\rangle=\tau(x_1^*c_1^*c_2x_2)=\langle\lambda(c_1)\eta_1,\lambda(c_2)\eta_2\rangle.$$

Similarly, we have

$$\begin{aligned}\langle\rho(b_1)\eta_1,\rho(b_2)\eta_2\rangle&=\tau(b_1^*x_1^*x_2b_2)=\tau(x_2b_2b_1^*x_1^*)\\&=\tau(x_2d_2d_1^*x_1^*)=\langle\rho(d_1)\eta_1,\rho(d_2)\eta_2\rangle.\ \blacksquare\end{aligned}$$

THEOREM 3.3. *The center of $\mathcal{M}$ lies in $\mathcal{D}$. In particular,*

$$Z(\mathcal{M})=\mathcal{D}\cap Z(\mathcal{B}).\tag{3.2}$$

Proof. It suffices to show $Z(\mathcal{M})\subseteq\mathcal{D}$, for then (3.2) follows readily.

Let $x\in Z(\mathcal{M})$. Let $\eta=\hat{x}-F(x)\hat{}$. Then

$$\eta\in\bigoplus_{\substack{k\geq1\\ i_1,\dots,i_k\geq1\\ i_j\neq i_{j+1}}}\mathcal{H}_{i_1}^{\mathrm{o}}\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}}\mathcal{H}_{i_k}^{\mathrm{o}}\otimes_{\mathcal{D}}L^2(\mathcal{D},\tau_{\mathcal{D}}).$$

For $N\in\mathbb{N}$, let η_N be the orthogonal projection of η onto the subspace

$$\bigoplus_{\substack{k\geq1\\ 1\leq i_1,\dots,i_k\leq N\\ i_j\neq i_{j+1}}}\mathcal{H}_{i_1}^{\mathrm{o}}\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}}\mathcal{H}_{i_k}^{\mathrm{o}}\otimes_{\mathcal{D}}L^2(\mathcal{D},\tau_{\mathcal{D}}).$$

Then η_N converges in $L^2(\mathcal{M},\tau)$ to η as $N\to\infty$. Suppose $b\in\mathcal{B}\cap\ker E$. Fix $N\in\mathbb{N}$ and let b_N denote the copy of b in the copy $\mathcal{B}_N\subseteq\mathcal{M}$ of $\mathcal{B}$. Then $\lambda(b_N)\eta_{N-1}$ and $\rho(b_N)\eta_{N-1}$ are orthogonal to each other, because they lie in the respective subspaces

$$\bigoplus_{\substack{k\geq1\\ 1\leq i_1,\dots,i_k\leq N-1\\ i_j\neq i_{j+1}}}\mathcal{H}_N^{\mathrm{o}}\otimes_{\mathcal{D}}\mathcal{H}_{i_1}^{\mathrm{o}}\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}}\mathcal{H}_{i_k}^{\mathrm{o}}\otimes_{\mathcal{D}}L^2(\mathcal{D},\tau_{\mathcal{D}}),\tag{3.3}$$

$$\bigoplus_{\substack{k\geq1\\ 1\leq i_1,\dots,i_k\leq N-1\\ i_j\neq i_{j+1}}}\mathcal{H}_{i_1}^{\mathrm{o}}\otimes_{\mathcal{D}}\cdots\otimes_{\mathcal{D}}\mathcal{H}_{i_k}^{\mathrm{o}}\otimes_{\mathcal{D}}\mathcal{H}_N^{\mathrm{o}}\otimes_{\mathcal{D}}L^2(\mathcal{D},\tau_{\mathcal{D}}).\tag{3.4}$$

On the other hand, $\lambda(b_N)F(x)\hat{}$ and $\rho(b_N)F(x)\hat{}$ lie in the subspace $\mathcal{H}_N^{\mathrm{o}}\otimes_{\mathcal{D}}L^2(\mathcal{D},\tau_{\mathcal{D}})$, which is orthogonal to both of the subspaces (3.3) and (3.4). Thus, we have

$$\begin{aligned}0&=(b_Nx-xb_N)\hat{}=(\lambda(b_N)-\rho(b_N))\hat{x}\\&=(\lambda(b_N)-\rho(b_N))(\eta_{N-1}+F(x)\hat{}+(\eta-\eta_{N-1}))\end{aligned}$$

and from the orthogonality relations noted above, we get

$$\begin{aligned}(3.5)\qquad \|\lambda(b_N)\eta_{N-1}\|_2^2 + \|\rho(b_N)\eta_{N-1}\|_2^2 \\ &\le \|\lambda(b_N)\eta_{N-1} - \rho(b_N)\eta_{N-1} + (\lambda(b_N) - \rho(b_N))F(x)\hat{}\|_2^2 \\ &= \|(\lambda(b_N) - \rho(b_N))(\eta - \eta_{N-1})\|_2^2 \\ &\le 4\|b\|^2\|\eta - \eta_{N-1}\|_2^2.\end{aligned}$$

Consider the elements $d_1 = E(b^*b)^{1/2}$ and $d_2 = E(bb^*)^{1/2}$ of $\mathcal{D}$. By Lemma 3.2, we have

$$\|\lambda(b_N)\eta_{N-1}\|_2 = \|\lambda(d_1)\eta_{N-1}\|_2, \qquad \|\rho(b_N)\eta_{N-1}\|_2 = \|\rho(d_2)\eta_{N-1}\|_2,$$

and from (3.5), we get

$$\|\lambda(d_1)\eta_{N-1}\|_2^2 + \|\rho(d_2)\eta_{N-1}\|_2^2 \le 4\|b\|^2\|\eta - \eta_{N-1}\|_2^2.$$

Letting $N \to \infty$, we get

$$\lambda(d_1)\eta = 0 = \rho(d_2)\eta. \tag{3.6}$$

Let $q = q(E) \in \mathcal{D} \cap Z(\mathcal{B})$ be the projection associated to the conditional expectation $E : \mathcal{B} \to \mathcal{D}$ as described in Lemma 3.1. From (3.6) and letting b run through all of $\ker E$, we get $\lambda(q)\eta = \rho(q)\eta = 0$. This yields $q(x - F(x)) = 0$, so $x - F(x) \in (1 - q)\mathcal{B} = (1 - q)\mathcal{D}$. But $x - F(x) \perp \mathcal{D}$, so we must have $x - F(x) = 0$ and $x \in \mathcal{D}$. ∎

The aim of the remainder of this section (realized in Corollary 3.6 below) is to characterize the normal traces on a von Neumann subalgebra whose compositions with a given conditional expectation are traces on the larger von Neumann algebra. The result is quite natural and is perhaps known. It may also be possible to prove it directly using state decompositions or averaging techniques, rather than free products. However, as we get it from the results above with very little extra effort, it seems worth doing it here. Furthermore, it is clearly related to the proof of our main result, Theorem 5.3, and indeed to the improved characterization of extremality of elements of $\mathrm{TQSS}(A)$, though we do not actually use it in the proof.

Let $\mathcal{D} \subseteq \mathcal{B}$ be a unital inclusion of finite von Neumann algebras with a faithful conditional expectation $E : \mathcal{B} \to \mathcal{D}$. Suppose there is a normal faithful tracial state ρ on $\mathcal{D}$ such that $\rho \circ E$ is a trace on $\mathcal{B}$. Let

$$\mathcal{C} = Z(\mathcal{B}) \cap \mathcal{D}. \tag{3.7}$$

Let $(\mathcal{M}, F)$ be the free product of infinitely many copies of $(\mathcal{B}, E)$ with amalgamation over $\mathcal{D}$, as in (3.1). Due to the existence of ρ, by Proposition 2.1, $\mathcal{M}$ is a finite von Neumann algebra. Let η be the center valued trace on $\mathcal{M}$ and let $\eta{\restriction}_{\mathcal{D}}$ denote its restriction to $\mathcal{D}$. By Theorem 3.3, the center of $\mathcal{M}$ is $\mathcal{C}$ as in (3.7).

Let α be a permutation of $\mathbb{N}$ that has no proper, nonempty, invariant subsets; thus, α results from the shift on $\mathbb{Z}$ after fixing a bijection from $\mathbb{N}$ to $\mathbb{Z}$. Let $\hat{\alpha}$ be the automorphism of $\mathcal{M}$ that permutes the copies of $\mathcal{B}$ in the free product construction (3.1) according to α.

Lemma 3.4. *We have $\eta = \eta \circ \hat{\alpha}$.*

Proof. Dixmier averaging says that for any $x \in \mathcal{M}$, $\eta(x)$ is the unique element in the intersection of $\mathcal{C}$ and the norm closed convex hull of the unitary conjugates of x. (See, for example, Section 8.3 of [5].) In symbols, this is

$$\{\eta(x)\} = \mathcal{C} \cap \overline{\mathrm{conv}\{uxu^* \mid u \in \mathcal{U}(\mathcal{M})\}}.$$

Since $\mathcal{C} \subseteq \mathcal{D}$, $\hat{\alpha}$ leaves every element of $\mathcal{C}$ fixed. Thus,

$$\begin{aligned}\{\eta(x)\} &= \hat{\alpha}(\{\eta(x)\}) = \hat{\alpha}(\mathcal{C}) \cap \hat{\alpha}(\overline{\mathrm{conv}\{uxu^* \mid u \in \mathcal{U}(\mathcal{M})\}}) \\ &= \mathcal{C} \cap \overline{\mathrm{conv}\{u\,\hat{\alpha}(x)u^* \mid u \in \mathcal{U}(\mathcal{M})\}}) = \{\eta(\hat{\alpha}(x))\}. \ \blacksquare\end{aligned}$$

Lemma 3.5. *We have $\eta = \eta \circ F$.*

Proof. It is well known and not difficult to check that for all $x \in \mathcal{M}$, the ergodic averages

$$\frac{1}{n}\sum_{k=0}^{n-1} \hat{\alpha}^k(x)$$

converge in $\|\cdot\|_2$-norm as $n \to \infty$, and thus also in strong operator topology, to $F(x)$. Because the center valued trace is normal, using Lemma 3.4, we get

$$\eta(F(x)) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \eta(\hat{\alpha}^k(x)) = \eta(x). \ \blacksquare$$

For a von Neumann algebra $\mathcal{N}$, we let $\mathrm{NTS}(\mathcal{N})$ denote the set of normal tracial states on $\mathcal{N}$.

Corollary 3.6. *The map*

$$\tau \mapsto \tau \circ \eta\restriction_{\mathcal{D}} \tag{3.8}$$

is a bijection from $\mathrm{NTS}(Z(\mathcal{B}) \cap \mathcal{D})$ *onto*

$$\{\rho \in \mathrm{NTS}(\mathcal{D}) \mid \rho \circ E \textit{ is a trace on } \mathcal{B}\}. \tag{3.9}$$

Proof. It is clear that the map (3.8) is injective.

We view $\mathcal{B}$ as embedded in $\mathcal{M}$ by identification of $\mathcal{B}$ with any of the copies arising in the free product construction (3.1). Since, by Lemma 3.5, $\eta = \eta \circ E = \eta\restriction_{\mathcal{D}} \circ E$, if $\tau \in \mathrm{NTS}(\mathcal{C})$ and $\rho = \tau \circ \eta\restriction_{\mathcal{D}}$, then $\rho \circ E = \tau \circ (\eta\restriction_{\mathcal{B}}) = (\tau \circ \eta)\restriction_{\mathcal{B}}$ is a trace on $\mathcal{B}$. Thus, the map (3.8) goes into the set (3.9).

To see that it is onto, suppose ρ belongs to the set (3.9). Since $\mathcal{M}$ is a finite von Neumann algebra, by a standard theory (see, for example, [5, Theorem 8.3.10]), the map $\tau \mapsto \tau \circ \eta$ is a bijection from $\mathrm{NTS}(\mathcal{C})$ onto $\mathrm{NTS}(\mathcal{M})$. By

Proposition 2.1, $\rho \circ F$ is a normal tracial state on $\mathcal{M}$, so equals $\tau \circ \eta$ for some $\tau \in \mathrm{NTS}(\mathcal{C})$. Thus, $\rho = \rho \circ F\restriction_{\mathcal{D}} = \tau \circ \eta\restriction_{\mathcal{D}}$, as required. ■

4. The conditional expectation onto the tail algebra in an amalgamated free product. Some of the main results of [4] depended on the characterization of quantum symmetric states as arising from amalgamated free products of C^*-algebras in a natural way. In this section, we prove a related result, which is unsurprising but of a slightly different flavor. We show, roughly speaking, that if a quantum symmetric state arises from an amalgamated free product of von Neumann algebras with faithful conditional expectations and states, then the tail algebra is contained in the von Neumann algebra over which one amalgamates, and the conditional expectation onto the tail algebra is the restriction of the one coming from the amalgamated free product construction.

Before embarking on the proof, we must review some facts about conditional expectations. The results of [4] about conditional expectations are for symmetric states, which are more general than quantum symmetric states. A state ψ on the universal free product C^*-algebra $\mathfrak{A} = *_1^\infty A$ is a symmetric state if it is invariant under the natural action of the permutation group S_∞ on $\mathfrak{A}$ that exchanges the copies of A. We then let $\mathcal{M}_\psi$ denote the von Neumann algebra generated by the image of $\mathfrak{A}$ under the GNS representation π_ψ of $\mathfrak{A}$ on $L^2(\mathfrak{A}, \psi)$, arising from ψ, and we let $\hat{\psi}$ denote the normal extension of ψ to $\mathcal{M}_\psi$, which is the vector state for the vector of $L^2(\mathfrak{A}, \psi)$ corresponding to the identity element of $\mathfrak{A}$. The tail algebra $\mathcal{T}_\psi$ is the von Neumann subalgebra

$$\mathcal{T}_\psi = \bigcap_{n\geq 1} W^*\Big(\bigcup_{k\geq n} \pi_\psi \circ \lambda_k(A)\Big) \subseteq \mathcal{M}_\psi$$

where $\lambda_k : A \to \mathfrak{A}$ is the embedding onto the kth copy of A in the universal free product C^*-algebra. Note that the action of the permutation group S_∞ on $\mathfrak{A}$ by permuting the embedded copies of A results in a ψ-preserving action of S_∞ on $\mathcal{M}_\psi$; we let $\mathcal{F}_\psi$ denote the fixed point subalgebra of this action, and we always have $\mathcal{T}_\psi \subseteq \mathcal{F}_\psi$ (see [4, Lemma 5.1.3]).

For most of this paper, we will be interested in the case when ψ is a trace. In this case, the state $\hat{\psi}$ on $\mathcal{M}_\psi$, since it has faithful GNS representation, must be faithful. Also, in Proposition 4.1 below, though in the interest of generality we do not assume ψ is a trace, the assumptions do imply that $\hat{\psi}$ is faithful on $\mathcal{M}_\psi$. In any case, assuming $\hat{\psi}$ is faithful on $\mathcal{M}_\psi$, Proposition 5.2.4 of [4] implies that $\mathcal{T}_\psi = \mathcal{F}_\psi$ and that there is a normal, faithful $\hat{\psi}$-preserving conditional expectation $E_\psi : \mathcal{M}_\psi \to \mathcal{T}_\psi$. The purpose of the next result and the remark that follows is to identify E_ψ and $\mathcal{T}_\psi$ in the case of an amalgamated free product.

Proposition 4.1. *Let $\mathcal{D} \subseteq \widetilde{\mathcal{B}}$ be a unital von Neumann subalgebra with $\widetilde{E} : \widetilde{\mathcal{B}} \to \mathcal{D}$ a normal, faithful, conditional expectation. Let*

$$(\widetilde{\mathcal{M}}, \widetilde{F}) \cong (*_{\mathcal{D}})_1^\infty (\widetilde{\mathcal{B}}, \widetilde{E})$$

be the amalgamated free product of von Neumann algebras. Suppose ρ is a normal faithful state on $\mathcal{D}$. Suppose A is a unital C^*-algebra and $\sigma : A \to \widetilde{\mathcal{B}}$ is a unital $*$-homomorphism. Let $\psi = \rho \circ \widetilde{F} \circ (*_1^\infty \sigma) : \mathfrak{A} = *_1^\infty A \to \mathbb{C}$. By [4, Proposition 3.1], $\psi \in \mathrm{QSS}(A)$. Then $\mathcal{M}_\psi$ is canonically identified with a von Neumann subalgebra of $\widetilde{\mathcal{M}}$ with the tail algebra $\mathcal{T}_\psi$ identified with a subalgebra of $\mathcal{D}$. Moreover, the normal state $\hat{\psi}$ on $\mathcal{M}_\psi$ is identified with the restriction of the state $\rho \circ \widetilde{F}$ to $\mathcal{M}_\psi$, which is faithful, and the normal conditional expectation $E_\psi : \mathcal{M}_\psi \to \mathcal{T}_\psi$ is identified with the restriction to $\mathcal{M}_\psi$ of $\widetilde{F}$.

Proof. Note that under the hypotheses, $\rho \circ \widetilde{F}$ is a faithful state on $\widetilde{\mathcal{M}}$ (by Ueda's result [9], as discussed in Section 2 above). Thus, the GNS Hilbert space $L^2(\mathfrak{A}, \psi)$ is a subspace of $L^2(\widetilde{\mathcal{M}}, \rho \circ \widetilde{F})$, and $\mathcal{M}_\psi$ is realized as the strong operator topology closure in $\widetilde{\mathcal{M}}$ of $(*_1^\infty \sigma)(\mathfrak{A})$ with $\hat{\psi}$ the restriction to $\mathcal{M}_\psi$ of $\rho \circ \widetilde{F}$. Now, by examining the free product structure of the Hilbert space $L^2(\widetilde{\mathcal{M}}, \rho \circ \widetilde{F})$, we see that the fixed point subalgebra $\mathcal{F}_\psi$ must lie in $\mathcal{D}$, and since $\hat{\psi}$ is faithful on $\mathcal{M}_\psi$, from [4], as discussed above, we have $\mathcal{T}_\psi = \mathcal{F}_\psi \subseteq \mathcal{D}$.

We must only show that the conditional expectation $E_\psi : \mathcal{M}_\psi \to \mathcal{T}_\psi$ equals the restriction to $\mathcal{M}_\psi$ of $\widetilde{F}$. Since both of these conditional expectations are normal, it will suffice to show their agreement on elements of $\pi_\psi(\mathfrak{A})$. For this, we appeal to the construction of the conditional expectation G_ψ found in [4, Theorem 5.1.10]; since $\hat{\psi}$ is faithful on $\mathcal{M}_\psi$, this conditional expectation G_ψ coincides with the restriction to $\pi_\psi(\mathfrak{A})$ of E_ψ. The $*$-endomorphism α appearing in the aforementioned construction of G_ψ must, by [4, Lemma 5.1.9], agree with the normal "shift" $*$-endomorphism $\tilde{\alpha}$ of $\widetilde{\mathcal{M}}$, that sends the ith copy of $\widetilde{\mathcal{B}}$ in $\widetilde{\mathcal{M}}$ to the $(i+1)$th copy (and which is easily seen to exist). Thus (see [4, Theorem 5.1.10]),

$$E_\psi(x) = \text{WOT-}\lim_{n\to\infty} \tilde{\alpha}^n(x)$$

for all $x \in \pi_\psi(\mathfrak{A})$, and by the structure of the free product Hilbert space $L^2(\widetilde{\mathcal{M}}, \rho \circ \widetilde{F})$, we conclude $E_\psi(x) = \widetilde{F}(x)$. ■

Remark 4.2. In the situation of the previous proposition, by the methods of [4, Section 7] (see in particular [4, Theorem 7.3 and Definition 7.1]) the tail algebra of ψ is equal to the smallest von Neumann subalgebra $\mathcal{D}_\infty$

of $\mathcal{D}$ that contains

$$\widetilde{F}(\sigma(a_1)d_1\sigma(a_2)\cdots d_{n-1}\sigma(a_n)) \tag{4.1}$$

for all $a_1,\dots a_n \in A$ and all $d_1,\dots,d_{n-1} \in \mathcal{D}_\infty$. Thus, letting $\mathcal{D}_0 = \mathbb{C}1$ and for $p \geq 1$ letting $\mathcal{D}_p$ be the von Neumann algebra generated by all expressions of the form (4.1) for $a_j \in A$ and $d_1,\dots,d_{n-1} \in \mathcal{D}_{p-1}$, we see that $\mathcal{D}_\infty$ equals the von Neumann algebra generated by $\bigcup_{p=0}^\infty \mathcal{D}_p$.

5. The simplex of tracial quantum symmetric states. Let A be a unital C^*-algebra and let $\mathrm{TQSS}(A)$ be the compact, convex set of tracial, quantum symmetric states on $\mathfrak{A} = *_1^\infty A$. We assume that A has a tracial state, so that $\mathrm{TQSS}(A)$ is nonempty, and we assume that $A \neq \mathbb{C}$.

The following is Definition 7.5 of [4].

Definition 5.1. For a unital C^*-algebra A, let $\mathcal{TW}(A)$ be the set of all equivalence classes of quintuples $W = (\mathcal{B},\mathcal{D},E,\sigma,\rho)$ where

(i) $\mathcal{B}$ is a von Neumann algebra,
(ii) $\mathcal{D}$ is a unital von Neumann subalgebra of $\mathcal{B}$,
(iii) $E : \mathcal{B} \to \mathcal{D}$ is a normal, faithful conditional expectation onto $\mathcal{D}$,
(iv) $\sigma : A \to \mathcal{B}$ is a unital $*$-homomorphism,
(v) $\sigma(A) \cup \mathcal{D}$ generates $\mathcal{B}$ as a von Neumann algebra,
(vi) $\mathcal{D}$ is the smallest unital von Neumann subalgebra of $\mathcal{B}$ that satisfies

$$E(x_0\sigma(a_1)x_1\cdots\sigma(a_n)x_n) \in \mathcal{D} \tag{5.1}$$

whenever $n \in \mathbb{N}$, $x_0,\dots,x_n \in \mathcal{D}$ and $a_1,\dots,a_n \in A$,

(vii) ρ is a normal faithful tracial state on $\mathcal{D}$, such that $\rho \circ E$ is a trace on $\mathcal{B}$,

under the obvious notion of equivalence.

(Naturally, to avoid set-theoretic difficulties, since conditions (v) and (vi) put a bound on the cardinality of $\mathcal{B}$, we may fix a Hilbert space and insist that $\mathcal{B}$ act on this Hilbert space.)

The following is Theorem 7.6 of [4].

Theorem 5.2. *There is a bijection $\mathcal{TW}(A) \to \mathrm{TQSS}(A)$ which to a quintuple $(\mathcal{B},\mathcal{D},E,\sigma,\rho)$ associates the quantum symmetric tracial state ψ defined as follows. One constructs the amalgamated free product von Neumann algebra*

$$(\mathcal{M},F) = (*_{\mathcal{D}})_1^\infty(\mathcal{B},E) \tag{5.2}$$

of infinitely many copies of $(\mathcal{B},E)$ and one takes the free product $*$-homomorphism $*_1^\infty\sigma : \mathfrak{A} \to \mathcal{M}$ arising from the universal property, sending the ith copy of A into the ith copy of $\mathcal{B}$. Then $\psi = \rho \circ F \circ (*_1^\infty\sigma)$. Moreover, $\mathcal{D} = \mathcal{T}_\psi$

is the tail algebra and $\mathcal{M} = \mathcal{M}_\psi$ *is the von Neumann algebra generated by the GNS representation of* ψ.

The extreme points of $\mathrm{TQSS}(A)$ were characterized in [4, Theorem 8.2] as corresponding to the set of quintuples $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ so that ρ is extreme among the set $R(E)$ of tracial states of $\mathcal{D}$ so that $\rho \circ E$ is a trace on $\mathcal{B}$. In fact, we arrive at a better characterization of the extreme tracial quantum symmetric states below.

Note that $\mathrm{TQSS}(A)$ is a closed convex subset of the tracial state space, $\mathrm{TS}(\mathfrak{A})$, of $\mathfrak{A}$. The tracial state space of any C^*-algebra is known to be a Choquet simplex (see, for example, [8, Theorem 3.1.18]) and the extreme points of it are the tracial states that are factor states.

THEOREM 5.3. $\mathrm{TQSS}(A)$ *is a Choquet simplex and is a face of* $\mathrm{TS}(\mathfrak{A})$. *Moreover, for* $\psi \in \mathrm{TQSS}(A)$ *with corresponding quintuple* $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$*, the following are equivalent:*

(i) ψ *is an extreme point of* $\mathrm{TQSS}(A)$,
(ii) ψ *is an extreme point of* $\mathrm{TS}(\mathfrak{A})$,
(iii) $\mathcal{D} \cap Z(\mathcal{B}) = \mathbb{C}1$.

Proof. The implication (i)⇒(ii), when proved, will imply that $\mathrm{TQSS}(A)$ is a face of $\mathrm{TS}(\mathfrak{A})$, and thus a Choquet simplex.

The implication (ii)⇒(i) is clearly true.

Let $(\mathcal{M}, F)$ be as in (5.2). By Theorem 3.3, condition (iii) is equivalent to factoriality of $\mathcal{M}$, and this is equivalent to condition (ii). Thus, conditions (ii) and (iii) are equivalent.

To finish the proof, it will suffice to show (i)⇒(iii). If (iii) fails to hold, then $\mathcal{D} \cap Z(\mathcal{B})$ has a projection p equal to neither 0 nor 1. Let $t = \rho(p)$. Since ρ is faithful, we have $0 < t < 1$ and we can write $\rho = t\rho_0 + (1-t)\rho_1$, where

$$\rho_0(x) = t^{-1}\rho(px), \qquad \rho_1(x) = (1-t)^{-1}\rho((1-p)x).$$

Since p lies in $\mathcal{D} \cap Z(\mathcal{B})$, we see that ρ_0 and ρ_1 are distinct normal tracial states on $\mathcal{D}$ and that $\rho_i \circ E$ is a trace on $\mathcal{B}$ $(i = 0, 1)$. Thus, ρ is not an extreme point of $R(E)$, and ψ is not extreme in $\mathrm{TQSS}(A)$. ∎

In Theorem 5.5, we will use multiplicative free Brownian motion (see [2]) to show that every quantum symetric state is a limit of extreme quantum symmetric states. This will show that $\mathrm{TQSS}(A)$ is the Poulsen simplex when A is separable and not a copy of $\mathbb{C}$.

Multiplicative free Brownian motion is the solution $(U_t)_{t \ge 0}$ of the linear stochastic differential equation

$$U_t = 1 - \frac{1}{2}\int_0^t U_s\, ds + \int_0^t i\, dS_s U_s = e^{-t/2} + \int_0^t i\, dS_s e^{-(t-s)/2} U_s,$$

where $(S_t)_{t\geq 0}$ is an additive free Brownian motion. Then each U_t is unitary (see [1]) and belongs to the von Neumann algebra $W^*(S_t, t>0)$, which is a copy of $L(\mathbb{F}_\infty)$. We will need the following lemma.

LEMMA 5.4. *Let $\mathcal{M}$ be a von Neumann algebra with normal, faithful, tracial state τ and suppose $\mathcal{N}\subseteq\mathcal{M}$ is a unital von Neumann subalgebra and $(U_t)_{t\geq 0}$ is a multiplicative free Brownian motion that is free from $\mathcal{N}$ with respect to τ. Then for every unital C^*-subalgebra $A\subseteq\mathcal{N}$ with* $\dim(A)>1$ *and for every $t>0$, we have*

$$(U_t^*AU_t)'\cap\mathcal{N}=\mathbb{C}1.$$

Proof. If $(U_t^*AU_t)'\cap\mathcal{N}$ is nontrivial, then it contains a projection $p\notin\{0,1\}$. Without loss of generality, we may assume A is a von Neumann subalgebra of $\mathcal{N}$, and thus contains a projection $q\notin\{0,1\}$.

From [10, Proposition 9.4 and Remark 8.10], the liberation Fisher information satisfies

$$\varphi^*(U_t^*AU_t:\mathcal{N})\leq F(U_t)<\infty,$$

for any $t>0$, where F is the Fisher information for unitaries. Thus, from [10, Remark 9.2(e)], we have

$$\varphi^*(W^*(U_t^*qU_t):W^*(p))\leq\varphi^*(U_t^*AU_t:\mathcal{N})<\infty.$$

As a consequence, the assumptions of [10, Lemma 12.5] are satisfied, and therefore $U_t^*qU_t$ and p are in general position, i.e.,

$$U_t^*qU_t\wedge p=0\quad\text{or}\quad U_t^*(1-q)U_t\wedge(1-p)=0,\tag{5.3}$$

and

$$U_t^*(1-q)U_t\wedge p=0\quad\text{or}\quad U_t^*qU_t\wedge(1-p)=0.\tag{5.4}$$

But this is not compatible with the assumption that $U_t^*qU_t$ and p commute. For example, if

$$U_t^*qU_t\wedge p=U_t^*(1-q)U_t\wedge p=0,$$

then

$$0=U_t^*qU_tp+U_t^*(1-q)U_tp=p,$$

contrary to hypothesis, and similarly if other cases of (5.3) and (5.4) hold. ∎

THEOREM 5.5. *For every unital C^*-algebra A with* $\dim(A)>1$*, the extreme points of* $\mathrm{TQSS}(A)$ *are dense in* $\mathrm{TQSS}(A)$*. Hence, if A is also separable, then* $\mathrm{TQSS}(A)$ *is the Poulsen simplex.*

Proof. If A is separable, then the free product algebra $\mathfrak{A}$ is also separable, and thus $\mathrm{TQSS}(A)$ is second countable. By Urysohn's metrization theorem, it is metrizable. Once the density of extreme points is shown, it will follow that $\mathrm{TQSS}(A)$ is the Poulsen simplex (see [7]).

We now show density of extreme points. Let $\psi \in \mathrm{TQSS}(A)$ and let $(\mathcal{B}, \mathcal{D}, E, \sigma, \rho)$ be its associated quintuple. We use the notation from the description at the beginning of this section. In particular, $\psi = \rho \circ F \circ (*_1^\infty \sigma)$, and we let $\hat{\psi} = \rho \circ F$ denote the normal extension of ψ to $\mathcal{M}$. Let

$$(\widetilde{\mathcal{M}}, \tau) = (\mathcal{M}, \hat{\psi}) * (L(\mathbb{F}_\infty), \tau_{\mathbb{F}_\infty})$$

be the free product of $\mathcal{M}$ with a copy of $L(\mathbb{F}_\infty)$. Then, since $(L(\mathbb{F}_\infty), \tau_{\mathbb{F}_\infty}) \cong *_1^\infty (L(\mathbb{F}_\infty), \tau_{\mathbb{F}_\infty}) \cong *_1^\infty (W^*(S_t, t > 0), \tau)$, for the von Neumann algebra of a free Brownian motion algebra $W^*(S_t, t > 0) \cong L(\mathbb{F}_\infty)$, letting

$$(\widetilde{\mathcal{B}}, \eta) = (\mathcal{B}, \rho \circ E) * (W^*(S_t, t > 0), \tau)$$

and letting $\widetilde{E} : \widetilde{\mathcal{B}} \to \mathcal{D}$ be the composition of the η-preserving conditional expectation $\widetilde{\mathcal{B}} \to \mathcal{B}$ arising from the free product construction with the conditional expectation $E : \mathcal{B} \to \mathcal{D}$, we find that $\widetilde{\mathcal{M}}$ is isomorphic to the von Neumann algebra free product with amalgamation,

$$(\widetilde{\mathcal{M}}, \widetilde{F}) \cong (*_{\mathcal{D}})_1^\infty (\widetilde{\mathcal{B}}, \widetilde{E}), \tag{5.5}$$

and the trace τ arises as $\rho \circ \widetilde{F}$.

Letting $(U_t)_{t \geq 0}$ be a multiplicative free Brownian motion in $W^*(S_t, t > 0)$, from the free L^∞ version of the Burkholder–Gundy inequalities [3, Theorem 3.2.1], we have the upper bound

$$\begin{aligned} \|U_t - 1\| &\leq (1 - e^{-t/2}) + 2\sqrt{2}\Big(\int_0^t \|U_s\|^2 e^{-(t-s)}\, ds\Big)^{1/2} \\ &= (1 - e^{-t/2}) + 2\sqrt{2(1 - e^{-t})}, \end{aligned} \tag{5.6}$$

which tends to zero as $t \to 0^+$.

Let $\sigma_t : A \to \widetilde{\mathcal{B}}$ be the $*$-homomorphism $U_t \sigma(\cdot) U_t^*$. Then $*_1^\infty \sigma_t$ is a $*$-homomorphism from $\mathfrak{A}$ into $\widetilde{\mathcal{M}}$. By freeness with amalgamation (see [4, Proposition 3.1]), the state $\psi_t := \rho \circ \widetilde{F} \circ (*_1^\infty \sigma_t) = \tau \circ (*_1^\infty \sigma_t)$ is a quantum symmetric state.

We will show that for every $t > 0$, ψ_t is an extreme point of $\mathrm{TQSS}(A)$. By Proposition 4.1, the tail algebra $\mathcal{T}_{\psi_t}$ of ψ_t is a von Neumann subalgebra of $\mathcal{D}$, and the conditional expectation E_{ψ_t} onto the tail algebra is the restriction of $\widetilde{F}$. In particular, see Remark 4.2 for description of generators for $\mathcal{D}$. Let $(\mathcal{B}_t, \mathcal{D}_t, E_t \sigma_t, \rho_t,)$ denote the quintuple corresponding to the quantum symmetric state ψ_t. Then $\mathcal{D}_t = \mathcal{T}_{\psi_t} \subseteq \mathcal{D}$ and $\mathcal{B}_t \supseteq \sigma_t(A)$. By Theorem 5.3, showing that ψ_t is an extreme point of $\mathrm{TQSS}(A)$ is equivalent to showing that $\mathcal{D}_t \cap Z(\mathcal{B}_t)$ is trivial. But $\mathcal{D}_t \cap Z(\mathcal{B}_t)$ is contained in $\mathcal{D}_t \cap (U_t^* \sigma(A) U_t)'$. By Lemma 5.4, the latter set is trivial, and we have proved that ψ_t is an extreme tracial quantum symmetric state.

From (5.6), we deduce that for every $x \in \mathfrak{A}$, $\lim_{t\to 0^+} |\psi_t(x) - \psi(x)| = 0$, working first with the case of x in the algebraic free product, and passing to the general case by norm approximation. ∎

REMARK 5.6. In contrast, the simplices ZQSS(A) and ZTQSS(A) of central quantum symmetric states and central tracial quantum symmetric states, respectively (see [4]), are Bauer simplices, meaning that their respective sets of extreme points are closed. This follows from the proof of [4, Theorem 9.2] and in particular the fact that the map $\phi \mapsto *_1^\infty \phi$ in [4, equation (35)] is a homeomorphism from $S(A)$ onto the extreme boundary (i.e., the set of extreme points) of ZQSS(A) and, by restricting to the tracial state space, yields a homeomorphism from TS(A) onto the extreme boundary of ZTQSS(A).

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