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Applications of the scarcity theorem in ordered Banach algebras

by

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In memory of my father, Daniel Rode (1943–2008)

Abstract. We apply Aupetit's scarcity theorem to obtain stronger versions of many spectral-theoretical results in ordered Banach algebras in which the algebra cone has generating properties.

1. Introduction. This paper is devoted to the role of B. Aupetit's scarcity theorem (Theorem 2.2) in ordered Banach algebras. This is a very deep result, which, in very general terms, states that if a function f is analytic on a domain D in the complex plane and with values in a Banach algebra, then either the subset of D on which the spectrum of f is finite is "very small" in some sense, or it is the whole of D, in which case the spectrum of f is even uniformly finite on D. In Chapter 5 of his book [2] Aupetit illustrated several applications of the scarcity theorem, and more can be found in, for example, the papers [12], [3] and [4], and later in [9], [8] and [7]. A corollary of the scarcity theorem (Corollary 2.5) was employed in [11] to solve the domination problem for radical elements in ordered Banach algebras. We now expand that line of thought to obtain stronger versions of many spectral-theoretical results in ordered Banach algebras in which the algebra cone is suitably well-behaved.

Relying heavily on the scarcity theorem, Lemma 4.1 and our main result Theorem 4.2 show that several spectral properties extend from certain subsets of the algebra cone C of an ordered Banach algebra to larger subsets of C, and from these subsets of C to their linear spans, respectively. The most interesting case where this applies is when the algebra cone is generating, such as in the algebra of all regular operators on a complex Banach

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lattice. It is also useful in cases where the algebra cone generates other subsets of the algebra, such as the set of quasinilpotent elements (see Section 3 for examples).

We apply our results to various aspects of spectral theory in ordered Banach algebras. A typical result is Theorem 4.4, where we show that for a semisimple ordered Banach algebra A with generating algebra cone to be finite-dimensional, it is sufficient (and, of course, necessary) that the spectrum of each positive invertible element is finite, and that A will be isomorphic to $\mathbb{C}$ provided that each of these spectra consists of one element only. Besides providing characterisations of finite-dimensional Banach algebras, we also investigate the centre (Theorem 4.6), rank one and finite rank elements (Theorems 4.8–4.9, 4.13–4.14 and 4.18–4.19) and the radical (Theorems 4.10 and 4.20).

In the interest of self-containedness, we provide an extensive preliminary section (Section 2), in which we give all the necessary background to our work. Section 3 is devoted to recalling some ordered Banach algebra theory, as well as providing a number of examples. We also show (see Proposition 3.4 for a stronger version) that algebra cones with certain natural properties have no interior points. This fact will be of interest in some of the results in Section 4.

2. Preliminaries. Throughout, A will be a complex Banach algebra with unit 1, in which we denote the set of all invertible elements by A^{-1}. By "ideal" we shall always mean "two-sided ideal". The spectrum of an element a in A will be denoted by $\sigma(a)$, the non-zero spectrum of a by $\sigma'(a)$, the connected hull of the spectrum of a by $\eta\sigma(a)$ and the spectral radius of a by $r(a)$. If $\sigma(a) = \{0\}$, we say that a is *quasinilpotent*. The set of all quasinilpotent elements of A is denoted by $\mathrm{QN}(A)$.

It is useful to observe the following:

LEMMA 2.1 ([12, Proposition 2.1(2)]). *Let A be a Banach algebra and $a \in A$. If $b \in A^{-1}$, then $b + a \notin A^{-1}$ if and only if $-1 \in \sigma(b^{-1}a)$.*

The (Jacobson) *radical* $\mathrm{Rad}(A)$ of A is the ideal defined as the intersection of all maximal left (or right) ideals of A, and A is said to be *semisimple* if $\mathrm{Rad}(A)$ consists of zero only. A is said to be *semiprime* if $I = \{0\}$ is the only ideal of A with the property that $I^2 = \{0\}$. By [6, Proposition 5, p. 155] a semisimple Banach algebra is semiprime.

If $\lambda \in \mathbb{C}$, the element $\lambda 1$ of A will be denoted by λ. The number of elements in a set $K \subseteq \mathbb{C}$ will be denoted by $\#K$ and the set of all non-negative real numbers by $\mathbb{R}^+$. The open disk, closed disk and circle in $\mathbb{C}$ with centre λ and radius ϵ will be denoted by $D(\lambda, \epsilon)$, $\overline{D}(\lambda, \epsilon)$ and $C(\lambda, \epsilon)$, respectively.

Let E be a complex Banach lattice and denote by $\mathcal{L}(E)$ the space of all bounded linear operators on E. An operator $T : E \to E$ is *regular* if it can be written as a linear combination over $\mathbb{C}$ of positive operators. The space of all regular operators on E is denoted by $\mathcal{L}^r(E)$; it is a subspace of $\mathcal{L}(E)$. When $\mathcal{L}^r(E)$ is provided with the r-norm

$$\|T\|_r = \inf\{\|S\| : S \in \mathcal{L}(E),\ |Tx| \leq S|x| \text{ for all } x \in E\},$$

it becomes a Banach algebra which contains the unit of $\mathcal{L}(E)$ ([16, IV, §1], [1]).

If A is a Banach algebra and D a domain in $\mathbb{C}$, then a map $g : A \to A$ will be called D-*analytic* if $g \circ f : D \to A$ is analytic for every analytic function $f : D \to A$. It is easy to see that the maps $g(x) = a + x$ and $g(x) = a(1 + x)$ (for a fixed $a \in A$), as well as every continuous linear map g, are D-analytic, for every domain $D \subseteq \mathbb{C}$.

The following famous result of B. Aupetit is known as the *scarcity theorem*:

Theorem 2.2 ([2, Theorem 3.4.25]). *Let $f : D \to A$ be analytic, where D is a domain in $\mathbb{C}$ and A is a Banach algebra. Then either the set of $\lambda \in D$ such that $\sigma(f(\lambda))$ is finite is a Borel set having zero capacity, or there exist an integer $n \geq 1$ and a closed discrete subset E of D such that $\#\sigma(f(\lambda)) = n$ for all $\lambda \in D \setminus E$ and $\#\sigma(f(\lambda)) < n$ for all $\lambda \in E$.*

Here, the *capacity* of a Borel set in the complex plane (see [2, pp. 177–180]) is in some sense a measure of its size, with compact sets having zero capacity being very small. For our purposes it suffices to know that balls and line segments have non-zero capacities.

It also follows from the scarcity theorem that if $\sigma(f(\lambda))$ is uniformly finite on a subset of D with non-zero capacity, then it is (uniformly) finite on the whole of D with the same bound:

Corollary 2.3. *Let $f : D \to A$ be analytic, where D is a domain in $\mathbb{C}$ and A is a Banach algebra. If $n \geq 1$ is such that $\#\sigma(f(\lambda)) \leq n$ for all λ in a subset of D with non-zero capacity, then $\#\sigma(f(\lambda)) \leq n$ for all $\lambda \in D$.*

In addition, we can say the following about the non-zero spectrum:

Corollary 2.4. *Let $g : D \to A$ be analytic, where D is a domain in $\mathbb{C}$ and A is a Banach algebra, and let $f(\lambda) = g(\lambda)a$ for some $a \in A$. If $n \geq 1$ is such that $\#\sigma'(f(\lambda)) \leq n$ for all λ in a subset D_1 of D with non-zero capacity such that $g(D_1) \subseteq A^{-1}$, then $\#\sigma'(f(\lambda)) \leq n$ for all $\lambda \in D$.*

Proof. Suppose that $\#\sigma'(f(\lambda)) \leq n$ for all $\lambda \in D_1 \subseteq D$, where D_1 has non-zero capacity and $g(D_1) \subseteq A^{-1}$.

If $a \in A^{-1}$, then $f(D_1) \subseteq A^{-1}$, so that $\#\sigma(f(\lambda)) \leq n$ for all $\lambda \in D_1$. It follows from Corollary 2.3 that $\#\sigma'(f(\lambda)) \leq \#\sigma(f(\lambda)) \leq n$, for all $\lambda \in D$.

If $a \notin A^{-1}$, then either $Aa \neq A$ or $aA \neq A$. Suppose that $Aa \neq A$. Then no element of Aa is invertible, so that $0 \in \sigma(f(\lambda))$ for all $\lambda \in D$. It follows that $\#\sigma(f(\lambda)) \leq n+1$ for all $\lambda \in D_1$, so that $\#\sigma(f(\lambda)) \leq n+1$ for all $\lambda \in D$ by Corollary 2.3, and hence $\#\sigma'(f(\lambda)) \leq n$ for all $\lambda \in D$. If instead $aA \neq A$, then $0 \in \sigma(h(\lambda))$ and $\sigma'(f(\lambda)) = \sigma'(h(\lambda))$ where $h(\lambda) = ag(\lambda)$, for all $\lambda \in D$, so that the previous argument, with h in place of f, establishes the result. ∎

Corollary 2.3, together with [2, Corollary 3.4.18], yields the following result about quasinilpotent elements:

Corollary 2.5 ([11, Corollary 2.3]). *Let f be an analytic function from a domain D of $\mathbb{C}$ into a Banach algebra A. If $\{\lambda \in D : \sigma(f(\lambda)) = \{0\}\}$ contains a ball or a line segment, then $\sigma(f(\lambda)) = \{0\}$ for all λ in D.*

In the following theorem we list a number of known characterisations of the radical. From the definition of the radical it follows easily (see [2, p. 36]) that $a \in \mathrm{Rad}(A)$ if and only if $Aa \subseteq \mathrm{QN}(A)$, and an application of Corollary 2.5 shows that the A on the left side of the inclusion sign may be replaced by an arbitrary open set (see [9, Theorem 2.2]). This gives (2) in the theorem below. Characterisations (3) and (4) are due to J. Zemánek and can be found in [18] and [19], respectively, (5) is due to Zemánek and Aupetit (see [18] and [5]), while (6)–(8) were recently given by R. Brits [9].

Theorem 2.6 ([2], [5], [9], [18], [19]). *Let A be a Banach algebra, $a \in A$ and G, G_0, G_a and G_1 an open set, a neighbourhood of 0, a neighbourhood of a and a neighbourhood of 1, respectively. Then the following are equivalent:*

(1) $a \in \mathrm{Rad}(A)$.
(2) $Ga \subseteq \mathrm{QN}(A)$.
(3) $\sigma(a+x) = \sigma(x)$ *for all* $x \in A$.
(4) $a(1+\mathrm{QN}(A)) \subseteq \mathrm{QN}(A)$.
(5) $a + (\mathrm{QN}(A) \cap G_0) \subseteq \mathrm{QN}(A)$.
(6) $a \in \mathrm{QN}(A)$ *and* $(\mathrm{QN}(A) \cap G_a)a \subseteq \mathrm{QN}(A)$.
(7) $a \notin A^{-1}$ *and* $\#\sigma(ax) \leq \#\sigma(x)$ *for all* $x \in G_1$.
(8) $a \notin A^{-1}$ *and* $\#\sigma(a+x) \leq \#\sigma(x)$ *for all* $x \in G_1$.

The following theorem gives necessary and sufficient conditions for an element of A to be equal to a scalar modulo the radical.

Theorem 2.7 ([2], [8]). *Let A be a Banach algebra, $a \in A$ and G_0 and G_1 neighbourhoods of 0 and 1, respectively. Then the following are equivalent:*

(1) *There exists* $\alpha \in \mathbb{C}$ *such that* $a - \alpha \in \mathrm{Rad}(A)$.
(2) $\#\sigma(a+q) = 1$ *for all* $q \in \mathrm{QN}(A) \cap G_0$.
(3) $\#\sigma(a+x) \leq \#\sigma(x)$ *for all* $x \in G_1$.

Characterisation (2) is a slightly stronger version of [2, Theorem 5.3.2], obtained by applying Corollary 2.3, while characterisation (3) is one of the results in the recent paper [8].

In order to formulate a theorem giving sufficient conditions for a Banach algebra to be finite-dimensional, we start by defining absorbing points and sets. A point a in a vector space X is said to be an *absorbing point* of a subset U of X if for all $x \in X$ there exists $r > 0$ such that $a + \lambda x \in U$ for all real λ with $|\lambda| \leq r$. A subset U of a vector space X is called an *absorbing set* if U contains an absorbing point. Open sets are absorbing, but not vice versa.

The second part of the following theorem follows from the proof of the given reference.

Theorem 2.8 ([2, Theorem 5.4.2]). *Let A be a Banach algebra. If A contains an absorbing subset U such that*

(1) *$\sigma(x)$ is finite for all $x \in U$, then $A/\mathrm{Rad}(A)$ is finite-dimensional,*
(2) *$\#\sigma(x) \leq n$ for all $x \in U$ and some fixed $n \in \mathbb{N}$, then $\dim A/\mathrm{Rad}(A) \leq n^6$.*

The following lemma is well-known:

Lemma 2.9. *Let A be a Banach algebra.*

(1) *If $\dim A < \infty$, then there exists $n \in \mathbb{N}$ such that x is algebraic of degree $\leq n$ for all $x \in A$.*
(2) *If $x \in A$ is algebraic of degree n, then $\#\sigma(x) \leq n$.*

The following theorem gives conditions under which an element in a Banach algebra A belongs to the centre $Z(A)$ modulo the radical of A:

Theorem 2.10 ([2, Theorem 5.2.1]). *Let A be a Banach algebra and $a \in A$. If $\#\sigma(ax - xa) = 1$ for all $x \in A$, then $a \in Z(A)$.*

Finally, we will discuss the socle and rank one elements in Banach algebras. In a semiprime Banach algebra A an element a is said to be *spatially rank one* if $a \neq 0$ and there exists a linear functional f_a on A such that $axa = f_a(x)a$ for all $x \in A$, and a is *spatially finite rank* if $a = 0$ or a is a finite sum of spatially rank one elements of A. On the other hand, an element a in A is defined to be *spectrally finite rank* if $\#\sigma'(xa) \leq n$ for all $x \in A$ and for some $n \in \mathbb{N}$, and a is *spectrally rank one* if $a \neq 0$ and $\#\sigma'(xa) \leq 1$ for all $x \in A$. The concepts of spatially and spectrally finite rank elements were introduced by J. Puhl [14] and Aupetit and H. du T. Mouton [3], respectively, and the terminology is due to R. Harte [10]. The proof of [12, Theorem 2.2] shows that an element is spatially rank one if and only if it is spectrally rank one, whenever A is semisimple. Therefore, in a semisimple Banach algebra, we can refer unambiguously to such an element as a *rank one* element.

If A has minimal left ideals (respectively minimal right ideals) then its *socle* $\mathrm{Soc}(A)$ is defined as the sum of the minimal left ideals (it is also equal to the sum of the minimal right ideals, so it is an ideal). If A is semisimple, then it is clear from [3, Theorem 2.1] that $\mathrm{Soc}(A)$ exists, and it follows from [14, p. 659] and the proof of [12, Theorem 3.1] that an element a is in $\mathrm{Soc}(A)$ if and only if a is spatially finite rank if and only if a is spectrally finite rank, so that we can refer unambiguously to such an element as a *finite rank* element (or an element of $\mathrm{Soc}(A)$). Together with Theorem 2.8 it follows that if A is semisimple, then $A = \mathrm{Soc}(A)$ if and only if $\dim A < \infty$.

In spectral-theoretical contexts, we usually think of the socle as follows:

THEOREM 2.11 ([12], [3, Theorem 2.1(1)]). *Let A be a semisimple Banach algebra. Then*

$$\{a \in A : \text{there exists } n \in \mathbb{N} \text{ such that } \#\sigma'(xa) \le n \text{ for all } x \in A\} \\ = \mathrm{Soc}(A) = \{a \in A : \#\sigma'(xa) < \infty \text{ for all } x \in A\}.$$

Alternative characterisations of rank one elements and of the socle are given in Theorems 2.12 and 2.13, respectively.

THEOREM 2.12 ([12, Theorem 2.2], [3, Theorem 2.2(1)]). *Let A be a semisimple Banach algebra and $0 \neq a \in A$. Then the following are equivalent:*

(1) *a is rank one.*
(2) *$\sigma(x + s_0 a) \cap \sigma(x + s_1 a) \subseteq \sigma(x)$ for all $s_0, s_1 \in \mathbb{C} \setminus \{0\}$ with $s_0 \neq s_1$ and all $x \in A$.*
(3) *$\eta\sigma(x+s_0 a) \cap \eta\sigma(x+s_1 a) \subseteq \eta\sigma(x)$ for all $s_0, s_1 \in \mathbb{C} \setminus \{0\}$ with $s_0 \neq s_1$ and all $x \in A$.*

THEOREM 2.13 ([12, Theorem 3.1], [3, Theorem 2.2(2)]). *Let A be a semisimple Banach algebra and $a \in A$. Then the following are equivalent:*

(1) *$a \in \mathrm{Soc}(A)$.*
(2) *There exists $n \in \mathbb{N}$ such that $\bigcap_{t \in F} \sigma(x + ta) \subseteq \sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \setminus \{0\}$ and all $x \in A$.*
(3) *There exists $n \in \mathbb{N}$ such that $\bigcap_{t \in F} \eta\sigma(x + ta) \subseteq \eta\sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \setminus \{0\}$ and all $x \in A$.*

3. Ordered Banach algebras. From [15, Section 3] we recall that, in a complex Banach algebra A with unit 1, a non-empty subset C of A is called a *space cone* if C is closed under addition and under non-negative real scalar multiplication, and we call C an *algebra cone* if C is a space cone containing 1 which is closed under multiplication. If A has an algebra cone C, then A (or (A, C)) is called an *ordered Banach algebra* (OBA). If, in addition, $C \cap -C = \{0\}$, then C is called *proper*, and if C contains the inverses of all its invertible elements, then C is called *inverse-closed.*

An algebra cone C of A induces an *ordering* "$\leq$" on A in the following way:

$$a \leq b \quad \text{if and only if} \quad b - a \in C$$

(where $a, b \in A$). This ordering is reflexive and transitive. Furthermore, C is proper if and only if the ordering has the additional property of being antisymmetric. Considering the ordering that C induces we find that $C = \{a \in A : a \geq 0\}$ and therefore we call the elements of C *positive*.

An algebra cone C of A is said to be *normal* if there exists a constant $\alpha > 0$ such that it follows from $0 \leq a \leq b$ in A that $\|a\| \leq \alpha \|b\|$. It is well known that if C is normal, then C is proper.

In addition, we shall need the following properties of positive elements:

THEOREM 3.1 ([15, Proposition 5.1]). *Let (A, C) be an OBA with C closed and normal. If $a \in C$, then $r(a) \in \sigma(a)$.*

PROPOSITION 3.2 ([13, Proposition 4.6]). *Let (A, C) be an OBA with C closed, and let $a \in C$. If $\lambda > r(a)$, then $(\lambda - a)^{-1} \geq 0$.*

LEMMA 3.3 ([13, Lemma 4.21]). *Let (A, C) be an OBA with a and b invertible elements of A such that $a \leq b$ and $a^{-1}, b^{-1} \geq 0$. Then $b^{-1} \leq a^{-1}$.*

If (A, C) is an OBA, then C is called *generating* if $\operatorname{span}(C) = A$, where $\operatorname{span}(X)$ denotes the linear span of a set X. More generally, if B is a subset of A, we say that C *generates* B if $\operatorname{span}(C \cap B) = B$. Note that, in this case, $C \cap B$ is a space cone of A.

If E is a complex Banach lattice, $C = \{x \in E : x = |x|\}$ and $K = \{T \in \mathcal{L}(E) : TC \subseteq C\}$, then $(\mathcal{L}^r(E), K)$ is an OBA with a closed, normal algebra cone (see [17, Lemma 3]), and by definition, K generates $\mathcal{L}^r(E)$.

Let $C(X)$ denote the Banach algebra of all continuous complex valued functions on a compact Hausdorff space X, and let C be the subset of $C(X)$ consisting of all functions which are real and non-negative at every point of X. Then $(C(X), C)$ is an OBA with a closed, normal, generating and inverse-closed algebra cone.

Let A_{n1} denote the Banach algebra of all complex $n \times n$ matrices (or all upper triangular complex $n \times n$ matrices), C_{n1} the subset of A_{n1} consisting of all matrices with only non-negative entries, A_2 the Banach algebra l^∞ of all bounded sequences of complex numbers and C_2 the subset of A_2 consisting of all sequences with only non-negative entries. Then (A_{n1}, C_{n1}) and (A_2, C_2) are OBAs with closed, normal algebra cones C_{n1} and C_2, respectively, and C_{n1} and C_2 generate A_{n1} and A_2, respectively. In addition, C_2 is inverse-closed. Moreover, if $A_{n3} = l^\infty(A_{n1})$ and $C_{n3} = l^\infty(C_{n1})$ denote the Banach algebra of all bounded sequences of (upper triangular) complex $n \times n$ matrices and its subset consisting of all such sequences having as entries only matrices with non-negative entries, respectively, then C_{n3} is a closed, normal algebra cone which generates A_{n3} (see [11]). In addition, for any OBAs (A_i, C_i) and (A_j, C_j), if C_i and C_j are closed, normal and generate A_i and A_j, respectively, then clearly $C = C_i \oplus C_j$ is closed, normal and generates $A = A_i \oplus A_j$.

In some OBAs the set of all quasinilpotent elements is generated by the algebra cone. In fact, it is easy to check that this is the case for (A_{n1}, C_{n1}), (A_2, C_2) and (A_{n3}, C_{n3}), as defined in the previous paragraph (only with A_{n1} the upper triangular matrices). Moreover, for any OBAs (A_i, C_i) and (A_j, C_j), if C_i and C_j generate $\mathrm{QN}(A_i)$ and $\mathrm{QN}(A_j)$, respectively, then $C = C_i \oplus C_j$ generates $\mathrm{QN}(A)$, where $A = A_i \oplus A_j$.

In the light of assumptions concerning absorbing sets that we will make in some of the results in Section 4, it is important to notice the following:

Proposition 3.4. *Let (A, C) be an OBA with C closed and normal. Then C is not an absorbing set.*

Proof. If C is an absorbing set with absorbing point a, then there exists $r_0 > 0$ such that $a + \lambda i \in C$ for all real λ with $|\lambda| \leq r_0$. If $r(a) = 0$ and $0 < \lambda_0 \leq r_0$, then $r(a + \lambda_0 i) = \lambda_0 \notin \sigma(a + \lambda_0 i)$, which contradicts Theorem 3.1. Otherwise, let $0 \neq \lambda_0 \in \mathbb{R}^+$ be such that $\lambda_0 < \min\{r_0, r(a)\}$. Then $a + \lambda_0 i \in C$, so that $r(a) \in \sigma(a)$ and $r(a + \lambda_0 i) \in \sigma(a + \lambda_0 i)$, by Theorem 3.1. Also, since $\lambda_0 < r(a)$, we have $0 \in D(\lambda_0 i, r(a))$, so that $C(\lambda_0 i, r(a))$ contains exactly one strictly positive real point μ_0.

Now, since $r(a+\lambda_0 i) \in \sigma(a+\lambda_0 i) \subseteq \bar{D}(\lambda_0 i, r(a))$, it follows that $r(a+\lambda_0 i) \leq \mu_0 = \sqrt{r(a)^2 - \lambda_0^2}$. But $r(a) \in \sigma(a)$ implies that $r(a) + \lambda_0 i \in \sigma(a + \lambda_0 i)$, so that $\sqrt{r(a)^2 + \lambda_0^2} = |r(a) + \lambda_0 i| \leq r(a + \lambda_0 i)$. Together with the previous inequality, this yields $\lambda_0 = 0$ — a contradiction. ∎

4. Applying the scarcity theorem in ordered Banach algebras. We start with the following lemma, which, roughly speaking, says that in an ordered Banach algebra, certain spectral properties extend from particular subsets of the algebra cone to larger subsets.

Lemma 4.1. *Let (A, C) be an OBA, B a subset of C which is a space cone of A, G a subset of A, $c_0 \in B$ an absorbing point of G and $g : A \to A$ a $\mathbb{C}$-analytic map.*

1. *If $\#\sigma(g(c)) < \infty$ for all $c \in B \cap G$, then there exists $m \in \mathbb{N}$ such that $\#\sigma(g(c + c_0)) \leq m$ for all $c \in B$.*
2. *If $n \in \mathbb{N}$ and $\#\sigma(g(c)) \leq n$ for all $c \in B \cap G$, then $\#\sigma(g(c+c_0)) \leq n$ for all $c \in B$.*
3. *If $\sigma(g(c)) = \{0\}$ for all $c \in B \cap G$, then $\sigma(g(c + c_0)) = \{0\}$ for all $c \in B$.*

Proof. Towards (2), let $c \in B$ and $f(\lambda) = g((1-\lambda)c + c_0)$. Then f is analytic on $\mathbb{C}$. Since c_0 is absorbing in G, there exists $r_c > 0$ such that $(1-\lambda)c + c_0 \in B \cap G$, and hence $\#\sigma(f(\lambda)) \leq n$, for all λ in the real interval $[1-r_c, 1]$. It follows from Corollary 2.3 that $\#\sigma(f(\lambda)) \leq n$ for all $\lambda \in \mathbb{C}$, so that (2) follows by taking $\lambda = 0$. The same argument, using Theorem 2.2 (respectively Corollary 2.5) instead of Corollary 2.3, proves (1) (respectively (3)). ∎

We now prove our main result, which we will afterwards apply to various situations in ordered Banach algebras.

THEOREM 4.2. *Let (A, C) be an OBA, B a subset of C which is a space cone of A, $c_0 \in B$ and $g : A \to A$ a $\mathbb{C}$-analytic map.*

(1) *If $\#\sigma(g(c + c_0)) < \infty$ for all $c \in B$, then there exists $m \in \mathbb{N}$ such that $\#\sigma(g(x)) \leq m$ for all $x \in \operatorname{span}(B)$.*
(2) *If $n \in \mathbb{N}$ and $\#\sigma(g(c + c_0)) \leq n$ for all $c \in B$, then $\#\sigma(g(x)) \leq n$ for all $x \in \operatorname{span}(B)$.*
(3) *If $\sigma(g(c + c_0)) = \{0\}$ for all $c \in B$, then $\sigma(g(x)) = \{0\}$ for all $x \in \operatorname{span}(B)$.*

Proof. Suppose that $\#\sigma(g(c + c_0)) \leq n$ for all $c \in B$ and let $m \in \mathbb{N}$ and $c_1, \ldots, c_m \in B$. If $\lambda_2, \ldots, \lambda_m$ are any fixed positive real numbers and λ_1 is a complex variable, then $f_1(\lambda_1) = \lambda_1 c_1 + \cdots + \lambda_m c_m + c_0$ is analytic on $\mathbb{C}$, and hence $g \circ f_1$ is analytic on $\mathbb{C}$. Also, for all $\lambda_1 \in [0, 1]$, we have $f_1(\lambda_1) \in B + c_0$, so that $\#\sigma(g(f_1(\lambda_1))) \leq n$. By Corollary 2.3 $\#\sigma(g(f_1(\lambda_1))) \leq n$ for all $\lambda_1 \in \mathbb{C}$. Hence, for any $\lambda_1 \in \mathbb{C}$ and $\lambda_2, \ldots, \lambda_m \in \mathbb{R}^+$, we have

$$\#\sigma(g(\lambda_1 c_1 + \cdots + \lambda_m c_m + c_0)) \leq n. \tag{4.3}$$

Now take any fixed $\lambda_1 \in \mathbb{C}$, any fixed $\lambda_3, \ldots, \lambda_m \in \mathbb{R}^+$ and let λ_2 be a complex variable. Then $f_2(\lambda_2) = \lambda_1 c_1 + \cdots + \lambda_m c_m + c_0$ is analytic on $\mathbb{C}$, so that $g \circ f_2$ is analytic on $\mathbb{C}$. Also, for all $\lambda_2 \in [0, 1]$, we have $\#\sigma(g(f_2(\lambda_2))) \leq n$, by 4.3. Again, it follows from Corollary 2.3 that $\#\sigma(g(f_2(\lambda_2))) \leq n$ for all $\lambda_2 \in \mathbb{C}$. Hence, for any $\lambda_1, \lambda_2 \in \mathbb{C}$ and $\lambda_3, \ldots, \lambda_m \in \mathbb{R}^+$, we deduce that $\#\sigma(g(\lambda_1 c_1 + \cdots + \lambda_m c_m + c_0)) \leq n$. After m steps it follows that $\#\sigma(g(\lambda_1 c_1 + \cdots + \lambda_m c_m + c_0)) \leq n$, for any $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. This proves (2).

The same argument, using Theorem 2.2 instead of Corollary 2.3 in the first step, proves (1), and the same argument, applying Corollary 2.5 instead of Corollary 2.3 in each step, proves (3). ∎

The above theorem applies, for instance, to $B = B_1 \cap C$ for any vector subspace B_1 of A. The most important case is when $B = C$, which shows that, when the algebra cone C is generating, then certain properties extend from C to all of A. In fact, by taking $B = C$, $c_0 = 0$ and $g(x) = ax$ for some $a \in A$, we note that Theorem 4.2(3) reduces to [11, Lemma 4.4].

Let us start illustrating the use of Theorem 4.2 by considering characterisations of finite-dimensional Banach algebras. Theorem 2.8 says that, for A a general Banach algebra, $A/\mathrm{Rad}(A)$ is finite-dimensional provided that the spectrum is finite on a very "small" part of A, namely on an absorbing set. Our next result shows that if A is an ordered Banach algebra with a generating algebra cone C, then in order for $A/\mathrm{Rad}(A)$ to be finite-dimensional, it is sufficient that the spectrum is finite on an even "smaller" part of A: for any subset G of A which contains a point of C which is absorbing in G, the spectrum only has to be finite at all positive elements of G.

Theorem 4.4. *Let (A,C) be an OBA such that C is generating, and let G be any subset of A which contains a point of C which is absorbing in G.*

(1) *If $\#\sigma(c) < \infty$ for all $c \in C \cap G$, then* $\dim A/\mathrm{Rad}(A) < \infty$.
(2) *If $n \in \mathbb{N}$ and $\#\sigma(c) \le n$ for all $c \in C \cap G$, then* $\dim A/\mathrm{Rad}(A) \le n^6$.
(3) *If $\#\sigma(c) = 1$ for all $c \in C \cap G$, then $A/\mathrm{Rad}(A) \cong \mathbb{C}$.*

Proof. If $n \in \mathbb{N}$ and $\#\sigma(c) \le n$ for all $c \in C \cap G$, then by taking $B = C$ and g the identity, it follows from Lemma 4.1(2) and Theorem 4.2(2) that $\#\sigma(x) \le n$ for all $x \in A$. The conclusion in (2) now follows by applying Theorem 2.8(2). The same argument, with Lemma 4.1(1) instead of Lemma 4.1(2), proves (1). ∎

For instance, by taking $G = A$ in Theorem 4.4, we see that a sufficient condition for $A/\mathrm{Rad}(A)$ to be finite-dimensional is that the spectrum of each positive element is finite. Moreover, by taking $G = A^{-1}$, we see that it is even enough that the spectrum of each invertible positive element is finite.

Corollary 4.5. *Let (A,C) be a semisimple OBA with C generating. Then the following are equivalent:*

(1) $\dim A < \infty$.
(2) $\#\sigma(c) < \infty$ *for all* $c \in C$.
(3) *There exists $n \in \mathbb{N}$ such that $\#\sigma(c) \le n$ for all $c \in C$.*
(4) *All elements of C are algebraic.*
(5) *There exists $n \in \mathbb{N}$ such that c is algebraic of degree $\le n$ for all $c \in C$.*

Proof. (2)⇒(3) follows from Theorem 4.2(1) by choosing $B = C$, $c_0 = 0$ and g the identity. (3)⇒(1) follows from Theorem 4.4(2) by choosing $G = A$. The proof is completed by applying Lemma 2.9. ∎

For OBAs with generating algebra cones, we have the following stronger version of Theorem 2.10 about the centre modulo the radical:

Theorem 4.6. *Let (A,C) be an OBA such that C is generating, and let G be any subset of A which contains a point of C which is absorbing in G. If $a \in A$ and $\#\sigma(ac - ca) = 1$ for all $c \in C \cap G$, then $a \in Z(A)$.*

Proof. The condition is equivalent to $\#\sigma(g(c)) \le 1$ for all $c \in C \cap G$, where $g(x) = ax - xa$. By Lemma 4.1(2) and Theorem 4.2(2), $\#\sigma(ax - xa) = 1$ for all $x \in A$, so that $a \in Z(A)$, by Theorem 2.10. ∎

In the following part of this section we will look at characterisations of the socle and of rank one elements in OBAs with generating algebra cones. Here, the most important applications of Theorem 4.2 are Theorems 4.8–4.10 and 4.18–4.19. We start with the following important observation:

PROPOSITION 4.7. *Let (A, C) be an OBA with C generating, and let G be any subset of A which contains a point of C which is absorbing in G. If $n \in \mathbb{N}$, then*

$$\{a \in A : \#\sigma(ca) < \infty\ \forall c \in C \cap G\} = \{a \in A : \#\sigma(xa) < \infty\ \forall x \in A\},$$
$$\{a \in A : \#\sigma(ca) \le n\ \forall c \in C \cap G\} = \{a \in A : \#\sigma(xa) \le n\ \forall x \in A\},$$
$$\{a \in A : \sigma(ca) = \{0\}\ \forall c \in C \cap G\} = \{a \in A : \sigma(xa) = \{0\}\ \forall x \in A\}.$$

Proof. Suppose that $\#\sigma(ca) \le n$ for all $c \in C \cap G$. Then by taking $B = C$ and $g(x) = xa$ in Lemma 4.1(2) and in Theorem 4.2(2), it follows that $\#\sigma(xa) \le n$ for all $x \in A$. The first and third statements follow similarly by using Lemma 4.1(1) and Theorem 4.2(1), and Lemma 4.1(3) and Theorem 4.2(3), respectively. ∎

Together with Theorem 2.11, we have the following characterisation of the socle in semisimple OBAs with generating algebra cones:

THEOREM 4.8. *Let (A, C) be a semisimple OBA with C generating, and let G be any subset of A which contains a point of C which is absorbing in G. Then*

$$\{a \in A : \text{there exists } n \in \mathbb{N} \text{ such that } \#\sigma'(ca) \le n \text{ for all } c \in C \cap G\}$$
$$= \operatorname{Soc}(A) = \{a \in A : \#\sigma'(ca) < \infty \text{ for all } c \in C \cap G\}.$$

For rank one elements we have:

THEOREM 4.9. *Let (A, C) be a semisimple OBA with $\dim A = \infty$ and C generating, and let G be any subset of A which contains a point of C which is absorbing in G. If $0 \ne a \in A$, then a is rank one if and only if $\#\sigma'(ca) \le 1$ for all $c \in C \cap G$.*

Proof. If $\#\sigma'(ca) \le 1$ for all $c \in C \cap G$, then $\#\sigma(xa) \le 2$ for all $x \in A$, by Proposition 4.7, so that $a \in \operatorname{Soc}(A)$, by Theorem 2.11 (or by Theorem 4.8). Since $\dim A = \infty$, $\operatorname{Soc}(A)$ is a proper ideal of A. Therefore $0 \in \sigma(xa)$ for all $x \in A$, so that $\#\sigma'(xa) \le 1$ for all $x \in A$, i.e. a is rank one. ∎

From the last statement in Proposition 4.7 we obtain the following characterisation of the radical, which is a slightly stronger version of [11, Theorem 4.17(3)] and an analogue of Theorem 2.6(2) in OBAs:

Theorem 4.10. *Let (A, C) be an OBA with C generating, and let G be any subset of A which contains a point of C which is absorbing in G. Then*

$$\mathrm{Rad}(A) = \{a \in A : (C \cap G)a \subseteq \mathrm{QN}(A)\}.$$

Having obtained a stronger version of Theorem 2.11 in the context of OBAs, we proceed to develop analogous versions of Theorems 2.12 and 2.13 in OBAs (see Theorems 4.13, 4.14, 4.18 and 4.19).

If B is a subset of a Banach algebra A, then we denote the set of inverses of invertible elements of B by B^{-1}, i.e. $B^{-1} = (B \cap A^{-1})^{-1}$. If the set B is inverse-closed, then $B^{-1} = B \cap A^{-1}$. The following result is a slightly sharper version of the reference, but the proof remains the same. We include it for completeness.

Lemma 4.11 ([12, Lemma 3.3]). *Let A be a Banach algebra, B a subset of A, $a \in A$ and $n \in \mathbb{N}$. If $\bigcap_{t \in F} \sigma(x + ta) \subseteq \sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \setminus \{0\}$ and all $x \in B^{-1}$, then $\#\sigma'(xa) \le n$ for all $x \in B \cap A^{-1}$.*

Proof. Suppose the given condition holds. Let $x \in B \cap A^{-1}$ and suppose that $\{t_i : i = 0, \ldots, n\}$ is a set of non-zero complex numbers. If $y = x^{-1}$, then $y \in B^{-1}$, and so $\bigcap_{i=0}^{n} \sigma(y + s_i a) \subseteq \sigma(y)$, where $s_i = -1/t_i$. Since $y \in A^{-1}$, there exists $i_0 \in \{0, \ldots, n\}$ such that $0 \notin \sigma(y + s_{i_0} a)$. It follows that $-s_{i_0} y(t_{i_0} - y^{-1}a) = y + s_{i_0} a \in A^{-1}$, so that $t_{i_0} - y^{-1}a \in A^{-1}$, i.e. $t_{i_0} \notin \sigma'(xa)$. ∎

For any invertible element x we can take $B = \{x^{-1}\}$, and obtain the following

Corollary 4.12. *Let A be a Banach algebra, $a \in A$ and $n \in \mathbb{N}$. If $x \in A^{-1}$ is such that $\bigcap_{t \in F} \sigma(x + ta) \subseteq \sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \setminus \{0\}$, then $\#\sigma'(x^{-1}a) \le n$.*

Lemma 4.11 (with $B = C$), together with Theorem 4.8 (with $G = A^{-1}$), and Theorem 2.13 yield the following analogue of the equivalence (1)⇔(2) in Theorem 2.13 in OBAs:

Theorem 4.13. *Let (A, C) be a semisimple OBA with C generating and let $a \in A$. Then $a \in \mathrm{Soc}(A)$ if and only if there exists $n \in \mathbb{N}$ such that $\bigcap_{t \in F} \sigma(x + ta) \subseteq \sigma(x)$ for all $(n+1)$-element subsets F of $\mathbb{C} \setminus \{0\}$ and all $x \in C^{-1}$.*

Similarly, Theorem 2.12, Theorem 4.9 and Lemma 4.11 yield an analogue of the equivalence (1)⇔(2) in Theorem 2.12 in OBAs:

Theorem 4.14. *Let (A, C) be a semisimple OBA with $\dim A = \infty$ and C generating, and let $0 \ne a \in A$. Then a is rank one if and only if $\sigma(x + s_0 a) \cap \sigma(x + s_1 a) \subseteq \sigma(x)$ for all $s_0, s_1 \in \mathbb{C} \setminus \{0\}$ with $s_0 \ne s_1$ and all $x \in C^{-1}$.*

In order to obtain OBA-versions of the equivalences (1)$\Leftrightarrow$(3) in Theorems 2.12 and 2.13, we take a slightly different approach. The proof of the following lemma (which we include for completeness) forms part of the proof of [3, Theorem 2.2(2)].

LEMMA 4.15. *Let A be a Banach algebra, $a \in A$ and $n \in \mathbb{N}$. If $x \in A$ is such that $\bigcap_{t\in F} \eta\sigma(x + ta) \subseteq \eta\sigma(x)$ for all $(n + 1)$-element subsets F of $\mathbb{C} \setminus \{0\}$, then $\#\sigma'((\mu - x)^{-1}a) \le n$ for all $\mu \notin \eta\sigma(x)$.*

Proof. Suppose that the condition holds and suppose that $\{t_i : i = 0, \dots, n\}$ is a set of non-zero complex numbers. Then $\bigcap_{i=0}^{n} \eta\sigma(x + s_i a) \subseteq \eta\sigma(x)$, where $s_i = 1/t_i$. If $\mu \notin \eta\sigma(x)$, there exists $i_0 \in \{0, \dots, n\}$ such that $\mu \notin \eta\sigma(x + s_{i_0}a)$. It now follows from Lemma 2.1 that $-1 \notin \sigma((\mu - x)^{-1}(-s_{i_0}a))$ and so $t_{i_0} \notin \sigma'((\mu - x)^{-1}a)$. ∎

The lemma remains true if $\eta\sigma$ is replaced by σ, in which case it is equivalent with Corollary 4.12.

LEMMA 4.16. *Let (A, C) be an OBA with C closed. If $c \in C \cap A^{-1}$ and $\lambda > 2r(c)$, then there exists $d \in C \cap A^{-1}$ such that $(\mu - d)^{-1} = c - \lambda$ (where $\mu = -1/\lambda$) and $\mu \notin \eta\sigma(d)$.*

Proof. Let $\mu = -1/\lambda$ and define $d = (\lambda - c)^{-1} + \mu$. Then $(\mu - d)^{-1} = c - \lambda$. Since $\lambda > r(c)$ and $c \in C$, it follows from Proposition 3.2 and Lemma 3.3 that $(\lambda - c)^{-1} \ge 1/\lambda$, which proves that $d \in C$. Since $c \in A^{-1}$ and $\lambda > 2r(c)$, it can easily be checked that $0 \notin \sigma(d)$, so that $d \in A^{-1}$, and that $r(d) < |\mu|$, so that $\mu \notin \eta\sigma(d)$. ∎

Lemmas 4.15 and 4.16 now yield the following

PROPOSITION 4.17. *Let (A, C) be a semisimple OBA with C closed and generating, $a \in A$ and $n \in \mathbb{N}$. If $\bigcap_{t\in F} \eta\sigma(c + ta) \subseteq \eta\sigma(c)$ for all $(n + 1)$-element subsets F of $\mathbb{C} \setminus \{0\}$ and all $c \in C \cap A^{-1}$, then $\#\sigma'(ca) \le n$ for all $c \in C \cap A^{-1}$.*

Proof. Suppose the condition holds. Let $c \in C \cap A^{-1}$ and let D_1 be the interval $[2r(c) + 1, 2r(c) + 2]$. Then it follows from Lemmas 4.16 and 4.15 that $\#\sigma'((c - \lambda)a) \le n$ for all $\lambda \in D_1$. Taking $g(\lambda) = c - \lambda$, it follows from Corollary 2.4 that $\#\sigma'((c - \lambda)a) \le n$ for all $\lambda \in \mathbb{C}$; in particular for $\lambda = 0$, which yields the result. ∎

The proposition remains true if $\eta\sigma$ is replaced by σ. Moreover, in the case that A in Lemma 4.11 is a semisimple OBA and $B = C$, where the algebra cone C is closed, generating and inverse-closed, Proposition 4.17 (with $\eta\sigma$ replaced by σ) coincides with Lemma 4.11.

Proposition 4.17, together with Theorem 4.8, and Theorem 2.13 now yield the following stronger version of Theorem 2.13 in the context of OBAs:

Theorem 4.18. *Let (A, C) be a semisimple OBA with C closed and generating and $a \in A$. Then the following are equivalent:*

(1) $a \in \mathrm{Soc}(A)$.
(2) *There exists $n \in \mathbb{N}$ such that $\bigcap_{t \in F} \sigma(c + ta) \subseteq \sigma(c)$ for all $(n+1)$-element subsets F of $\mathbb{C} \setminus \{0\}$ and all $x \in C \cap A^{-1}$.*
(3) *There exists $n \in \mathbb{N}$ such that $\bigcap_{t \in F} \eta\sigma(c + ta) \subseteq \eta\sigma(c)$ for all $(n+1)$-element subsets F of $\mathbb{C} \setminus \{0\}$ and all $x \in C \cap A^{-1}$.*

Similarly, a stronger version of Theorem 2.12 for OBAs is obtained by applying Theorem 2.12, Theorem 4.9 and Proposition 4.17:

Theorem 4.19. *Let (A, C) be a semisimple OBA with $\dim A = \infty$ and C closed and generating, and let $0 \neq a \in A$. Then the following are equivalent:*

(1) *a is rank one.*
(2) *$\sigma(c + s_0 a) \cap \sigma(c + s_1 a) \subseteq \sigma(c)$ for all $s_0, s_1 \in \mathbb{C} \setminus \{0\}$ with $s_0 \neq s_1$ and all $c \in C \cap A^{-1}$.*
(3) *$\eta\sigma(c + s_0 a) \cap \eta\sigma(c + s_1 a) \subseteq \eta\sigma(c)$ for all $s_0, s_1 \in \mathbb{C} \setminus \{0\}$ with $s_0 \neq s_1$ and all $c \in C \cap A^{-1}$.*

Finally, we consider some applications of Lemma 4.1 and Theorem 4.2 to OBAs in which the algebra cone generates the quasinilpotents (see the examples preceding Proposition 3.4). The following result provides a number of characterisations of the radical in these cases:

Theorem 4.20. *Let (A, C) be an OBA with the property that C generates the quasinilpotents. Let $a \in A$ and let G_0 and G_1 be neighbourhoods of 0 and 1, respectively. Then the following are equivalent:*

(1) $a \in \mathrm{Rad}(A)$.
(2) $a(1 + (C \cap \mathrm{QN}(A))) \subseteq \mathrm{QN}(A)$.
(3) $a + (C \cap \mathrm{QN}(A) \cap G_0) \subseteq \mathrm{QN}(A)$.
(4) *$a \in \mathrm{QN}(A)$ and $(C \cap \mathrm{QN}(A))a \subseteq \mathrm{QN}(A)$.*
(5) *$a \notin A^{-1}$ and $\#\sigma(ac) \leq \#\sigma(c)$ for all $c \in C \cap G_1$.*

Proof. Taking $g(x) = a(1 + x)$, $g(x) = a + x$ and $g(x) = xa$, respectively, with $B = \mathrm{QN}(A) \cap C$ and $c_0 = 0$ in Theorem 4.2(3) and $G = G_0$ in Lemma 4.1, characterisations (2), (3) and (4) follow from Theorem 2.6(4), (5) and (6), respectively. Towards the non-trivial part of the equivalence (1)$\Leftrightarrow$(5), we note that the second condition in (5) implies that, given $c_q \in C \cap \mathrm{QN}(A)$, there exists $r > 0$ such that $\#\sigma(a(1 + \lambda c_q)) \leq \#\sigma(1 + \lambda c_q) = 1$ for all $0 \leq \lambda \leq r$. Therefore, with $f(\lambda) = a(1 + \lambda c_q)$, it follows from Corollary 2.3 that $\#\sigma(a(1 + c_q)) = 1$. The first condition now implies that $\sigma(a(1 + c_q)) = \{0\}$, and since $c_q \in C \cap \mathrm{QN}(A)$ was arbitrary, the result follows from (2). ■

We note that characterisation (5) in the above theorem is an OBA-analogue of characterisation (7) in Theorem 2.6.

The following theorem gives necessary and sufficient conditions for elements of certain OBAs to be equal to a scalar modulo the radical.

THEOREM 4.21. *Let (A, C) be an OBA with the property that C generates the quasinilpotents. Let $a \in A$ and let G_0 be a neighbourhood of* 0. *Then there exists $\alpha \in \mathbb{C}$ such that $a - \alpha \in \mathrm{Rad}(A)$ if and only if $\#\sigma(a + c_q) = 1$ for all $c_q \in C \cap \mathrm{QN}(A) \cap G_0$.*

Proof. By choosing $B = \mathrm{QN}(A) \cap C$, $G = G_0$, $c_0 = 0$ and $g(x) = a + x$, this follows from Lemma 4.1(2) and Theorem 4.2(2), together with Theorem 2.7(2). ∎

We conclude with the following result, where, in (3), conditions are given under which two elements a and b will be equal modulo the radical:

THEOREM 4.22. *Let (A, C) be an OBA with the property that C generates the quasinilpotents. Let $a \in A$, $b \in -C$ and let G_1 and G_{-b} be neighbourhoods of* 1 *and $-b$, respectively. Then the following holds:*

(1) $\mathrm{Rad}(A) = \{a \notin A^{-1} : \#\sigma(a + c) \leq \#\sigma(c)$ *for all* $c \in C \cap G_1\}$.
(2) *There exists $\alpha \in \mathbb{C}$ such that $a - \alpha \in \mathrm{Rad}(A)$ if and only if $\#\sigma(a + c) \leq \#\sigma(c)$ for all $c \in C \cap G_1$.*
(3) *$a - b \in \mathrm{Rad}(A)$ if and only if $\sigma(a + c) = \sigma(b + c)$ for all $c \in C \cap G_{-b}$.*

Proof. (1) The $\subseteq$ inclusion follows from Theorem 2.6(3). For the reverse inclusion we note that if $a \notin A^{-1}$ and $\#\sigma(a + c) \leq \#\sigma(c)$ for all $c \in C \cap G_1$, then $\sigma(a) = \{0\}$. Moreover, if $c_q \in C \cap \mathrm{QN}(A)$, then there exists $r > 0$ such that $\#\sigma(a + 1 + \lambda c_q) = 1$ for all $0 \leq \lambda \leq r$. Hence it follows from Corollary 2.3 that $\#\sigma(a + 1 + c_q) = 1$ for all $c_q \in C \cap \mathrm{QN}(A)$, so that, by Theorem 4.21, $a + 1 - \alpha \in \mathrm{Rad}(A)$ for some $\alpha \in \mathbb{C}$. Since $\sigma(a) = \{0\}$, it follows that $a \in \mathrm{Rad}(A)$.

(2) follows from (1).

(3) For the non-trivial implication, we note that the assumption implies that $\sigma(a - b) = \{0\}$ and that $\#\sigma(a + c) = \#\sigma(b + c)$ for all $c \in C \cap G_{-b}$. Hence, given $c_q \in C \cap \mathrm{QN}(A)$, there exists $r > 0$ such that $\#\sigma(a - b + \lambda c_q) = \#\sigma(\lambda c_q) = 1$ for all $0 \leq \lambda \leq r$. It follows from Corollary 2.3 that $\#\sigma(a - b + c_q) = 1$ for all $c_q \in C \cap \mathrm{QN}(A)$, so that Theorem 4.21 yields $a - b - \alpha \in \mathrm{Rad}(A)$ for some $\alpha \in \mathbb{C}$. Therefore $\{0\} = \sigma(a - b + \mathrm{Rad}(A)) = \sigma(\alpha + \mathrm{Rad}(A)) = \{\alpha\}$, so that $a - b \in \mathrm{Rad}(A)$. ∎

We note that (1), (2) and (3) in the above theorem are OBA-analogues of (1)⇔(8) in Theorem 2.6, (1)⇔(3) in Theorem 2.7 and [7, Theorem 2.1(ii)], respectively.

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