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Invariant means on a class of von Neumann algebras related to ultraspherical hypergroups

by

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Abstract. Let K be an ultraspherical hypergroup associated to a locally compact group G and a spherical projector π and let $\mathrm{VN}(K)$ denote the dual of the Fourier algebra $A(K)$ corresponding to K. In this note, invariant means on $\mathrm{VN}(K)$ are defined and studied. We show that the set of invariant means on $\mathrm{VN}(K)$ is nonempty. Also, we prove that, if H is an open subhypergroup of K, then the number of invariant means on $\mathrm{VN}(H)$ is equal to the number of invariant means on $\mathrm{VN}(K)$. We also show that a unique topological invariant mean exists precisely when K is discrete. Finally, we show that the set $\mathrm{TIM}(\widehat{K})$ becomes uncountable if K is nondiscrete.

1. Introduction. Let G be a locally compact group and let $A(G)$ and $\mathrm{VN}(G)$ denote the Fourier algebra and its Banach space dual respectively. Invariant means on $\mathrm{VN}(G)$ were defined and studied by Renaud [9]. He proved that a locally compact group G is discrete if and only if $\mathrm{VN}(G)$ admits a unique invariant mean. Cho-Ho Chu and A. T. M. Lau [3] have extended the results of Renaud to the case of homogeneous spaces.

Let K be an ultraspherical hypergroup associated to a locally compact group G and a spherical projector π. Let $A(K)$ denote the Fourier algebra corresponding to the hypergroup K and let $\mathrm{VN}(K)$ be its Banach space dual. In this paper, a systematic study of invariant means on $\mathrm{VN}(K)$ is carried out. As a result, we extend some of the results of Renaud [9] to the case of ultraspherical hypergroups.

In Section 3, we define and study means on $\mathrm{VN}(K)$. Invariant means on $\mathrm{VN}(K)$ are defined in Section 4, and some of their basic properties are derived. In Section 5, we prove that if H is an open subhypergroup of K, then the number of invariant means on $\mathrm{VN}(H)$ is equal to the number of invariant means on $\mathrm{VN}(K)$. We use this to prove that a unique invariant mean exists precisely when K is discrete. Finally, in Section 6, we show that

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when K is nondiscrete, the number of invariant means is actually uncountable.

We begin with some preliminaries in the next section.

2. Preliminaries. Let G be a locally compact group. Fix a left Haar measure m_G on G. Let σ be a unitary representation of G on a Hilbert space $\mathcal{H}_\sigma$. For $u, v \in \mathcal{H}_\sigma$, let $\sigma_{u,v}$ denote the coefficient function corresponding to σ, u and v. The *Fourier–Stieltjes algebra* of G, introduced by Eymard [5, p. 192] and denoted $B(G)$, is defined as the collection of all coefficient functions arising from all the unitary representations. Eymard showed that it is also the dual of the group C^*-algebra $C^*(G)$. With the dual norm, $B(G)$ becomes a commutative Banach algebra with pointwise addition and multiplication.

The left regular representation ρ of G on the Hilbert space $L^2(G)$ is given by $\rho(x)(f)(y) = f(x^{-1}y)$. Via integration, ρ extends to a representation of $L^1(G)$ given as $\rho(f)(g) = f * g$. The closed linear span in $B(G)$ of all coefficient functions arising only from the left regular representation is called the *Fourier algebra* of G, denoted $A(G)$. For more on the Fourier algebra and the Fourier–Stieltjes algebra, we refer to the fundamental paper of Eymard [5].

We shall now define the notion of a spherical projector on a locally compact group [8, Definition 2.1].

Definition 2.1. A map $\pi : C_c(G) \to C_c(G)$ is called a *spherical projector* if for all $f, g \in C_c(G)$:

1. We have
   - (i) $\pi^2 = \pi$ and π is positivity preserving;
   - (ii) $\pi(\pi(f)g) = \pi(f)\pi(g)$;
   - (iii) $\langle \pi(f), g\rangle = \langle f, \pi(g)\rangle$;
   - (iv) $\int_G \pi(f)(x)\,dx = \int_G f(x)\,dx$.
2. $\pi(\pi(f) * \pi(g)) = \pi(f) * \pi(g)$.
3. Let $\pi^* : M(G) \to M(G)$ denote the transpose of π and let $\mathcal{O}_x = \operatorname{supp}(\pi^*(\delta_x))$, $x \in G$. Then for all $x, y \in G$:
   - (i) either $\mathcal{O}_x \cap \mathcal{O}_y = \emptyset$ or $\mathcal{O}_x = \mathcal{O}_y$;
   - (ii) $x \in \mathcal{O}_y \Rightarrow\ y^{-1} \in \mathcal{O}_{x^{-1}}$;
   - (iii) $\mathcal{O}_{xy} = \mathcal{O}_e \Rightarrow \mathcal{O}_y = \mathcal{O}_{x^{-1}}$;
   - (iv) the map $x \mapsto \mathcal{O}_x$ from G to $\mathcal{K}(G)$ is continuous, where $\mathcal{K}(G)$ denotes the space of all nonempty compact subsets of G equipped with the Michael topology.

Note that π extends to a norm decreasing linear map on various function spaces, including $L^p(G)$, $1 \le p \le \infty$, and $A(G)$. A function f is called π-*radial* if $\pi(f) = f$, and a measure μ is called π-*radial* if $\pi^*(\mu) = \mu$.

Let $K = \{\mathcal{O}_x : x \in G\}$ with the natural quotient topology under the quotient map $p : G \to K$. We identify $M(K)$ with the space of all π-radial measures on G. Restricting the convolution on $M(G)$ to $M(K)$ makes $M(K)$ a Banach algebra. With this convolution structure, K becomes a hypergroup, called a *spherical hypergroup* [8]. See [1, 6] for more details on hypergroups.

A spherical hypergroup is called *ultraspherical* if the modular function on G is π-radial. The most common example of an ultraspherical hypergroup is the double coset hypergroup $G//C$ corresponding to the spherical projector π given by

$$\pi(f)(x) = \int_C \int_C f(c'xc)\, dc\, dc',$$

where C is a compact subgroup of a locally compact group G. A *Haar measure* on a hypergroup K is a regular measure μ such that $\delta_x * \mu = \mu$ for all $x \in K$. On an ultraspherical hypergroup, a Haar measure always exists [8].

The Fourier algebra of an ultraspherical hypergroup K, denoted $A(K)$, was defined and studied by Muruganandam [8]. The Fourier algebra $A(K)$ is defined as the range of π. Thus a function in $A(K)$ can be treated as a function on both G and K. The algebra $A(K)$ is a commutative Banach algebra with the Gelfand spectrum homeomorphic to K [8].

As in the group case, the Fourier–Stieltjes algebra, denoted $B(K)$, can be defined as the closed linear span of positive definite functions on K. Note that $B(K)$ can be identified with the algebra of all π-radial functions in $B(G)$. It is shown in [7] that $B(K)$ is the dual of the C^*-algebra $C^*_\rho(K)$. For definition and details on $C^*_\rho(K)$ see [7]. Just as in the group case, $A(K)$ is also an ideal in $B(K)$.

For a locally compact group G, there is a naturally associated von Neumann algebra, called the *group von Neumann algebra* and denoted $\mathrm{VN}(G)$; it is the weak operator topology closure of the span of $\{\rho(x) : x \in G\}$. By [5, p. 210], the dual of $A(G)$ is isometrically isomorphic to $\mathrm{VN}(G)$. Observe that $\mathrm{VN}(G)$ is also equal to the weak operator topology closure of the span of $\{\rho(f) : f \in L^1(G)\}$. Let $\mathrm{VN}(K)$ be the weak operator topology closure of the span of $\{\rho(f) : f \in L^1(K)\}$. The algebra $\mathrm{VN}(K)$ is a von Neumann algebra and by [7], it is isometrically isomorphic to the dual of $A(K)$.

For $\varphi \in B(K)$ and $T \in \mathrm{VN}(K)$, define $\varphi.T \in \mathrm{VN}(K)$ by

$$\langle \psi, \varphi.T\rangle := \langle \varphi\psi, T\rangle \quad \forall \psi \in A(K).$$

With this action, $\mathrm{VN}(K)$ becomes a $B(K)$-module. Further, if $m \in \mathrm{VN}(K)^*$ and $\varphi \in B(K)$, define $\varphi.m \in \mathrm{VN}(K)^*$ by

$$\langle T, \varphi.m\rangle := \langle \varphi.T, m\rangle \quad \forall T \in \mathrm{VN}(K).$$

This action makes $\mathrm{VN}(K)^*$ also into a $B(K)$-module.

We now define a multiplication on $\mathrm{VN}(K)^*$, called *Arens multiplication*, as follows. For $m \in \mathrm{VN}(K)^*$ and $T \in \mathrm{VN}(K)$ define $m \odot T \in \mathrm{VN}(K)$ by

$$\langle \psi, m \odot T\rangle := \langle \psi.T, m\rangle \quad \forall \psi \in A(K).$$

For $m, n \in \mathrm{VN}(K)^*$ define $m \odot n \in \mathrm{VN}(K)^*$ by

$$\langle T, m \odot n\rangle = \langle n \odot T, m\rangle \quad \forall T \in \mathrm{VN}(K).$$

This multiplication makes $\mathrm{VN}(K)^*$ into a Banach algebra.

Throughout this paper, K will denote an ultraspherical hypergroup associated with a locally compact group G and a spherical projector π, and $p : G \to K$ will denote the canonical quotient map. Also, for any $x \in G$, $\dot{x}$ will denote the corresponding element of K. We shall denote by ι the canonical inclusion of $A(K)$ into its double dual $\mathrm{VN}(K)^*$, and by j the natural inclusion map $j : A(K) \to A(G)$.

3. Means on $\mathrm{VN}(K)$**.** In this section, we define the notion of a mean on the space $\mathrm{VN}(K)$ and prove some of its properties. The main aim of this section is to prove Theorem 3.5.

Definition 3.1. A linear functional m on $\mathrm{VN}(K)$ is called a *mean* if

$$\|m\| = m(I) = 1.$$

Note that, by [10, p. 38], m is a positive linear functional on $\mathrm{VN}(K)$. Let M denote the set of all means on $\mathrm{VN}(K)$. Notice that M is a weak* compact convex subset of $\mathrm{VN}(K)^*$.

Let

$$M_{A(K)} := \{\varphi \in A(K) : \|\varphi\|_{A(K)} = \varphi(\dot{e}) = 1\}.$$

Similarly, let

$$M_{B(K)} := \{\varphi \in B(K) : \|\varphi\|_{B(K)} = \varphi(\dot{e}) = 1\}.$$

The next lemma lists some trivial properties of $M_{A(K)}$ and $M_{B(K)}$.

Lemma 3.2. *Let K be an ultraspherical hypergroup.*

(i) *$M_{A(K)}$ and $M_{B(K)}$ are convex subsets of $A(K)$ and $B(K)$ respectively.*
(ii) *$M_{A(K)}$ and $M_{B(K)}$ are abelian semigroups under pointwise multiplication.*
(iii) *If $\varphi \in M_{B(K)}$ and $m \in M$, then $\varphi.m \in M$.*
(iv) *If ι denotes the canonical inclusion of $A(K)$ into its second dual, then*

$$\iota(M_{A(K)}) \subset M.$$

Proof. It is enough to prove (iii) as others are clear. Let $\varphi \in M_{B(K)}$ and $m \in M$. Then

$$1 = \|m\| = \|\varphi\|_{B(K)}\|m\| \geq \|\varphi.m\| \geq \langle I, \varphi.m\rangle$$
$$= \langle \varphi.I, m\rangle = \langle \varphi(\dot{e})I, m\rangle = \langle I, m\rangle = \|m\|.$$

Thus $\|\varphi.m\| = \varphi.m(I) = 1$. ∎

PROPOSITION 3.3. *There exists a mean $m \in M$ such that $\varphi.m = m$ for all $\varphi \in M_{A(K)}$.*

Proof. By Lemma 3.2, the conclusion follows from the Markov–Kakutani fixed point theorem, as $M_{A(K)}$ acts on the dual of VN(K) as an abelian semigroup of weak* continuous affine operators. ∎

We now show the existence of a certain kind of function.

PROPOSITION 3.4. *Let $\widetilde{V}$ be a neighbourhood of $\dot{e}$ in K. Then there exists a function $\varphi \in A(K)$ such that:*

(a) $0 \leq \varphi \leq 1$;
(b) $\|\varphi\|_{A(K)} = \varphi(\dot{e}) = 1$;
(c) $\mathrm{supp}(\varphi) \subset \widetilde{V}$.

Proof. Let $\widetilde{U}$ be a symmetric, relatively compact neighbourhood of $\dot{e}$ in K such that $\widetilde{U} \subset \widetilde{V}$. Then $\varphi = \frac{1}{m_K(\widetilde{U})}\chi_{\widetilde{U}} * \chi_{\widetilde{U}}$ satisfies the requirements. ∎

THEOREM 3.5. *If $m \in M$ is as in Proposition* 3.3, *then $\varphi.m = \varphi(\dot{e})m$ for all $\varphi \in B(K)$.*

Proof. (i) If $\phi \in B(K)$ is such that $\phi = 1$ on a neighbourhood $\widetilde{V}$ of $\dot{e}$ in K, let ψ be the function as in Proposition 3.4 corresponding to the neighbourhood $\widetilde{V}$. Then $\psi\phi = \psi$. Therefore,

$$\phi.m = \phi.(\psi.m) = (\phi\psi).m = \psi.m = m.$$

(ii) Let $\phi \in A(K)$ be such that $\phi(\dot{e}) = 0$. By [4, Lemma 3.8 and Theorem 3.1], $\{\dot{e}\}$ is a set of spectral synthesis and hence there exists a sequence $\{\phi_n\} \in j_{A(K)}(\{\dot{e}\})$ such that $\|\phi_n - \phi\|_{A(K)} \to 0$. Further, by (i), $m = (1-\phi_n).m = m - \phi_n.m$, and hence $\phi_n.m = 0$. Therefore,

$$\|\phi.m\| = \|(\phi - \phi_n).m\| \leq \|\phi - \phi_n\|\,\|m\| \to 0.$$

Hence, $\phi.m = 0$.

(iii) Let $\varphi \in B(K)$ be such that $\varphi(\dot{e}) \neq 0$. Let $\phi \in M_{A(K)}$ and let $\psi \in A(K)$ be such that $\psi = 1$ on some neighbourhood $\widetilde{V}$ of $\dot{e}$. As $\frac{\varphi\phi}{\varphi(\dot{e})} - \psi = 0$ on $\dot{e}$, by (ii), $\frac{\varphi\phi}{\varphi(\dot{e})}.m = \psi.m$. Then by (i),

$$\frac{\varphi}{\varphi(\dot{e})}.m = \frac{\varphi}{\varphi(\dot{e})}.(\phi.m) = \left(\frac{\varphi\phi}{\varphi(\dot{e})}\right).m = \psi.m = m.$$

(iv) It remains to prove the assertion for $\varphi \in B(K)$ such that $\varphi(\dot{e}) = 0$. Choose $\phi \in A(K)$ such that $\phi(\dot{e}) = 1$. Then $((1-\varphi)\phi)(\dot{e}) = 1$ and hence from (iii), $((1-\varphi)\phi).m = m$, and so $\varphi.(\phi.m) = 0$. Thus, again by (iii),

$$\varphi.m = \varphi.(\phi.m) = 0 = \varphi(\dot{e})m,$$

which is what we intended to show. ∎

4. Invariant means. In this section, invariant means are defined and their basic properties are studied in the spirit of [9].

DEFINITION 4.1. A linear functional m on $\mathrm{VN}(K)$ is said to be *topologically invariant* if $\varphi.m = \varphi(\dot{e})m$ for all $\varphi \in A(K)$, i.e.,

$$\langle T, \varphi.m\rangle = \langle \varphi.T, m\rangle = \varphi(\dot{e})\langle T, m\rangle \quad \forall T \in \mathrm{VN}(K), \forall \varphi \in A(K).$$

We denote by $\mathrm{TIM}(\widehat{K})$ the set of all topological invariant means on $\mathrm{VN}(K)$. Note that, by Theorem 3.5, the set $\mathrm{TIM}(\widehat{K})$ is nonempty.

Before we move on to the main theorems, we note the action of $A(G)$ on $\mathrm{VN}(K)$ in the following lemma.

LEMMA 4.2. *If $j : A(K) \to A(G)$ denotes the canonical inclusion map, then, for $T' \in \mathrm{VN}(K)$ and $\varphi \in A(G)$, we have $j^*(\varphi.\pi^*(T')) = \pi(\varphi).T'$.*

The following theorem gives some properties of an invariant functional.

THEOREM 4.3.

(i) *For $m \in \mathrm{VN}(K)^*$, m is invariant if and only if $\iota(\varphi) \odot m = \varphi(\dot{e})m$ for all $\varphi \in A(K)$.*
(ii) *Let $m, n \in \mathrm{VN}(K)^*$. If m is invariant, then so is $m \odot n$.*
(iii) *If $m \in \mathrm{VN}(K)^*$ is invariant, then so is $j^{**}(m)$ as an element of $\mathrm{VN}(G)^*$.*
(iv) *Let $m \in \mathrm{TIM}(\widehat{G})$ and $m' \in \mathrm{VN}(K)^*$ be invariant. Then $m \odot j^{**}(m')$ is an invariant element of $\mathrm{VN}(G)$ and $\|m \circ j^{**}(m')\| = \|m'\|$.*

Proof. (i) This follows from the fact that if $\varphi \in A(K)$ and $m \in \mathrm{VN}(K)^*$, then for $T \in \mathrm{VN}(K)$ we have

$$\langle T, \iota(\varphi) \odot m\rangle = \langle m \odot T, \iota(\varphi)\rangle = \langle \varphi, m \odot T\rangle = \langle \varphi.T, m\rangle.$$

(ii) This follows from (i) if we observe that, for $m, n \in \mathrm{VN}(K)^*$ and $\varphi \in A(K)$,

$$\iota(\varphi) \odot (m \odot n) = (\iota(\varphi) \odot m) \odot n.$$

(iii) This follows from the fact that if $\varphi \in A(G)$ and $T \in \mathrm{VN}(G)$, then by Lemma 4.2,

$$\langle \varphi.T, j^{**}(m)\rangle = \langle \pi(\varphi).j^*(T), m\rangle.$$

(iv) By (ii) and (iii), $m \odot j^{**}(m') \in \mathrm{VN}(G)^*$ and is in fact invariant. Further, $\|m \odot j^{**}(m')\| \le \|m\|\,\|j^{**}(m')\|$. Hence it is enough to prove the opposite inequality.

Notice that, by Lemma 4.2, for $\varphi \in A(G)$ we have

$$\begin{aligned}\langle \varphi, j^{**}(m') \odot \pi^*(T')\rangle &= \langle \varphi.\pi^*(T'), j^{**}(m')\rangle\\ &= \langle j^*(\varphi.\pi^*(T')), m'\rangle\\ &= \langle \pi(\varphi).T', m'\rangle \qquad \text{(by Lemma 4.2)}\\ &= \pi(\varphi)(\dot{e})\langle T', m'\rangle \qquad \text{(by definition)}\\ &= \langle T', m'\rangle \rho(\delta_{\dot{e}})(\pi(\varphi)).\end{aligned}$$

Thus $j^{**}(m') \odot \pi^*(T') = \langle T', m'\rangle \rho(e)$.

Let $\epsilon > 0$. Choose $T' \in \mathrm{VN}(K)$ such that $\|T'\| \le 1$ and $|\langle T, m'\rangle| \ge \|m'\| - \epsilon$. Then

$$\begin{aligned}|\langle \pi^*(T'), m \odot j^{**}(m')\rangle| &= |\langle j^{**}(m') \odot \pi^*(T'), m\rangle|\\ &= \big|\big\langle \langle T', m'\rangle \rho(e), m\big\rangle\big|\\ &= |\langle T', m'\rangle| \ge \|m'\| - \epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, $\|m \odot j^{**}(m')\| \ge \|m'\|$. ∎

We now collect some properties of invariant means. In the proof, some of the ideas are adapted from [9].

THEOREM 4.4.

(i) *If* $\mathrm{TIM}(\widehat{K})$ *contains more than one element then so does* $\mathrm{TIM}(\widehat{G})$.
(ii) *If* K *is discrete, then* $\mathrm{TIM}(\widehat{K}) \cap A(K) \neq \emptyset$.
(iii) *If* $\mathrm{TIM}(\widehat{K}) \cap A(K) \neq \emptyset$ *then* K *is discrete.*
(iv) *If* K *is not discrete and* $m \in \mathrm{TIM}(\widehat{K})$, *then* $m(T) = 0$ *for all* $T \in C^*_\rho(K)$.
(v) *If* K *is nondiscrete,* $\phi \in M_{A(K)}$ *and* $m \in \mathrm{TIM}(\widehat{K})$ *then* $\|\phi - m\| = 2$.

Proof. (i) Let $m \in \mathrm{TIM}(\widehat{K})$. By Theorem 4.3(iv), the map $m' \mapsto m \odot j^{**}(m')$ from $\mathrm{TIM}(\widehat{K})$ to $\mathrm{TIM}(\widehat{K})$ is an isometry, from which the statement follows.

(ii) Since K is discrete, $\chi_{\{\dot{e}\}} \in L^2(K)$ and hence by [7], $\varphi = \chi_{\{\dot{e}\}} * \chi_{\{\dot{e}\}} \in A(K)$ and $\|\varphi\|_{A(K)} = 1$. By Lemma 3.2(iv), $\iota(\varphi) \in M$ and is in fact invariant.

(iii) Let $\varphi \in A(K)$ be such that $\iota(\varphi)$ is a topologically invariant mean on $\mathrm{VN}(K)$. Suppose that K is not discrete. Then there exists $\dot{x} \neq \dot{e}$ such that $\varphi(\dot{x}) \neq 0$. Let $\widetilde{V}$ be a compact neighbourhood of $\dot{e}$ such that $\dot{x} \notin \widetilde{V}$. Let ψ be the function as in Proposition 3.4 corresponding to $\widetilde{V}$. Then

$$\psi\varphi = \psi.\iota(\varphi) = \iota(\varphi) = \varphi$$

and $\varphi(\dot{x}) = \psi(\dot{x})\varphi(\dot{x}) = 0$, which is a contradiction.

(iv) Let $T \in C^*_\rho(K)$. Let $\epsilon > 0$. Since $L^1(K)$ is dense in $C^*_\rho(K)$, there exists $f \in L^1(K)$ such that $\|T - \rho(f)\| < \epsilon/2$. Let $\widetilde{U}$ be a neighbourhood of $\dot{e}$ such that $\|f\chi_{\widetilde{U}}\| < \epsilon/2$. Let $\phi \in M_{A(K)}$ be such that $\mathrm{supp}(\phi) \subset \widetilde{U}$. Thus

$$\begin{aligned}|\langle T, m\rangle| = |\langle T, \phi.m\rangle| &= |\langle \phi.T, m\rangle| \le \|\phi.T\| \\ &\le \|\phi.(T-\rho(f))\| + \|\phi.\rho(f)\| < \epsilon/2 + \|\phi.\rho(f.\chi_{\widetilde{U}})\| < \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $m(T) = 0$.

(v) Let $\phi \in M_{A(K)}$. As $M_{A(K)} \subset B_\rho(K)$, by [8, Theorem 3.15], ϕ can be considered as a positive linear functional on $C^*_\rho(K)$. By [10, Corollary 3.5], $C^*_\rho(K)$ has an approximate identity and hence, for any $\epsilon > 0$, there exists $S \in C^*_\rho(K)$ such that $0 \le S \le I$ and $\langle S, \phi\rangle \ge 1-\epsilon$. Let $T = 2S - I \in \mathrm{VN}(K)$. Then for $m \in \mathrm{TIM}(\widehat{K})$,

$$\langle T, \iota(\phi) - m\rangle = 2\langle S, \iota(\phi) - m\rangle \ge 2\langle S, \phi\rangle \ge 2(1-\epsilon).$$

Since $\epsilon > 0$ is arbitrary and $\|T\| \le 1$, we have $\|\phi - m\| = 2$. ∎

5. Open subhypergroups and invariant means. In this section we prove that the cardinalities of the sets of all invariant means on $\mathrm{VN}(H)$ and on $\mathrm{VN}(K)$ are equal. At the end of this section, we prove a necessary and sufficient condition for the uniqueness of invariant means, which is our main aim in this section.

We first state some functorial properties of $A(K)$ in the spirit of [5]. As the proofs follows the same lines as in [5], we omit them.

LEMMA 5.1. *Let H be a closed subhypergroup of K. For $\varphi \in A(H)$ let φ° denote the function on K that is φ on H and vanishes outside H.*

(i) *If H is open, then $\varphi \mapsto \varphi^\circ$ is an isometric isomorphism of $A(H)$ onto $A(K)^\circ = \{\varphi^\circ : \varphi \in A(H)\}$*
(ii) *The restriction map from $A(K)$ to $A(H)$ is a contractive homomorphism.*

Let $r_K : A(K) \to A(H)$ and $e_K : A(H) \to A(G)$ denote the restriction and extension maps, respectively, of the above lemma. Notice that $r_K e_K$ is the identity on $A(H)$. In the remaining part of this section, H will denote an open subhypergroup of K.

We now prove some lemmas which will be used in the proof of Theorem 5.6.

LEMMA 5.2. *For $\varphi \in A(K)$ and $T \in \mathrm{VN}(K)$, we have*

$$e^*_K(\varphi.T) = r_K(\varphi).e^*_K(T).$$

Proof. For any $\psi \in A(K)$,

$$\begin{aligned}\langle \psi, e_K^*(\varphi.T)\rangle &= \langle e_K(\psi), \varphi.T\rangle = \langle \varphi e_K(\psi), T\rangle \\ &= \langle e_K(r_K(\varphi)\psi), T\rangle = \langle r_K(\varphi)\psi, e_K^*(T)\rangle \\ &= \langle \psi, r_K(\varphi).e_K^*(T)\rangle. \ \blacksquare\end{aligned}$$

LEMMA 5.3. *For $\psi \in A(K)$ and $T \in \mathrm{VN}(K)$, we have*

$$r_K^*(\psi.T) = e_K(\psi).r_K^*(T).$$

Proof. For any $\varphi \in A(K)$,

$$\begin{aligned}\langle \varphi, r_K^*(\psi.T)\rangle &= \langle r_K(\varphi), \psi.T\rangle = \langle \psi r_K(\varphi), T\rangle \\ &= \langle r_K(e_K(\psi)\varphi), T\rangle = \langle e_K(\psi)\varphi, r_K^*(T)\rangle \\ &= \langle \varphi, e_K(\psi).r_K^*(T)\rangle. \ \blacksquare\end{aligned}$$

LEMMA 5.4. *For $\varphi \in A(H)$ and $T \in \mathrm{VN}(K)$, we have*

$$r_K^*(\varphi.e_K^*(T)) = e_K(\varphi).T.$$

Proof. For any $\psi \in A(K)$,

$$\begin{aligned}\langle \psi, r_K^*(\varphi.e_K^*(T))\rangle &= \langle r_K(\psi), \varphi.e_K^*(T)\rangle = \langle \varphi r_K(\psi), e_K^*(T)\rangle \\ &= \langle e_K(\varphi.r_K(\psi)), T)\rangle = \langle e_K(\varphi)\psi, T\rangle \\ &= \langle \psi, e_K(\varphi).T\rangle. \ \blacksquare\end{aligned}$$

LEMMA 5.5. *The second adjoint $e_K^{**} : A(H)^{**} \to A(K)^{**}$ is an isometry.*

Proof. By Lemma 5.1, the restriction map $r_K : A(K) \to A(H)$ is a contraction and hence $\|r_K^{**}\| = \|r_K\| \le 1$. Since $r_K^{**}e_K^{**}$ is the identity map on $A(H)^{**}$, we have $\|m\| = \|r_K^{**}e_K^{**}(m)\|$ for any $m \in A(K)^{**}$. Suppose $\|e_K^{**}(m)\| < \|m\|$ for some $m \in A(K)^{**}$. Then

$$\|m\| = \|r_K^{**}e_K^{**}(m)\| \le \|e_K^{**}(m)\| < \|m\|,$$

which is a contradiction. ∎

We now proceed to prove the main results of this section.

THEOREM 5.6. *Let H be an open subhypergroup of K. Then*

$$e_K^{**}(\mathrm{TIM}(\widehat{H})) = \mathrm{TIM}(\widehat{K}).$$

Proof. (i) We first prove $e_K^{**}(\mathrm{TIM}(\widehat{H})) \subseteq \mathrm{TIM}(\widehat{K})$. In fact, let m be an invariant mean. Let $\varphi \in A(K)$ and $T \in \mathrm{VN}(K)$. Then, by Lemma 5.2,

$$\begin{aligned}\langle \varphi.T, e_K^{**}(m)\rangle &= \langle e_K^*(\varphi.T), m\rangle = \langle r_K(\varphi).e_K^*(T), m\rangle \\ &= r_K(\varphi)(\dot{e})\langle e_K^*(T, m)\rangle = \varphi(\dot{e})\langle T, e_K^{**}(m)\rangle.\end{aligned}$$

Hence the claim.

(ii) We now claim that $r_K^{**}(\mathrm{TIM}(\widehat{K})) = \mathrm{TIM}(\widehat{H})$. Since $r_K^{**}e_K^{**}$ is the identity map on $A(H)^{**}$, by (i) we have

$$\mathrm{TIM}(\widehat{H}) = r_K^{**}e_K^{**}(\mathrm{TIM}(\widehat{H})) \subseteq r_K^{**}(\mathrm{TIM}(\widehat{K})).$$

We now prove the reverse inclusion. If $m \in \mathrm{TIM}(\widehat{K})$, then for $\psi \in A(H)$ and $T \in \mathrm{VN}(H)$ we have, by Lemma 5.3,

$$\begin{aligned}\langle \psi.T, r_K^{**}(m)\rangle &= \langle r_K^*(\psi.T), m\rangle = \langle e_K(\psi).r_K^*(T), m\rangle \\ &= e_K(\psi)(\dot{e})\langle r_K^*(T), m\rangle = \psi(\dot{e})\langle T, r_K^{**}(m)\rangle.\end{aligned}$$

Hence the claim.

(iii) We next claim that $e_K^{**}r_K^{**}(m) = m$ for any $m \in \mathrm{TIM}(\widehat{K})$. Indeed, let $m \in \mathrm{TIM}(\widehat{K})$. By (ii), $m' = r_K^{**}(m) \in \mathrm{TIM}(\widehat{H})$. Let $\varphi \in A(H)$ be such that $\varphi(\dot{e}) = 1$ and let $T \in \mathrm{VN}(K)$. By Lemma 5.4,

$$\begin{aligned}\langle T, m\rangle &= \varphi(\dot{e})\langle T, m\rangle = e_K(\varphi)(\dot{e})\langle T, m\rangle \\ &= \langle e_K(\varphi).T, m\rangle = \langle r_K^*(\varphi.e_K^*(T)), m\rangle \\ &= \langle \varphi.e_K^*(T), r_K^{**}(m)\rangle = \langle e_K^*(T), r_K^{**}(m)\rangle \\ &= \langle T, e_K^{**}r_K^{**}(m)\rangle.\end{aligned}$$

Thus $e_K^{**}r_K^{**}(m) = m$.

(iv) We now prove the remaining inclusion of the theorem. By (ii), $m' = r_K^{**}(m) \in \mathrm{TIM}(\widehat{H})$. By (iii), $e_K^{**}(m') = m$ and hence the reverse inclusion follows. ∎

Here is the promised result on the cardinality of the sets of invariant means, whose proof is immediate from Theorem 5.6. Here $\#X$ denotes the cardinality of the set X.

Corollary 5.7. *If H is an open subhypergroup of K, then*

(a) $\#\,\mathrm{TIM}(\widehat{H}) = \#\,\mathrm{TIM}(\widehat{K})$;

(b) $\mathrm{TIM}(\widehat{H})$ *is separable if and only if* $\mathrm{TIM}(\widehat{K})$ *is separable.*

The following corollary generalizes Theorem 1 of [9].

Corollary 5.8. *If K is discrete, then there exists a unique topological invariant mean on* $\mathrm{VN}(K)$.

Proof. Choose $H = \{\dot{e}\}$ in Corollary 5.7(a). ∎

The converse to the above corollary is the next theorem which also generalizes Theorem 11 of [9]. Moreover, the proof of the theorem below is a modification of the proof given for the case of locally compact groups in [9].

Theorem 5.9. *Let K be a second countable ultraspherical hypergroup. If* $\mathrm{VN}(K)$ *admits a unique topological invariant mean, then K is discrete.*

Proof. Let $\mathscr{U}$ be a neighbourhood base of $\dot{e}$ such that each element of $\mathscr{U}$ is a compact set. Since K is second countable, without loss of generality, we can even assume that $\mathscr{U}$ is countable. So let $\mathscr{U}$ be the sequence $\{\widetilde{U}_n\}$ such that $\widetilde{U}_n \to \{\dot{e}\}$. For each $n \in \mathbb{N}$, let $\psi_n \in M_{A(K)}$ with $\mathrm{supp}(\psi_n) \subseteq \widetilde{U}_n$.

Let $\psi \in M_{A(K)}$ and $\epsilon > 0$. Note that the set of compactly supported elements in $M_{A(K)}$ is dense in $M_{A(K)}$. Hence there exists $\psi' \in M_{A(K)}$ with compact support such that $\|\psi-\psi'\| < \epsilon/2$. By regularity of $A(K)$ [8, Proposition 2.22], there exists $\varphi \in A(K)$ such that φ is 1 on $\mathrm{supp}(\psi')$. Since $\psi'(\dot{e}) = 1$ and $\dot{e} \in \mathrm{supp}(\psi')$, we have $(\psi' - \varphi)(\dot{e}) = 0$. By [4, Theorem 3.1 and Lemma 3.8], $\{\dot{e}\}$ is a set of spectral synthesis and hence there exists $\chi \in A(K)$ such that $\|\psi' - \varphi - \chi\| < \epsilon/2$ and $\chi(\widetilde{W}) = 0$ for some neighbourhood $\widetilde{W}$ of $\dot{e}$. Further, for any $n \in \mathbb{N}$ such that $\widetilde{U}_n \subset \widetilde{W} \cap \mathrm{supp}(\psi')$, we have $\psi_n\varphi = \psi_n$ and $\psi_n\chi = 0$. Thus, by a standard $\epsilon/2$ argument, it follows that $\|\psi\psi_n - \psi_n\| < \epsilon$.

Note that every weak* accumulation point of $\{\psi_n\}$ in $A(K)^{**}$ is a topological invariant mean. By the assumption that the topological invariant mean is unique and as the set of topological invariant means on $\mathrm{VN}(K)$ is nonempty, let m be the unique topological invariant mean on $\mathrm{VN}(K)$. Also $A(K)$ is the predual of the von Neumann algebra $\mathrm{VN}(K)$ and hence, by [10, Corollary 5.2], $A(K)$ is weakly sequentially complete. Thus $\{\psi_n\}$ converges to m weakly in $A(K)$, which means that $m \in A(K)$. Hence by Theorem 4.4, it follows that K is open. ■

6. Cardinality of the set of invariant means. In this section, we take up the case of K nondiscrete. We prove that the number of invariant means is then uncountable.

Definition 6.1. A net $\{\psi_\alpha\}$ in $M_{A(K)}$ is called a *TI-net* if

$$\lim_\alpha \|\psi\psi_\alpha - \psi_\alpha\| = 0 \quad \text{for all } \psi \in M_{A(K)}.$$

Remark. It follows from the first half of the proof of Theorem 5.9 that if the ultraspherical hypergroup K is second countable, then TI-sequences exist. This proof can also be imitated to show that a TI-net always exists in $M_{A(K)}$ for every ultraspherical hypergroup K.

Lemma 6.2. *If $\{\psi_\alpha\}$ is a TI-net and $\psi \in M_{A(K)}$, then $\|\psi - \psi_\alpha\| \to 2$.*

Proof. Suppose that $\|\psi - \psi_\alpha\|$ does not converge to 2 for some ψ in $M_{A(K)}$. Then there exists a subnet $\{\psi_\beta\} \subset \{\psi_\alpha\}$ and an $\epsilon > 0$ such that $\|\psi - \psi_\beta\| \le 2-\epsilon$. If m is the weak* limit of $\{\psi_\beta\}$, then $\|\psi - m\| \le 2-\epsilon$, which contradicts Theorem 4.4(v). ■

Proposition 6.3. *Let K be a second countable ultraspherical hypergroup such that K is not discrete. Let $\{\psi_n\}$ be a TI-sequence in $M_{A(K)}$. There exist positive integers $n_1 < n_2 < \cdots$ and a sequence $\{\varphi_j\} \subset M_{A(K)}$ such that*

(i) $\lim_j \|\psi_{n_j} - \varphi_j\| = 0$,
(ii) *the φ_j's are mutually orthogonal, i.e., $\|\varphi_i - \varphi_j\| = \|\varphi_i\| + \|\varphi_j\|$ whenever $i \neq j$,*
(iii) *$\{\varphi_j\}$ is a TI-sequence.*

Proof. This follows from [2, Theorem 2.4] and the previous lemma. ■

Before we state the main result of this section, here is some notation. Let $\mathscr{F} = \{\mathcal{O} \in (\ell^\infty)^* : \mathcal{O}(f) = 0 \text{ if } f \in \ell^\infty \text{ and } \lim_n f(n) = 0\}$ and $\mathscr{F}_1 = \{\mathcal{O} \in \mathscr{F} : \mathcal{O} \geq 0 \text{ and } \|\mathcal{O}\| = 1\}$. The theorem below is a generalization of [2, Theorem 3.3].

THEOREM 6.4. *Let K be a nondiscrete second countable ultraspherical hypergroup. Let $\{\varphi_n\}$ be an orthogonal TI-sequence in $M_{A(K)}$. Let $\sigma : \mathrm{VN}(K) \to \ell^\infty$ be defined by*

$$\sigma(T)(n) = \langle T, \varphi_n \rangle,\ T \in \mathrm{VN}(K), n \in \mathbb{N}.$$

Then

(i) *σ is a positive linear mapping of $\mathrm{VN}(K)$ onto ℓ^∞ with $\|\sigma\| = 1$.*
(ii) *Its adjoint σ^* is a linear isometry of $(\ell^\infty)^*$ into $\mathrm{VN}(K)^*$.*
(iii) *If $\mathcal{O} \in \mathscr{F}$, then $\sigma^*(\mathcal{O})$ is topologically invariant.*
(iv) *If $\mathcal{O} \in \mathscr{F}_1$, then $\sigma^*(\mathcal{O}) \in \mathrm{TIM}(\widehat{K})$.*

Proof. (i) It is clear that σ is a positive linear mapping with $\|\sigma\| = 1$. It remains to prove that σ is an onto map. Let $\{a_n\} \in \ell^\infty$. By the assumption that the projections P_n of φ_n are mutually orthogonal, the series

$$\sum_{n=1}^{\infty} a_n P_n$$

converges in the weak* topology of $\mathrm{VN}(K)$, say to $T \in \mathrm{VN}(K)$. Since $\varphi_n \in A(K)$, it follows that

$$\sigma(T)(n) = \langle T, \varphi_n \rangle = \sum_{n=1}^{\infty} a_n \langle P_n, \varphi_n \rangle = a_n.$$

Thus σ is onto.

The proofs of (ii)–(iv) are clear. ■

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