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A property of ergodic flows

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Abstract. We introduce a property of ergodic flows, called Property B. We prove that an ergodic hyperfinite equivalence relation of type III_0 whose associated flow has this property is not of product type. A consequence is that a properly ergodic flow with Property B is not approximately transitive. We use Property B to construct a non-AT flow which—up to conjugacy—is built under a function with the dyadic odometer as base automorphism.

1. Introduction. A remarkable result of Krieger [9] establishes a complete correspondence between orbit equivalence classes of ergodic hyperfinite equivalence relations of type III_0, conjugacy classes of properly ergodic flows and isomorphism classes of approximately finite-dimensional factors of type III_0. Product type equivalence relations are hyperfinite equivalence relations, which, up to orbit equivalence, are generated by product type odometers. In order to show that there exist ergodic non-singular automorphisms not orbit equivalent to any product type odometer, Krieger [7] introduced a property of non-singular automorphisms, called Property A. He proved that any product type odometer has this property [8], and he also constructed an ergodic non-singular automorphism that does not have this property, and therefore is not of product type. It was shown in [11] that there exist non-singular automorphisms which have Property A but are not of product type.

To characterize the ITPFI factors among all approximately finite-dimensional factors, Connes and Woods [1] introduced a property of ergodic actions, called approximate transitivity, abbreviated AT. They showed that an approximately finite-dimensional factor of type III_0 is an ITPFI factor if and only if its flow of weights is AT. Equivalently, their result says that an ergodic hyperfinite equivalence relation $\mathcal{R}$ of type III_0 is of product type if and only if the associated flow of $\mathcal{R}$ is AT.

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In this paper we introduce a property of ergodic flows, called Property B. We show that no properly ergodic flow with this property is AT and we construct a flow which has this property. The non-AT flow corresponding to the non-ITPFI factor constructed in [4] does not have Property B, and so the property of a flow to be non-AT is not equivalent to Property B.

The paper is organized as follows. In Section 2, we recall some notation and definitions. In Section 3 we define Property B, we show that this property is invariant under conjugacy of flows and we characterize this property for a flow built under a function. In Section 4 we prove that a hyperfinite ergodic equivalence relation $\mathcal{R}$ of type III_0 whose associated flow has Property B is not of product type and we show that a properly ergodic flow which has Property B is not AT. In Section 5, we show that there exists a flow with Property B. This flow is built under a function with the dyadic odometer as base automorphism.

2. Preliminaries. Throughout this paper, $(X, \mathfrak{B}, \mu)$ will be a standard σ-finite measure space. A *measurable flow* on $(X, \mathfrak{B}, \mu)$ is a one-parameter group $\{F_t\}_{t\in\mathbb{R}}$ of non-singular automorphisms of $(X, \mathfrak{B}, \mu)$ such that the mapping $X \times \mathbb{R} \ni (x, t) \mapsto F_t(x) \in X$ is measurable. Two flows $\{F_t\}_{t\in\mathbb{R}}$ and $\{F'_t\}_{t\in\mathbb{R}}$ on $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}', \mu')$ respectively are *conjugate* if there exists an isomorphism $T : (X, \mathfrak{B}, \mu) \to (X', \mathfrak{B}', \mu')$ such that $F'_t(T(x)) = T(F_t(x))$ for all $t \in \mathbb{R}$ and μ-almost all $x \in X$. We say that $\{F_t\}_{t\in\mathbb{R}}$ is *ergodic* if every F_t-invariant measurable set is either null or conull.

Let $\mathcal{R}$ be an equivalence relation on $(X, \mathfrak{B}, \mu)$. We say that $\mathcal{R}$ is a *countable measured equivalence relation* if the equivalence classes $\mathcal{R}(x)$, $x \in X$, are countable, $\mathcal{R}$ is a measurable subset of $X \times X$, and the saturation of any set of measure zero has measure zero. $\mathcal{R}$ is called *ergodic* if every invariant set is either null or conull. Recall that if ν_l and ν_r are the left and the right counting measures on $\mathcal{R}$ then $\nu_l \sim \nu_r$ and $\delta(x, y) = \frac{d\nu_l}{d\nu_r}(x, y)$ is the Radon–Nikodym cocycle of μ with respect to $\mathcal{R}$. We say that the measure μ is *lacunary* if there exist $\varepsilon > 0$ such that $\delta(x, y) = 0$ or $|\delta(x, y)| > \varepsilon$, for all $(x, y) \in \mathcal{R}$. The *full group* $[\mathcal{R}]$ of $\mathcal{R}$ is the group of all non-singular automorphisms V of $(X, \mathfrak{B}, \mu)$ with $(x, Vx) \in \mathcal{R}$ for μ-a.e. $x \in X$.

A countable measured equivalence relation $\mathcal{R}$ is called *finite* if $\mathcal{R}(x)$ is finite for almost all $x \in X$. We say that $\mathcal{R}$ is *hyperfinite* if there are finite relations $\mathcal{R}_n$ with $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ and $\bigcup \mathcal{R}_n = \mathcal{R}$, up to a set of measure zero. We recall that $\mathcal{R}$ is hyperfinite if and only if there exists a non-singular automorphism T on $(X, \mathcal{B}, \mu)$ such that, up to a set of measure zero, $\mathcal{R}$ is equal to the equivalence relation $\mathcal{R}_T = \{(x, T^n x) : x \in X,\ n \in \mathbb{Z}\}$ generated by T, that is, $\mathcal{R}(x) = \{T^n x : n \in \mathbb{Z}\}$ for μ-a.e. $x \in X$.

Two countable measured equivalence relations $\mathcal{R}$ and $\mathcal{R}'$ on $(X, \mathfrak{B}, \mu)$ and $(X', \mathfrak{B}',\ \mu')$ respectively are called *orbit equivalent* if there exists an isomorphism $S : (X, \mathfrak{B}, \mu) \to (X', \mathfrak{B}', \mu')$ such that $S(\mathcal{R}(x)) = \mathcal{R}'(Sx)$ for μ-a.e. $x \in X$.

Let $(k_n)_{n\geq 1}$ be a sequence of positive integers, with $k_n \geq 2$. Consider the infinite product probability space $(X, \mu) = \prod_{n=1}^{\infty}(X_n, \mu_n)$, where $X_n = \{0, 1, \ldots, k_n - 1\}$ and each μ_n is a probability measure on X_n such that $\mu_n(x) > 0$ for all $x \in X_n$. We recall that the *tail equivalence relation* $\mathcal{T}$ on (X, μ) is defined for $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1}$ by

$$(x, y) \in \mathcal{T} \text{ if and only if there exists } n \geq 1 \text{ such that } x_i = y_i \text{ for all } i > n.$$

It can be easily observed that, up to a set of measure zero, $\mathcal{T}$ is generated by the odometer defined on (X, μ). A countable measured equivalence relation is said to be *of product type* if it is orbit equivalent to the tail equivalence relation on an infinite product probability space as above, or equivalently, if it is orbit equivalent to the equivalence relation generated by a product type odometer.

An ergodic equivalence relation $\mathcal{R}$ is *of type III* if there is no σ-finite $\mathcal{R}$-invariant measure ν equivalent to μ. The type III equivalence relations are further classified into subtypes III_λ, where $0 \leq \lambda \leq 1$. Up to orbit equivalence, for $\lambda \neq 0$, there is only one hyperfinite equivalence of type III_λ, and it is of product type.

The orbit equivalence classes of ergodic hyperfinite equivalence relations of type III_0 are completely classified by the conjugacy class of their associated flow. For more details we refer the reader to [3], [5] and [6].

In order to show that there exist ergodic non-singular automorphisms not orbit equivalent to any product odometer, Krieger introduced a property of non-singular automorphisms, called Property A. This property can be defined for equivalence relations (see [10]) as follows. Suppose that $\mathcal{R}$ is a hyperfinite equivalence relation on $(X, \mathfrak{B}, \mu)$. Let ν be a σ-finite measure on X, equivalent to μ, and δ_ν the corresponding Radon–Nikodym cocycle. For $x \in A$, define

$$\begin{aligned}\Lambda_{\nu,A,\mathcal{R}}(x) &= \{\log \delta_\nu(y, x) : (x, y) \in \mathcal{R} \text{ and } y \in A\} \\ &= \left\{\log \frac{d\nu \circ \phi}{d\nu}(x) : \phi \in [\mathcal{R}],\ (x, \phi(x)) \in \mathcal{R} \text{ and } \phi(x) \in A\right\}.\end{aligned}$$

For a σ-finite measure $\nu \sim \mu$, $A \in \mathcal{B}$ of positive measure and $s, \zeta > 0$, set

$$\begin{aligned}K_{\nu,\mathcal{R}}(A, s, \zeta) = \{x \in A &: (e^{s-\zeta}, e^{s+\zeta}) \cap \Lambda_{\nu,A,\mathcal{R}}(x) \neq \emptyset\} \\ &\cup \{x \in A : (-e^{s+\zeta}, -e^{s-\zeta}) \cap \Lambda_{\nu,A,\mathcal{R}}(x) \neq \emptyset\}.\end{aligned}$$

DEFINITION 2.1. Let $\mathcal{R}$ be a hyperfinite equivalence relation on $(X, \mathfrak{B}, \mu)$. Then $\mathcal{R}$ has *Property A* if there exists a σ-finite measure $\nu \sim \mu$ and $\eta, \zeta > 0$ such that every set $A \in \mathfrak{B}$ of positive measure contains a set $B \in \mathfrak{B}$ of positive measure such that

$$\limsup_{s\to\infty} \nu(K_{\nu,\mathcal{R}}(B,s,\zeta)) > \eta\cdot\nu(B).$$

If $\mathcal{R}$ is a hyperfinite equivalence relation and T is a non-singular automorphism such that $\mathcal{R} = \mathcal{R}_T$ up to a null set, it can be easily observed that $\mathcal{R}$ has Property A if and only if T has Property A (see [11]). We mention the following result (see [8] and [11]) that will be used later.

PROPOSITION 2.2. *Assume that $\mathcal{R}$ has Property A. Then there exist $\eta, \delta > 0$ such that for all $\lambda \sim \mu$ and all $\epsilon > 0$, every measurable set A of positive measure contains a measurable set B of positive measure with*

$$\limsup_{s\to\infty} \lambda(K_{\lambda,\mathcal{R}}(B,s,\delta+\epsilon)) > e^{-\epsilon}\eta\cdot\lambda(B).$$

We recall that Krieger's result from [8] can be reformulated as follows [11]:

THEOREM 2.3. *Any ergodic equivalence relation of product type and of type III has Property A.*

3. Property B. In this section we define a property of measurable flows that we call Property B, we show that it is an invariant for conjugacy of flows, and we characterize this property for a flow built under a function.

Let $\{F_t\}_{t\in\mathbb{R}}$ be a flow of automorphisms of $(X, \mathfrak{B}, \mu)$. For $A \in \mathfrak{B}$ of positive measure and $s, \delta > 0$ we define

$$\Lambda_{F,\delta,s}(A) = \{x \in A : \exists t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}),\ F_t(x) \in A\}.$$

DEFINITION 3.1. We say that $\{F_t\}_{t\in\mathbb{R}}$ has *Property B* if there exists a measurable set $A \subseteq X$ of positive measure such that for all $\delta > 0$,

$$\limsup_{s\to\infty} \mu(\Lambda_{F,\delta,s}(A)) = 0. \tag{3.1}$$

Since the proof of the lemma below uses standard arguments, we omit it.

LEMMA 3.2. *Let $\{F_t\}_{t\in\mathbb{R}}$ be a flow on $(X, \mathfrak{B}, \mu)$ which has Property B.*

(i) *If μ' is a σ-finite measure on $(X, \mathfrak{B})$ equivalent to μ, then the flow $\{F_t\}_{t\in\mathbb{R}}$ on $(X, \mathfrak{B}, \mu')$ has Property B.*
(ii) *If $T : (X', \mathfrak{B}', \mu') \to (X, \mathfrak{B}, \mu)$ is an isomorphism and $\{F'_t\}_{t\in\mathbb{R}}$ is a flow on $(X', \mathfrak{B}', \mu')$ such that $T(F'_t(x)) = F_t(Tx)$ for all $t \in \mathbb{R}$ and μ'-almost all $x \in X'$, then F'_t has Property B.*

The following proposition is a consequence of the above lemma.

PROPOSITION 3.3. *Property B is an invariant for conjugacy of flows.*

Let T be an automorphism of $(X_0, \mathfrak{B}_0, \mu_0)$ and $\xi : X_0 \to \mathbb{R}$ be a positive measurable function. Consider $Y = \{(x,t) \in X_0 \times \mathbb{R} : 0 \le t < \xi(x)\}$ and let

ν be the measure on Y that is the restriction of the product measure $\mu_0 \times \lambda$, where λ is the usual Lebesgue measure on $\mathbb{R}$. Let $\{F_t\}_{t\in\mathbb{R}}$ be the flow built under the function ξ with base automorphism T; it is defined on (Y, ν), and for $t > 0$ it is given by

$$F_t(x,s) = \begin{cases} (x, t+s) & \text{if } 0 \le t+s < \xi(x), \\ (T(x), t+s-\xi(x)) & \text{if } \xi(x) \le t+s < \xi(T(x)) + \xi(x), \\ \ldots . \end{cases}$$

For a measurable set $A \subseteq X_0$ we define

$\Delta_{F,\delta,s}(A) = \{x \in A : \exists t \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}), F_t(x,0) \in A \times \{0\}\}$.

With this notation we have the following result:

PROPOSITION 3.4. *The flow $\{F_t\}_{t\in\mathbb{R}}$ has Property B if and only if there exists a measurable set $A_0 \subseteq X_0$ of positive measure such that, for all $\delta > 0$,*

$$\limsup_{s\to\infty} \mu_0(\Delta_{F,\delta,s}(A_0)) = 0. \tag{3.2}$$

Proof. For the "if" part notice that if A_0 satisfies the conditions of the proposition then $A = A_0 \times [0,1] \cap Y$ satisfies (3.1) for every $\delta > 0$. The "only if" is a consequence of the following: for every measurable set $A \subseteq Y$ of positive measure we can find a measurable set $A_0 \subseteq X_0$ of positive measure such that $\mu(A_x \cap I) > \alpha$ for some positive α and some bounded interval I (here $A_x = \{y \in \mathbb{R} : (x,y) \in A\}$). ∎

4. Property B implies non-AT. In this section we show that if $\mathcal{R}$ is an ergodic hyperfinite equivalence relation of type III_0 whose associated flow has Property B, then $\mathcal{R}$ does not have Krieger's Property A and therefore is not of product type. A consequence is that a properly ergodic flow with Property B is not approximately transitive. Note that if $\mathcal{R}$ is of type III_λ, $\lambda \neq 0$, then the associated flow of $\mathcal{R}$ does not have Property B.

Consider an ergodic hyperfinite equivalence relation $\mathcal{R}$ of type III_0 on $(X, \mathfrak{B}, \mu)$ and let δ be the Radon–Nikodym cocycle of μ with respect to $\mathcal{R}$. Replacing μ with an equivalent measure if necessary, we can assume that μ is a finite lacunary measure (see [6, Proposition 2.3 and the explanations leading to it]). Define

$$\xi(x) = \min\{\log \delta(x', x) : (x', x) \in \mathcal{R},\ \log \delta(x', x) > 0\}$$

and consider the equivalence relation $\mathcal{S}$ on X given by

$$(x,y) \in \mathcal{S} \quad \text{if and only if} \quad (x,y) \in \mathcal{R} \text{ and } \delta(x,y) = 1.$$

Let $\mathfrak{B}(\mathcal{S})$ be the σ-algebra of sets in $\mathfrak{B}$ that are $\mathcal{S}$-invariant. Let X_0 be the quotient space $X/\mathfrak{B}(\mathcal{S})$, that is, the space of ergodic components of $\mathcal{S}$. We denote the quotient map from X onto X_0 by π, where $\pi(x)$ is the element of X_0 containing x. On X_0, consider the measure $\mu_0 = \mu\circ\pi^{-1}$. Note that $\xi(x)$ is

$\mathfrak{B}(\mathcal{S})$-measurable, and therefore ξ can be regarded as a function on X_0. We have an ergodic automorphism T on X_0 defined by $T(\pi(x)) = \pi(x')$ where $(x, x') \in \mathcal{R}$ and $\log \delta(x', x) = \xi(\pi(x))$. Then the associated flow $\{F_t\}_{t\in\mathbb{R}}$ of $\mathcal{R}$ can be realized as the flow built under the ceiling function ξ with base automorphism T (see for example [5] or [6]). As a direct consequence we obtain the following lemma.

LEMMA 4.1. *Let $(x, x') \in \mathcal{R}$, $z = \pi(x)$ and $z' = \pi(x')$. Then*

$$F_{\log \delta(x',x)}(z, 0) = (z', 0).$$

THEOREM 4.2. *With the above notation, if the associated flow $\{F_t\}_{t\in\mathbb{R}}$ of $\mathcal{R}$ has Property B, then $\mathcal{R}$ does not have Property A.*

Proof. By Proposition 3.4, we can find a measurable set $A_0 \subseteq X_0$ of positive measure such that, for all $\delta > 0$,

$$\limsup_{s\to\infty} \mu_0(\Delta_{F,\delta,s}(A_0)) = 0.$$

Let $C = \pi^{-1}(A_0) \subseteq X$ and $\delta > 0$. Consider $s > 0$ and $x \in K_{\mu,\mathcal{R}}(C, s, \delta)$. Thus, $x \in C$ and there exists $y \in C$ such that $(x, y) \in \mathcal{R}$ and $\log \delta(y, x) \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta})$. From Lemma 4.1, we have

$$F_{\log \delta(y,x)}(\pi(x), 0) = (\pi(y), 0).$$

Hence, $\pi(x) \in \Delta_{F,\delta,s}(A_0)$ and so $x \in \pi^{-1}(\Delta_{F,\delta,s}(A_0))$. Therefore,

$$K_{\mu,\mathcal{R}}(C, s, \delta) \subseteq \pi^{-1}(\Delta_{F,\delta,s}(A_0)),$$

and consequently

$$\mu(K_{\mu,\mathcal{R}}(C, s, \delta)) \le \mu \circ \pi^{-1}(\Delta_{F,\delta,s}(A_0)) = \mu_0(\Delta_{F,s,\delta}(A_0)).$$

This clearly implies that

$$\limsup_{s\to\infty} \mu(K_{\mu,\mathcal{R}}(C, s, \delta)) = 0$$

and so, by Proposition 2.2, $\mathcal{R}$ does not have Property A. ∎

The following is an immediate consequence of the above theorem.

COROLLARY 4.3.

(i) *An equivalence relation $\mathcal{R}$ whose associated flow has Property B is not of product type.*
(ii) *A properly ergodic flow with Property B is not AT.*

REMARK 4.4. The converse of the second statement of Corollary 4.3 is not true: there exist ergodic flows which are not AT and do not have Property B. To give an example, consider the type III_0 factor N whose flow of weights is AT(2) but not AT, constructed by Giordano and Handelman in [4, Section VI]. Since, up to isomorphism, N is the von Neumann algebra associated to an ergodic hyperfinite equivalence relation $\mathcal{R}$, the flow of weights of N is, up to conjugacy, the associated flow of $\mathcal{R}$. According to [11], the equivalence relation $\mathcal{R}$ has Property A, and so Theorem 4.2 implies that the associated flow of $\mathcal{R}$ does not have Property B.

The following result gives a sufficient condition for a non-singular automorphism to be non-AT.

COROLLARY 4.5. *Let T be a non-singular automorphism of $(X, \mathfrak{B}, \mu)$. Assume that there exists $A \subset X$ of positive measure such that*

$$\limsup_{s\to\infty} \mu(\{x \in A : \exists n \in (e^{s-\delta}, e^{s+\delta}) \cup (-e^{s+\delta}, -e^{s-\delta}),\ T^n x \in A\}) = 0.$$

Then T is not AT.

Proof. Proposition 3.4 implies that $\{F_t\}_{t\in\mathbb{R}}$, the flow built under the constant function $f = 1$ with base automorphism T, has Property B, and Corollary 4.3 implies that $\{F_t\}_{t\in\mathbb{R}}$ is not AT. From [1, Lemma 2.5], we conclude that T is not AT. ∎

5. An ergodic flow with Property B. In this section we construct a properly ergodic flow which has Property B and therefore is not AT. The flow that we construct is built under a function with a product odometer (conjugate to the dyadic odometer) as base automorphism.

Let $(z_n)_{n\geq 1}$ be the sequence of integers given by $z_n = 2^n - 1$ for $n \geq 1$. Consider the product space $X = \prod_{n\geq 1}\{0, 1, \ldots, z_n\}$ endowed with the usual product σ-algebra and the product measure $\mu = \bigotimes_{n\geq 1} \mu_n$, where each μ_n is the probability measure on $\{0, 1, \ldots, z_n\}$ given by $\mu_n(i) = 1/2^n$ for $i = 0, 1, \ldots, z_n$ and $n \geq 1$. Let $T : X \to X$ be the product odometer defined on X. We recall that T is the non-singular automorphism defined for almost every $x \in X$ by

$$(Tx)_n = \begin{cases} 0 & \text{if } n < N(x), \\ x_n + 1 & \text{if } n = N(x), \\ x_n & \text{if } n > N(x), \end{cases} \tag{5.1}$$

where $N(x) = \min\{n \geq 1 : x_n < z_n\}$. Notice that T is measure conjugate to the dyadic odometer.

Let $(K_n)_{n\geq 4}$ be the sequence given by

$$K_n = 1!2!\cdots n! \quad \text{for } n \geq 4,$$

and let $f : X \to \mathbb{R}$ be the function defined for almost every $x \in X$ by

$$f(x) = K_{2^{N+1}+x_{N+1}} \tag{5.2}$$

where $x = (x_n)_{n\geq 1}$ and $N = N(x)$.

For $n \geq 1$, let $Z_n = \{x \in X : x_i = z_i \text{ for } i = 1, \ldots, n\}$. Let $C_i^{n+1} = \{x \in Z_{n-1} : x_n \neq z_n,\, x_{n+1} = i\}$ for $i = 0, 1, \ldots, 2^{n+1} - 1$. Then

$$\int_X f\, d\mu \geq \sum_{i=0}^{z_{n+1}} \int_{C_i^{n+1}} f\, d\mu > \sum_{i=0}^{2^{n+1}-1} \frac{(2^n - 1)K_{2^{n+1}+i}}{2 \cdot 2^2 \cdots 2^n \cdot 2^{n+1}}$$
$$> \frac{1}{2 \cdot 2^2 \cdots 2^n \cdot 2^{n+1}}(2^n - 1)2^{n+1}K_{2^{n+1}} > 2^{n+1}.$$

Therefore

$$\int_X f\, d\mu = \infty.$$

Proposition 5.1. *Let $n \geq 4$ be a positive integer, $m = [\log_2 n]$ and $l = n - 2^m$. For almost every $x \in X$, we have:*

(i) *If there exists an integer $k \geq 1$ such that*

$$K_n \leq \sum_{i=0}^{k-1} f(T^i x) < K_{n+1}, \tag{5.3}$$

then $x_m = l$.

(ii) *If there exists an integer $k \geq 1$ such that*

$$K_n \leq \sum_{i=1}^{k} f(T^{-i} x) < K_{n+1}, \tag{5.4}$$

then $x_m = l$.

Proof. (i) Let $x \in X$ be such that $K_n \leq \sum_{i=0}^{k-1} f(T^i x) < K_{n+1}$ for some integer $k \geq 1$. Let

$$p = \max\{N(T^i x) : 0 \leq i \leq k - 1\}.$$

Hence, there exists $0 \leq j < k$ such that $N(T^j x) = p$. By (5.2) we have $f(T^j x) = K_{2^{p+1}+x_{p+1}}$. Notice that $(T^i x)_q = x_q$ for $q > p$ and $1 < i \leq k$. Indeed, if there exist $i \in \{1, \ldots, k\}$ and $q > p$ such that $T(T^{i-1}x)_q = (T^i x)_q \neq x_q$, then (5.1) implies that $N(T^{i-1}x) > p$, which is a contradiction. We also have $2 \cdot 2^2 \cdots 2^p > k$. If we assume that $k \geq 2 \cdot 2^2 \cdots 2^p$ then by (5.1) there exists $j \leq k-1$ such that $(T^j x)_i = z_i$ for all $i = 1, \ldots, p$, which implies that $N(T^j x) > p$. Hence,

$$K_{2^{p+1}+x_{p+1}} \leq \sum_{i=0}^{k-1} f(T^i x) < 2 \cdot 2^2 \cdots 2^p \cdot K_{2^{p+1}+x_{p+1}} < K_{2^{p+1}+x_{p+1}+1}.$$

We claim that $n = 2^{p+1} + x_{p+1}$. Indeed, if $n < 2^{p+1} + x_{n+1}$ we have $K_{n+1} \leq K_{2^{p+1}+x_{n+1}} \leq \sum_{i=0}^{k-1} f(T^i x)$, which contradicts (5.3). If $n > 2^{p+1} + x_{p+1}$, then $K_n \geq K_{2^{p+1}+x_{p+1}+1} > \sum_{i=0}^{k-1} f(T^i x)$, which again contradicts (5.3). Therefore $n = 2^{p+1} + x_{p+1}$, and so $m = p + 1$ and $x_m = l$.

(ii) Let $x \in X$ be such that $K_n \le \sum_{i=1}^{k} f(T^{-i}x) < K_{n+1}$ for some positive integer k. Let

$$p = \max\{N(T^{-i}x) : 1 \le i \le k\}$$

and note that $(T^{-i}x)_q = x_q$ for $q > p$ and $1 \le i \le k$. The conclusion follows as in case (i); we leave the details to the reader. ∎

Let $\{F_t\}_{t\in\mathbb{R}}$ be the flow built under the function f with base automorphism T. Since $\int_X f\,d\mu = \infty$ and T is measure perserving it follows that the flow $\{F_t\}_{t\in\mathbb{R}}$ is infinite measure preserving. Notice also that it is properly ergodic. The following lemma is immediate from the definition of $\{F_t\}_{t\in\mathbb{R}}$.

Lemma 5.2.

(i) *If $t > 0$ then $F_t(x,0) \in X \times \{0\}$ if and only if there exists an integer $k \ge 1$ such that $t = \sum_{i=0}^{k-1} f(T^{i}x)$.*
(ii) *If $t < 0$ then $F_t(x,0) \in X \times \{0\}$ if and only if there exists an integer $k \ge 1$ such that $t = -\sum_{i=1}^{k} f(T^{-i}x)$.*

Proposition 5.3. *For any $\delta > 0$,*

$$\lim_{s\to\infty} \mu(\Delta_{F,\delta,s}(X)) = 0. \tag{5.5}$$

Proof. By Lemma 5.2 we have

$$\begin{aligned}\mu(\Delta_{F,\delta,s}) = \mu\Big(\Big\{x \in X : \exists k \in \mathbb{N},\ e^{s-\delta} < \sum_{i=0}^{k-1} f(T^{i}x) < e^{s+\delta}\Big\}\\ \cup \Big\{x \in X : \exists k \in \mathbb{N},\ e^{s-\delta} < \sum_{i=1}^{k} f(T^{-i}x) < e^{s+\delta}\Big\}\Big).\end{aligned} \tag{5.6}$$

Proposition 5.1 implies that

$$\mu\Big(\Big\{x \in X : \exists k \in \mathbb{N},\ K_n \le \sum_{i=0}^{k-1} f(T^{i}x) < K_{n+1}\Big\}\Big) \le \frac{1}{2^{[\log_2 n]}}, \tag{5.7}$$

$$\mu\Big(\Big\{x \in X : \exists k \in \mathbb{N},\ K_n \le \sum_{i=1}^{k} f(T^{-i}x) < K_{n+1}\Big\}\Big) \le \frac{1}{2^{[\log_2 n]}}. \tag{5.8}$$

Notice that for s sufficiently large, $(e^{s-\delta}, e^{s+\delta})$ intersects at most two consecutive intervals $[K_n, K_{n+1})$. This together with (5.6)–(5.8) implies (5.5). ∎

From Propositions 3.4 and 5.3 we can deduce:

Corollary 5.4. *The flow $\{F_t\}_{t\in\mathbb{R}}$ constructed above has Property B.*

Comment. As already mentioned, the flow with Property B constructed in this paper is infinite measure preserving. It would be of interest to construct a finite measure preserving flow that would have Property B.

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