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Order theory and interpolation in operator algebras

by

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Abstract. In earlier papers we have introduced and studied a new notion of positivity in operator algebras, with an eye to extending certain C^*-algebraic results and theories to more general algebras. Here we continue to develop this positivity and its associated ordering, proving many foundational facts. We also give many applications, for example to noncommutative topology, noncommutative peak sets, lifting problems, peak interpolation, approximate identities, and to order relations between an operator algebra and the C^*-algebra it generates. In much of this it is not necessary that the algebra have an approximate identity. Many of our results apply immediately to function algebras, but we will not take the time to point these out, although most of these applications seem new.

1. Introduction. An *operator algebra* is a closed subalgebra of $B(H)$, for a Hilbert space H. In a series of papers (see e.g. [14, 15, 17, 18]) we have studied such algebras. In many of these works our operator algebras have contractive approximate identities (cai's), and we call such algebras *approximately unital*. In particular in [14, 15, 37] we introduced and studied a new notion of positivity in operator algebras. We have shown elsewhere that the 'completely positive' maps on C^*-algebras or operator systems in our new sense are precisely the completely positive maps in the usual sense; however, the new notion of positivity allows the development of useful order theory for more general spaces and algebras. Our main goals are to extend certain useful C^*-algebraic results and theories to more general algebras; and also to develop 'noncommutative function theory' in the sense of generalizing certain parts of the classical theory of function spaces and algebras [25]. Simultaneously we are developing applications (see also e.g. [11, 12] with Matthew Neal, and [8]).

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With the same goals in mind, in the present paper, we continue the development of foundational aspects of this positivity and of the associated ordering for operator algebras. We also give many applications, for example to noncommutative topology, noncommutative peak sets, lifting problems, peak interpolation, approximate identities, and to order relations between an operator algebra and the C^*-algebra it generates.

Before proceeding further, we make an editorial/historical note: approximately half of the present paper was formerly part of a preprint [16]. The latter has been split into several papers, each of which has taken on a life of its own, e.g. the present paper which focuses on order in operator algebras, and [13] which covers the more general setting of Banach algebras. The reader is encouraged to browse the latter paper for complementary theory; we will not prove results here that may be found in [13] except if there is a much simpler proof in the operator algebra setting.

As in the aforementioned papers, a central role is played by the set $\mathfrak{F}_A = \{a \in A : \|1 - a\| \leq 1\}$ (here 1 is the identity of the unitization A^1 if A is nonunital). We will be interested in four 'cones' or notions of 'positivity' in A, and the relations between them. The biggest of these is the set of *accretive* operators

$$\mathfrak{r}_A = \{a \in A : \mathrm{Re}(a) = a + a^* \geq 0\},$$

namely the elements of A whose numerical range in A^1 is contained in the closed right half-plane. This has as a dense subcone

$$\mathfrak{c}_A = \mathbb{R}_+ \mathfrak{F}_A$$

(see e.g. [15, Theorem 3.3]). In turn the latter cone contains as a dense subcone (see Lemma 2.15) the cone of sectorial operators of angle $\rho < \pi/2$, which we use less frequently. By sectorial angle ρ we mean that the numerical range is contained in the sector S_ρ consisting of numbers $re^{i\theta}$ with argument θ such that $|\theta| < \rho$ (cf. e.g. [26, 39]). The fourth notion, 'near positivity', is more subtle. If in the statement of a result an element of A is described as 'nearly positive', this means that if $\epsilon > 0$ is given one can choose x in the statement to be in the previous three cones, but also sectorial with angle ρ so small that x is within distance ϵ of an actual positive operator. Note that if an operator x is sectorial with acute angle ρ so small that $\|x\| \sin \rho < \epsilon$ for example, then $\mathrm{Re}(x) \geq 0$ and

$$\|x - \mathrm{Re}(x)\| = \|\mathrm{Im}(x)\| = \sup\{|\mathrm{Im}\langle x\zeta, \zeta\rangle| : \zeta \in \mathrm{Ball}(H)\} < \epsilon,$$

so that x is within distance ϵ of the positive operator $\mathrm{Re}(x)$. Such nearly positive operators usually arise because $\mathfrak{r}_A$ is closed under taking (principal) roots, and the nth root of an accretive operator is sectorial with angle as small as desired for n large enough. We will also usually require our nearly positive operators to be in $\frac{1}{2}\mathfrak{F}_A$ too.

Elements of these 'cones', and their roots, play the role in many situations of positive elements in a C^*-algebra. There are some remarkable relationships between operator algebras and the classical theory of ordered linear spaces (due to Krein, Ando, Alfsen, and many others). We mention some examples of this (and see [13], particularly Section 6 there, for more): In the language of ordered Banach spaces, an operator algebra is approximately unital iff $\mathfrak{r}_A$ and $\mathfrak{c}_A$ are *generating* cones (this is sometimes called *positively generating* or *directed* or *co-normal*). That is, iff $A = \mathfrak{r}_A - \mathfrak{r}_A$, for example. Read's theorem states that any approximately unital operator algebra has a cai in $\frac{1}{2}\mathfrak{F}_A$ (see [37], although there are now several much shorter proofs [8, 13]), and indeed, by taking roots, nearly positive. We will show that A is *cofinal* in any C^*-algebra B which it generates. Indeed, given any $b \in B_+$ and $\epsilon > 0$ there exists nearly positive $a \in A$ with $b \preccurlyeq a \preccurlyeq (\|b\| + \epsilon)1$ in the ordering induced by the cone above. We will also investigate the relationship between such results and 'noncommutative peak interpolation'.

Turning to the layout of our paper, in Section 2 we study general properties of these cones and the related ordering. This is a collection of results on positivity, some of which are used elsewhere in this paper, or in other papers, and some of which are of independent interest. In particular we prove several surprising order-theoretic properties, some of which are new relations between an operator algebra and the C^*-algebra it generates. Many of these order-theoretic properties turn out to be equivalent to the existence of a cai. The short Section 3 studies 'strictly positive' elements, a topic that is quite important for C^*-algebras.

The lengthy Section 4 concerns applications to noncommutative topology, noncommutative peak sets, lifting problems, and peak interpolation. First we present versions of some of our previous Urysohn lemmas and peak interpolation results for operator algebras (see e.g. [14, 12]), but now insisting that the 'interpolating element' is 'nearly positive' in the sense defined above (and also in $\frac{1}{2}\mathfrak{F}_A$). This also solves the problems raised at the end of [12]. We also prove a Tietze extension theorem for operator algebras, and a strict form of the Urysohn lemma for operator algebras, generalizing the usual strict form of the Urysohn lemma from topology, and also generalizing the Brown–Pedersen strict noncommutative Urysohn lemma (see [36]). See [20] for a recent paper containing 'Urysohn lemmas' for function algebras; our Urysohn lemma applied to the algebras considered there is more general (see the discussion after Theorem 4.2). Indeed, many results in our paper apply immediately to function algebras (uniform algebras), that is, to uniformly closed subalgebras of $C(K)$, since these are special cases of operator algebras. We will not take the time to point these out, although most of these applications seem new.

We now turn to notation and some background facts (for more details the reader should consult our previous papers in this series, e.g. [14, 15, 17, 18, 13, 10]). In this paper H will always be a Hilbert space, usually the Hilbert space on which our operator algebra is acting, or is completely isometrically represented.

We recall that by a theorem due to Ralf Meyer, every operator algebra A has a unitization A^1 which is unique up to completely isometric homomorphism (see [10, Section 2.1]). Below 1 always refers to the identity of A^1 if A has no identity. We almost always set $A^1 = A$ if A already has an identity.

We write $\mathrm{oa}(x)$ for the operator algebra generated by x in A, the smallest closed subalgebra containing x. We will often use C^*-algebras generated by an operator algebra A (or containing A completely isometrically as a subalgebra). For example, the disk algebra $A(\mathbb{D})$ generates $C(\mathbb{T}), C(\bar{\mathbb{D}})$, and the Toeplitz C^*-algebra (here $\mathbb{T}$ and $\mathbb{D}$ represent the circle and open unit disk respectively). However, we want anything we say about an operator algebra A to be independent of which particular generated C^*-algebra was used.

A *state* of an approximately unital operator algebra A is a functional with $\|\varphi\| = \lim_t \varphi(e_t) = 1$ for some (or any) cai (e_t) for A. These extend to states of A^1. They also extend to a state on any C^*-algebra B generated by A, and conversely any state on B restricts to a state of A. See [10, Section 2.1] for details. If A is not approximately unital then we define a state on A to be a norm 1 functional that extends to a state on A^1. We write $S(A)$ for the collection of such states; this is the *state space* of A. These extend further by the Hahn–Banach theorem to a state on any C^*-algebra generated by A^1, and therefore restrict to a positive functional on any C^*-algebra B generated by A. The latter restriction is actually a state, since it has norm 1 (even on A). Conversely, every state on B extends to a state on B^1, and this restricts to a state on A^1. From these considerations it is easy to see that states on an operator algebra A may equivalently be defined to be norm 1 functionals that extend to a state on any C^*-algebra B generated by A.

For us a *projection* is always an orthogonal projection, and an *idempotent* merely satisfies $x^2 = x$. If X, Y are sets, then XY denotes the closure of the span of products of the form xy for $x \in X$ and $y \in Y$. We write X_+ for the positive operators (in the usual sense) that happen to belong to X. We write $M_n(X)$ for the space of $n \times n$ matrices over X, and of course $M_n = M_n(\mathbb{C})$. The second dual A^{**} is also an operator algebra with its (unique) Arens product; this is also the product inherited from the von Neumann algebra B^{**} if A is a subalgebra of a C^*-algebra B. Note that A has a cai iff A^{**} has an identity $1_{A^{**}}$ of norm 1, and then A^1 is sometimes identified with $A + \mathbb{C}1_{A^{**}}$. We write $1_{A^{**}}$ very often as e.

For an operator algebra, not necessarily approximately unital, we recall that $\frac{1}{2}\mathfrak{F}_A = \{a \in A : \|1 - 2a\| \leq 1\}$. Here 1 is the identity of the unitization A^1 if A is nonunital. As we said, A^1 is uniquely defined, and can be viewed as $A + \mathbb{C}I_H$ if A is completely isometrically represented as a subalgebra of $B(H)$. Hence so is $A^1 + (A^1)^*$ uniquely defined, by e.g. 1.3.7 in [10]. We define $A + A^*$ to be the obvious subspace of $A^1 + (A^1)^*$. This is well defined independently of the particular Hilbert space H on which A is represented, as shown at the start of Section 3 in [15]. Thus a statement such as $a + b^* \geq 0$ makes sense whenever $a, b \in A$, and is independent of the particular H on which A is represented. This gives another way of seeing that the set $\mathfrak{r}_A = \{a \in A : a + a^* \geq 0\}$ is independent of the particular representation too.

Note that $x \in \mathfrak{c}_A = \mathbb{R}_+\mathfrak{F}_A$ iff there is a positive constant C with $x^*x \leq C(x + x^*)$.

We recall that an *r-ideal* is a right ideal with a left cai, and an *ℓ-ideal* is a left ideal with a right cai. We say that an operator algebra D with cai, which is a subalgebra of another operator algebra A, is a HSA (hereditary subalgebra) in A, if $DAD \subset D$. See [9] for the basic theory of HSA's. HSA's in A are in an order preserving, bijective correspondence with the r-ideals in A, and with the ℓ-ideals in A. Because of this symmetry we will usually restrict our results to the r-ideal case; the ℓ-ideal case will be analogous. There is also a bijective correspondence with the *open projections* $p \in A^{**}$ (also called open projections for A), by which we mean that there is a net $x_t \in A$ with $x_t = px_t \to p$ weak* in A^{**}, or equivalently with $x_t = px_tp \to p$ weak* (see [9, Theorem 2.4]). These are also the open projections p in the sense of Akemann [1] in B^{**}, where B is a C^*-algebra containing A, such that $p \in A^{\perp\perp}$. If A is approximately unital then the complement $p^\perp = 1_{A^{**}} - p$ of an open projection for A is called a *closed projection* for A. A closed projection q for which there exists an $a \in \mathrm{Ball}(A)$ with $aq = qa = q$ is called *compact* for A. This is equivalent to q being a closed projection with respect to A^1, if A is approximately unital. See [12, 15] for the theory of compact projections in operator algebras.

If $x \in \mathfrak{r}_A$ then it is shown in [15, Section 3] that the operator algebra $\mathrm{oa}(x)$ generated by x in A has a cai, which can be taken to be a normalization of $(x^{1/n})$, and the weak* limit of $(x^{1/n})$ is the support projection $s(x)$ for x. This is an open projection, and in a separable operator algebra these are the only open projections. For a unital operator algebra the complement of an open projection (different from 1) for A is a peak projection, thus for A separable unital operator algebra the peak projections are exactly the closed projections for A. There are many equivalent definitions of peak projections (see e.g. [28, 9, 12, 15]). For any operator algebra A we recall that if x is in $\frac{1}{2}\mathfrak{F}_A$ then the *peak projection* associated with x is $u(x) = w^* \lim_n x^n$. This is the weak* limit in A^{**}, which always exists for x in $\frac{1}{2}\mathfrak{F}_A$, and it is nonzero if x has norm 1 [12, Corollary 3.3] (or equivalently if it has 1 in its spectrum, or attains value 1 on some state). For other contractions x this weak* limit may not exist or may be zero, but if this weak* limit does exist and is nonzero then it is a peak projection, and every peak projection is of this form (indeed even equals $w^* \lim_n a^n$ for some $a \in \frac{1}{2}\mathfrak{F}_A$). We have $u(x^{1/n}) = u(x)$ for $x \in \frac{1}{2}\mathfrak{F}_A$ (see [12, Corollary 3.3]). Compact projections for approximately unital algebras are precisely the infima (or decreasing weak* limits) of collections of such peak projections [12]. We will say more about peak projections around Lemma 4.3.

In this paper we will sometimes use the word 'cigar' for the wedge-shaped region consisting of numbers $re^{i\theta}$ with argument θ such that $|\theta| < \rho$ (for some fixed small $\rho > 0$), which are also inside the circle $|z - 1/2| \leq 1/2$. If ρ is small enough so that $|\operatorname{Im}(z)| < \epsilon/2$ for all z in this region, then we will call this a 'horizontal cigar of height $< \epsilon$ centered on the line segment $[0, 1]$ in the x-axis'.

By *numerical range*, we will mean the one defined by states, while the literature we quote usually uses the one defined by vector states on $B(H)$. However, since the former range is the closure of the latter, as is well known, this will cause no difficulties. For any operator $T \in B(H)$ whose numerical range does not include strictly negative numbers, and for any $\alpha \in [0, 1]$, there is a well-defined 'principal' root T^α, which obeys the usual law $T^\alpha T^\beta = T^{\alpha+\beta}$ if $\alpha + \beta \leq 1$ (see e.g. [33, 30]). If the numerical range is contained in a sector $S_\psi = \{re^{i\theta} : 0 \leq r, \text{ and } -\psi \leq \theta \leq \psi\}$ where $0 \leq \psi < \pi$, then things are better still. For fixed $\alpha \in (0, 1]$ there is a constant $K > 0$ with $\|T^\alpha - S^\alpha\| \leq K\|T - S\|^\alpha$ for operators S, T with numerical range in S_ψ (see [33, 30]). Our operators T will in fact be accretive (that is, $\psi \leq \pi/2$), and then these powers obey the usual laws such as $T^\alpha T^\beta = T^{\alpha+\beta}$ for all $\alpha, \beta > 0$, $(T^\alpha)^\beta = T^{\alpha\beta}$ for $\alpha \in (0, 1]$ and any $\beta > 0$, and $(T^*)^\alpha = (T^\alpha)^*$. We shall see in Lemma 2.15 that if $\psi < \pi/2$ then $T \in \mathfrak{c}_{B(H)}$. The numerical range of T^α lies in $S_{\alpha\pi/2}$ for any $\alpha \in (0, 1)$. Indeed, if $n \in \mathbb{N}$ then $T^{1/n}$ is the unique nth root of T with numerical range in $S_{\pi/(2n)}$. See e.g. [39, Chapter IV, Section 5], [26], and [30] for all of these facts. Some of the following facts are no doubt also in the literature; since we do not know of a reference we sketch short proofs.

LEMMA 1.1. *For an accretive operator $T \in B(H)$ we have:*

(1) $(cT)^\alpha = c^\alpha T^\alpha$ *for positive scalars c, and $\alpha \geq 0$.*
(2) $\alpha \mapsto T^\alpha$ *is continuous from $(0, \infty)$ into $B(H)$ with the norm topology.*
(3) $T^\alpha \in \operatorname{oa}(T)$, *the operator algebra generated by T, if $\alpha > 0$.*

Proof. (1) This is obvious if $\alpha = 1/n$ for $n \in \mathbb{N}$ by the uniqueness of nth roots discussed above. In general it can be proved e.g. by a change of variable in the Balakrishnan representation for powers (see e.g. [26]), or by the continuity in (2).

(2) By a triangle inequality argument, and the inequality for $\|T^\alpha - S^\alpha\|$ above, we may assume that $T \in \mathfrak{c}_{B(H)}$. By (1) we may assume that $T \in \frac{1}{2}\mathfrak{F}_{B(H)}$. Define

$$f(z) = ((1-z)/2)^\alpha - ((1-z)/2)^\beta, \qquad z \in \mathbb{C},\ |z| \leq 1.$$

Via the relation $T^\alpha T^\beta = T^{\alpha+\beta}$ above, we may assume that $\beta \in (0,1]$. Fix such β. We leave it as an exercise using calculus and manipulations with powers of complex numbers that $|f(z)| \leq g(|\alpha - \beta|)$ on the unit disk, for a function g with $\lim_{t\to 0^+} g(t) = 0$. By von Neumann's inequality, used as in [15, Proposition 2.3], we have

$$\|T^\alpha - T^\beta\| = \|f(1-2T)\| \leq g(|\alpha - \beta|).$$

Now let $\alpha \to \beta$.

(3) We proved this in the second paragraph of [15, Section 3] if $\alpha = 1/n$ for $n \in \mathbb{N}$. Hence for $m \in \mathbb{N}$ we have by the paragraph above the lemma that $T^{m/n} = (T^{1/n})^m \in \mathrm{oa}(T)$. The general case for $\alpha > 0$ then follows by the continuity in (2). ■

As in [33, Theorem 1] and [13, Lemma 3.8], if $\alpha \in (0,1)$ then there exists a constant K such that if $a, b \in \mathfrak{r}_{B(H)}$ for a Hilbert space H, and $ab = ba$, then $\|(a^\alpha - b^\alpha)\zeta\| \leq K\|(a-b)\zeta\|^\alpha$ for $\zeta \in H$.

2. Positivity in operator algebras. Let A be an operator algebra, not necessarily approximately unital for the present. Note that $\mathfrak{r}_A = \{a \in A : a + a^* \geq 0\}$ is a closed cone in A, hence is Archimedean, but it is not proper (hence is what is sometimes called a *wedge*). On the other hand $\mathfrak{c}_A = \mathbb{R}_+\mathfrak{F}_A$ is not closed in general, but it is a proper cone (that is, $\mathfrak{c}_A \cap (-\mathfrak{c}_A) = (0)$). Indeed, suppose $a \in \mathfrak{c}_A \cap (-\mathfrak{c}_A)$. Then $\|1 - ta\| \leq 1$ and $\|1 + sa\| \leq 1$ for some $s, t > 0$. By convexity we may assume $s = t$ (by replacing them by $\min\{s,t\}$). It is well known that in any Banach algebra with an identity of norm 1, the identity is an extreme point of the ball. Applying this in A^1 we deduce that $a = 0$ as desired.

As we said earlier without proof, for any operator algebra A, $x \in \mathfrak{r}_A$ iff $\mathrm{Re}(\varphi(x)) \geq 0$ for all states φ of A^1. Indeed, such φ extend to states on $C^*(A^1)$. So we may assume that A is a unital C^*-algebra, in which case the result is well known ($x + x^* \geq 0$ iff $2\,\mathrm{Re}(\varphi(x)) = \varphi(x + x^*) \geq 0$ for all states φ). We remark though that for an operator algebra which is not approximately unital, it is not true that $x \in \mathfrak{r}_A$ iff $\mathrm{Re}(\varphi(x)) \geq 0$ for all states φ of A, with states defined as in the introduction. An example would be $\mathbb{C} \oplus \mathbb{C}$, with the second summand given the zero multiplication.

The $\mathfrak{r}$-*ordering* is simply the order $\preccurlyeq$ induced by the above closed cone; that is, $b \preccurlyeq a$ iff $a - b \in \mathfrak{r}_A$. If A is a subalgebra of an operator algebra B, it is clear from a fact mentioned in the introduction (or at the start of [15, Section 3]) that the positivity of $a + a^*$ may be computed with reference to any containing C^*-algebra, that is, $\mathfrak{r}_A \subset \mathfrak{r}_B$. If A, B are approximately unital subalgebras of $B(H)$ then it follows from [15, Corollary 4.3(2)] that $A \subset B$ iff $\mathfrak{r}_A \subset \mathfrak{r}_B$. As in [14, Section 8], $\mathfrak{r}_A$ contains no idempotents which are not orthogonal projections, and no nonunitary isometries u (since by the analogue of [14, Corollary 2.8] we would have $uu^* = s(uu^*) = s(u^*u) = I$). In [15] it is shown that $\overline{\mathfrak{c}_A} = \mathfrak{r}_A$.

The following begins to illustrate the interesting order theory that exists in an operator algebra A and its generated C^*-algebra B. Note particularly how the order-theoretic results (3)–(7) flow out of the new ‘cofinality of A in B result’ (item (2) or (2′)). See [13] (particularly Section 6 there) for more interesting connections to, and remarkable relationships with, the classical theory of ordered linear spaces. In Section 4 we shall see the relationship between (2′) and ‘noncommutative peak interpolation’.

THEOREM 2.1. *Let A be an operator algebra which generates a C^*-algebra B, and let $\mathcal{U}_A$ denote the open unit ball $\{a \in A : \|a\| < 1\}$. The following are equivalent:*

(1) *A is approximately unital.*
(2) *For any positive $b \in \mathcal{U}_B$ there exists $a \in \mathfrak{r}_A$ with $b \preccurlyeq a$.*
(2′) *Same as (2), but also $a \in \frac{1}{2}\mathfrak{F}_A$ and nearly positive.*
(3) *For any pair $x, y \in \mathcal{U}_A$ there exist nearly positive $a \in \frac{1}{2}\mathfrak{F}_A$ with $x \preccurlyeq a$ and $y \preccurlyeq a$.*
(4) *For any $b \in \mathcal{U}_A$ there exist nearly positive $a \in \frac{1}{2}\mathfrak{F}_A$ with $-a \preccurlyeq b \preccurlyeq a$.*
(5) *For any $b \in \mathcal{U}_A$ there exist $x, y \in \frac{1}{2}\mathfrak{F}_A$ with $b = x - y$.*
(6) *$\mathfrak{r}_A$ is a generating cone (that is, $A = \mathfrak{r}_A - \mathfrak{r}_A$).*
(7) *$A = \mathfrak{c}_A - \mathfrak{c}_A$.*

Proof. (1)⇒(2′). Let (e_t) be a cai for A in $\frac{1}{2}\mathfrak{F}_A$ (by Read's theorem stated in the Introduction). By [10, 2.1.6], (e_t) is a cai for B, and hence so is (e_t^*), and $f_t = \mathrm{Re}(e_t)$. By the proof of Cohen's factorization theorem, as adapted in e.g. [13, Lemma 4.8], we may write $b^2 = zwz$, where $0 \le w \le 1$ and

$$z = \sum_{k=1}^{\infty} 2^{-k} f_{t_k} = \mathrm{Re}\Big(\sum_{k=1}^{\infty} 2^{-k} e_{t_k}\Big),$$

where $\{f_{t_k}\}$ are some of the f_t. If $a = \sum_{k=1}^{\infty} 2^{-k} e_{t_k} \in \frac{1}{2}\mathfrak{F}_A$, then $z = \mathrm{Re}(a)$. Then $b^2 \le z^2$, so that $b \le z$ and $b \preccurlyeq a$. We also have $b \preccurlyeq a^{1/n}$ for each $n \in \mathbb{N}$ by [7, Proposition 4.7], which gives the ‘nearly positive’ assertion.

(2′)⇒(3). By C^*-algebra theory there exists a positive $b \in \mathcal{U}_B$ with x and y 'dominated' by b. Then apply (2′).

(3)⇒(4). Apply (3) to b and $-b$.

(4)⇒(6). $b = \frac{a+b}{2} - \frac{a-b}{2} \in \mathfrak{r}_A - \mathfrak{r}_A$.

(6)⇒(1). This is in [15, Section 4], but we give a variant of the argument. First suppose that A is a weak* closed subalgebra of $B(H)$. Each $x \in \mathfrak{r}_A$ has a support projection $p_x \in B(H)$ by the discussion in [7, Section 3], which is just the weak* limit of $(x^{1/n})$, and hence is in A. Then $p = \bigvee_{x \in \mathfrak{r}_A} p_x$ is in A, and for any $x \in \mathfrak{r}_A$ we have

$$px = ps(x)x = s(x)x = x.$$

Since $\mathfrak{r}_A$ is generating, we have $px = x$ for all $x \in A$. Similarly, $xp = x$. So A is unital. In the general case, we can use the fact from the theory of ordered spaces [5] that if the order in A is generating, then the order in A^* is normal, and then the order in A^{**} is generating. The latter forces A^{**} to be unital, and hence A is approximately unital by e.g. [10, Proposition 2.5.8].

(1)⇒(5). Apply [13, Theorem 6.1].

It is obvious that (2′) implies (2), and that (5) implies (7), which implies (6).

(2)⇒(6). If $a \in A$ then by C^*-algebra theory and (2) there exist $b \in B_+$ and $x \in \mathfrak{r}_A$ with $-x \preccurlyeq -b \preccurlyeq a \preccurlyeq b \preccurlyeq x$. Thus $a = \frac{a+x}{2} - \frac{x-a}{2} \in \mathfrak{r}_A - \mathfrak{r}_A$. ∎

Remarks. 1) One cannot expect to be able to choose the a in (2) with $\|a\| = \|b\|$. Indeed, suppose that $A = \{f \in A(\mathbb{D}) : f(1) = 0\}$ and $B = \{f \in C(\mathbb{T}) : f(1) = 0\}$, with $b = 1$ on a nontrivial arc. If $b \leq \mathrm{Re}(a) \leq |a| \leq 1$ on that arc, then $\mathrm{Re}(a) = a = 1$ on that arc too. But this implies that $a = 1$ always, a contradiction.

Similarly, in (3) one cannot replace $\mathcal{U}_A$ by $\mathrm{Ball}(A)$, even if A is a C^*-algebra (consider for example the universal nonunital C^*-algebra generated by two projections [38]). However, perhaps one can replace $\mathcal{U}_A$ by $\mathrm{Ball}(A)$ in (3) (and also perhaps in (4)) if B is commutative. Some remarks on (5) may be found in [13] after Theorem 6.1.

2) Another proof that (1) implies (2): if $b \in B_+$ with $\|b\| < 1$ then it is immediate from [8, Lemma 2.1] that there exists $x \in -\mathfrak{F}_A$ such that $b \leq -x^*x - 2\,\mathrm{Re}(x)$. Hence $b \preccurlyeq a$, where $a = -2x \in 2\mathfrak{F}_A$.

This leads to a quick proof that (1) implies (2′) if b commutes with $\mathrm{Re}(a)$. Namely, first choose $\epsilon > 0$ such that $(1+\epsilon)\|b\| < 1$. Let $c = (1+\epsilon)b$, and suppose that $m \in \mathbb{N}$, and choose by the last paragraph $a \in 2\mathfrak{F}_A$ with $c^m \leq \mathrm{Re}(a)$. Hence if $n \in \mathbb{N}$ we have $1 \leq \mathrm{Re}(z)$ where $z = (c^m + 1/n)^{-1}(a + 1/n)$. It follows from a result on p. 181 of [26] that $1 \leq \mathrm{Re}(z^{1/m})$. Thus $(c^m + 1/n)^{1/m} \leq \mathrm{Re}((a+1/n)^{1/m})$. Letting $n \to \infty$ we obtain $c \leq \mathrm{Re}(a^{1/m})$.

For $m \geq m_0$ say, we have

$$a^{1/m} = 4^{1/m}\left(\frac{a}{4}\right)^{1/m} \in \frac{1+\epsilon}{2}\mathfrak{F}_A.$$

Dividing by $1+\epsilon$ and taking m large enough we obtain (2′).

3) Of course all parts of the theorem are trivial if A is unital.

2.1. Non-approximately unital operator algebras. Most of the results in this section apply to approximately unital operator algebras. We offer a couple of results that are useful in applying the approximately unital case to algebras with no approximate identity. We will use the space $A_H = \overline{\mathfrak{r}_A A \mathfrak{r}_A}$ studied in [15, Section 4]; it is actually a HSA in A (and will be an ideal if A is commutative).

Corollary 2.2. *For any operator algebra A, the largest approximately unital subalgebra of A is*

$$A_H = \mathfrak{r}_A - \mathfrak{r}_A = \mathfrak{c}_A - \mathfrak{c}_A.$$

In particular these spaces are closed, and form a HSA of A.

If A is a weak closed operator algebra then $A_H = qAq$ where q is the largest projection in A. In this case A_H is weak* closed.*

Proof. In the language of [15, Section 4], and using [15, Corollary 4.3], $\mathfrak{r}_A = \mathfrak{r}_{A_H}$, and the largest approximately unital subalgebra of A is the HSA

$$A_H = \mathfrak{r}_{A_H} - \mathfrak{r}_{A_H} = \mathfrak{r}_A - \mathfrak{r}_A,$$

by Theorem 2.1(6). A similar argument works in the $\mathfrak{c}_A$ case, with $\mathfrak{r}_{A_H}$ replaced by $\mathfrak{c}_{A_H}$ in view of Theorem 2.1(7) and facts from [15, Section 4] about $\mathfrak{F}_{A_H}$.

To see the final assertion, note that if p is as in the proof of (6)⇒(1) in Theorem 2.1, then certainly $q \leq p$ since $q = s(q) \in \mathfrak{r}_A$. However, $p \leq q$ since p is a projection in A. So $p = q$, and this acts as the identity on $\mathfrak{r}_A - \mathfrak{r}_A = A_H$. So $A_H \subset qAq$, and conversely $qAq \subset A_H$ since A_H is a HSA, or because A_H is the largest (approximately) unital subalgebra of A. ∎

Lemma 2.3. *Let A be any operator algebra. Then for every $n \in \mathbb{N}$,*

$$M_n(A_H) = M_n(A)_H, \quad \mathfrak{r}_{M_n(A)} = \mathfrak{r}_{M_n(A_H)}, \quad \mathfrak{F}_{M_n(A)} = \mathfrak{F}_{M_n(A_H)}$$

(*these are the matrix spaces*).

Proof. Clearly $M_n(A_H)$ is an approximately unital subalgebra of $M_n(A)$. So $M_n(A_H)$ is contained in $M_n(A)_H$, since the latter is the largest approximately unital subalgebra of $M_n(A)$. To show that $M_n(A)_H \subset M_n(A_H)$ it suffices, by Corollary 2.2, to show that $\mathfrak{r}_{M_n(A)} \subset M_n(A_H)$. So suppose that $a = [a_{ij}] \in M_n(A)$ with $a + a^* \geq 0$. Then $a_{ii} + a_{ii}^* \geq 0$ for each i. We also have $\sum_{i,j} \bar{z}_i(a_{ij} + a_{ji}^*)z_j \geq 0$ for all scalars $z_1, \ldots, z_n$. So $\sum_{i,j} \bar{z}_i a_{ij} z_j \in \mathfrak{r}_A$.

Fix i, j, which we will assume to be $1, 2$ for simplicity. Set all $z_k = 0$ if $k \notin \{i, j\} = \{1, 2\}$, to deduce

$$\bar{z}_1 z_2 a_{12} + \bar{z}_2 z_1 a_{21} = \sum_{i,j=1}^{2} \bar{z}_i a_{ij} z_j - (|z_1|^2 a_{11} + |z_2|^2 a_{22}) \in \mathfrak{r}_A - \mathfrak{r}_A = A_H.$$

Choose $z_1 = 1$; if $z_2 = 1$ then $a_{12} + a_{21} \in A_H$, while if $z_2 = i$ then $i(a_{12} - a_{21}) \in A_H$. So $a_{12}, a_{21} \in A_H$. A similar argument shows that $a_{ij} \in A_H$ for all i, j. Thus $M_n(A_H) = M_n(A)_H$, from which we deduce by [15, Corollary 4.3(1)] that

$$\mathfrak{r}_{M_n(A)} = \mathfrak{r}_{M_n(A)_H} = \mathfrak{r}_{M_n(A_H)}.$$

Similarly $\mathfrak{F}_{M_n(A)} = \mathfrak{F}_{M_n(A)_H} = \mathfrak{F}_{M_n(A_H)}$. ∎

The last result is used in [7].

If $S \subset \mathfrak{r}_A$ for an operator algebra A, and if $xy = yx$ for all $x, y \in S$, write $\mathrm{oa}(S)$ for the smallest closed subalgebra of A containing S.

Proposition 2.4. *If S is any subset of $\mathfrak{r}_A$ for an operator algebra A, then* $\mathrm{oa}(S)$ *has a cai.*

Proof. Let $C = \mathrm{oa}(S)$. Then $\mathfrak{r}_C = C \cap \mathfrak{r}_A$, so that

$$C \subset \overline{\mathfrak{r}_C C \mathfrak{r}_C} = C_H \subset C.$$

Hence $C = C_H$, which is approximately unital. ∎

2.2. The $\mathfrak{F}$-transform and existence of an increasing approximate identity. In [15] the sets $\frac{1}{2}\mathfrak{F}_A$ and $\mathfrak{r}_A$ were related by a certain transform. We now establish a few more basic properties of this transform. The Cayley transform $\kappa(x) = (x - I)(x + I)^{-1}$ of an accretive $x \in A$ exists since $-1 \notin \mathrm{Sp}(x)$, and is well known to be a contraction. Indeed, it is well known (see e.g. [39]) that if A is unital then the Cayley transform maps $\mathfrak{r}_A$ bijectively onto the set of contractions in A whose spectrum does not contain 1, and the inverse transform is $T \mapsto (I + T)(I - T)^{-1}$. The Cayley transform maps the accretive elements x with $\mathrm{Re}(x) \geq \epsilon 1$ for some $\epsilon > 0$ onto the set of elements $T \in A$ with $\|T\| < 1$ (see e.g. [10, 2.1.14]). The $\mathfrak{F}$-transform $\mathfrak{F}(x) = 1 - (x + 1)^{-1} = x(x + 1)^{-1}$ may be written as $\mathfrak{F}(x) = \frac{1}{2}(1 + \kappa(x))$. Equivalently, $\kappa(x) = -(1 - 2\mathfrak{F}(x))$.

Lemma 2.5. *For any operator algebra A, the $\mathfrak{F}$-transform maps $\mathfrak{r}_A$ bijectively onto the set of elements of $\frac{1}{2}\mathfrak{F}_A$ of norm < 1. Thus $\mathfrak{F}(\mathfrak{r}_A) = \mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$.*

Proof. First assume that A is unital. By the last equations $\mathfrak{F}(\mathfrak{r}_A)$ is contained in the set of elements of $\frac{1}{2}\mathfrak{F}_A$ whose spectrum does not contain 1. The inverse of the $\mathfrak{F}$-transform on this domain is $T(I - T)^{-1}$. To see for example that $T(I - T)^{-1} \in \mathfrak{r}_A$ if $T \in \frac{1}{2}\mathfrak{F}_A$ note that $2\,\mathrm{Re}(T(I - T)^{-1})$

equals

$$(I-T^*)^{-1}(T^*(I-T)+(I-T^*)T)(I-T)^{-1} \\ = (I-T^*)^{-1}(T+T^*-2T^*T)(I-T)^{-1},$$

which is positive since T^*T is dominated by $\mathrm{Re}(T)$ if $T \in \frac{1}{2}\mathfrak{F}_A$. Hence for any (possibly nonunital) operator algebra A the $\mathfrak{F}$-transform maps $\mathfrak{r}_{A^1}$ bijectively onto the set of elements of $\frac{1}{2}\mathfrak{F}_{A^1}$ whose spectrum does not contain 1. However, this equals the set of elements of $\frac{1}{2}\mathfrak{F}_{A^1}$ of norm < 1. Indeed, if $\|\mathfrak{F}(x)\| = 1$ then $\left\|\frac{1}{2}(1+\kappa(x))\right\| = 1$, and so $1-\kappa(x)$ is not invertible by [3, Proposition 3.7]. Hence $1 \in \mathrm{Sp}_{A^1}(\kappa(x))$ and $1 \in \mathrm{Sp}_A(\mathfrak{F}(x))$. Since $\mathfrak{F}(x) \in A$ iff $x \in A$, we are done. ∎

Thus in some sense we can identify $\mathfrak{r}_A$ with the strict contractions in $\frac{1}{2}\mathfrak{F}_A$. This for example induces an order on this set of strict contractions.

We recall that the positive part of the open unit ball of a C^*-algebra is a directed set, and indeed is a net which is a positive cai for B (see e.g. [35]). The following generalizes this to operator algebras:

PROPOSITION 2.6. *If A is an approximately unital operator algebra, then $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is a directed set in the $\preccurlyeq$ ordering, and with this ordering $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is an increasing cai for A.*

Proof. We know $\mathfrak{F}(\mathfrak{r}_A) = \mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ by Lemma 2.5. By Theorem 2.1(3), $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is directed by $\preccurlyeq$. So we may view $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ as a net (e_t). Given $x \in \frac{1}{2}\mathfrak{F}_A$, choose n such that $\|\mathrm{Re}(x^{1/n})x - x\| < \epsilon$ (note that as in the first few lines of the proof of Theorem 2.1, $(\mathrm{Re}(x^{1/n}))$ is a cai for $C^*(\mathrm{oa}(x))$. If $z \in \mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ with $x^{1/n} \preccurlyeq z$ then

$$x^*|1-z|^2x \le x^*(1-\mathrm{Re}(z))x \le x^*(1-\mathrm{Re}(x^{1/n}))x \le \epsilon.$$

Thus $e_t x \to x$ for all $x \in \frac{1}{2}\mathfrak{F}_A$. ∎

Note that $\mathcal{U}_A \cap \mathfrak{r}_A$ is directed, by Theorem 2.1(3), but we do not know if it is a cai in this ordering.

The following is a variant of [13, Corollary 2.10]:

COROLLARY 2.7. *Let A be an approximately unital operator algebra, and B a C^*-algebra generated by A. If $b \in B_+$ with $\|b\| < 1$ then there is an increasing cai for A in $\frac{1}{2}\mathfrak{F}_A$, every term of which dominates b (where ‘increasing’ and ‘dominates’ are in the $\preccurlyeq$ ordering).*

Proof. Since $\mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A$ is a directed set, $\{a \in \mathcal{U}_A \cap \frac{1}{2}\mathfrak{F}_A : b \preccurlyeq a\}$ is a subnet of the increasing cai in the last result. ∎

We remark that any operator algebra A with a countable cai, and in particular any separable approximately unital A, has a commuting cai which is increasing (for the $\preccurlyeq$ ordering), and also in $\frac{1}{2}\mathfrak{F}_A$ and nearly positive.

Namely, by [14, Corollary 2.18] we have $A = \overline{xAx}$ for some $x \in \frac{1}{2}\mathfrak{F}_A$, so that $(x^{1/n})$ is a commuting cai which is increasing by [7, Proposition 4.7]. For a related fact see Lemma 3.6 below.

2.3. Real positive maps and real states. An $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$ (resp. $\mathbb{C}$-linear $T : A \to B$) will be said to be *real positive* if $\varphi(\mathfrak{r}_A) \subset [0, \infty)$ (resp. $T(\mathfrak{r}_A) \subset \mathfrak{r}_B$). By the usual trick, for any $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$, there is a unique $\mathbb{C}$-linear $\tilde{\varphi} : A \to \mathbb{C}$ with $\mathrm{Re}(\tilde{\varphi}) = \varphi$, and clearly φ is real positive (resp. bounded) iff $\tilde{\varphi}$ is real positive (resp. bounded).

COROLLARY 2.8. *Let A be an approximately unital operator algebra, and B a C^*-algebra generated by A. Then every real positive $\varphi : A \to \mathbb{R}$ extends to a real positive real functional on B. Also, φ is bounded.*

Proof. Theorem 2.1(2) says that the ordering in A is dominating or 'cofinal' in B in the language of ordered spaces (see e.g. [29]). The first assertion is a well known consequence in the theory of ordered spaces of this cofinal property (see e.g. [23] or [29, Theorem 1.6.1]). Similarly the final assertion follows from a general principle for an ordered Banach space $(X, \leq)$ whose order is generating: if $f : X \to \mathbb{R}$ is positive but (by way of contradiction) unbounded then by a theorem of Ando (see e.g. [6, Theorem II.1.2]), f is unbounded on $\mathrm{Ball}(X_+)$. So there exist $x_k \in X_+$ of norm $\leq$ but with $f(x_k) > 2^k$. So $n < \sum_{k=1}^{n} 2^{-k} f(x_k) \leq f(\sum_{k=1}^{\infty} 2^{-k} x_k)$ for all n. This is the desired contradiction. ∎

COROLLARY 2.9. *Let $T : A \to B$ be a $\mathbb{C}$-linear map between approximately unital operator algebras, and suppose that T is real positive* (*resp. suppose that the nth matrix amplifications T_n are each real positive; cf.* [7, Definition 2.1]). *Then T is bounded* (*resp. completely bounded*).

Proof. First suppose that $B = \mathbb{C}$. Then $\mathrm{Re}(T)$ is real positive, hence bounded by Corollary 2.8. It is then obvious that T is bounded.

In the general case, we can assume B is a unital C^*-algebra. Let $\psi \in S(B)$, and $\varphi = \psi \circ T$. Then φ is real positive, hence bounded. Thus there exists a constant K such that for all $x \in \mathrm{Ball}(A)$ we have $|\psi(T(x))| = |\varphi(x)| \leq K$. By the 'Jordan decomposition' in B^*, it follows that $|\psi(T(x))| \leq 4K$ for all $\psi \in \mathrm{Ball}(B^*)$. Thus T is bounded. In the 'respectively' case, applying the above at each matrix level shows that the nth amplifications T_n are each bounded. The proof in [7, Section 2] shows that T extends to a completely positive map on an operator system, and it is known that completely positive maps are completely bounded. ∎

REMARK. It follows from this that in the 'Extension and Stinespring dilation theorem for real completely positive maps' from [7], it is unnecessary to assume that the RCP maps defined in [7, Definition 2.1] are (completely) bounded. One only needs T to be linear and real positive, and similarly at each matrix level.

We will write $\mathfrak{c}_{A^*}^{\mathbb{R}}$ for the real dual cone of $\mathfrak{r}_A$, the set of continuous $\mathbb{R}$-linear $\varphi : A \to \mathbb{R}$ such that $\varphi(\mathfrak{r}_A) \subset [0,\infty)$. Since $\overline{\mathfrak{c}_A} = \mathfrak{r}_A$, we see that $\mathfrak{c}_{A^*}^{\mathbb{R}}$ is also the real dual cone of $\mathfrak{c}_A$. It is a proper cone, for if $\rho, -\rho \in \mathfrak{c}_{A^*}^{\mathbb{R}}$ then $\rho(a) = 0$ for all $a \in \mathfrak{r}_A$, hence $\rho = 0$ by the fact above that the norm closure of $\mathfrak{r}_A - \mathfrak{r}_A$ is A.

Lemma 2.10. *Suppose that A is an approximately unital operator algebra. The real dual cone $\mathfrak{c}_{A^*}^{\mathbb{R}}$ equals $\{t\,\mathrm{Re}(\psi) : \psi \in S(A),\ t \in [0,\infty)\}$. It also equals the set of restrictions to A of the real parts of positive functionals on any C^*-algebra containing (a copy of) A as a closed subalgebra. The prepolar of $\mathfrak{c}_{A^*}^{\mathbb{R}}$, which equals its real predual cone, is $\mathfrak{r}_A$; and the polar of $\mathfrak{c}_{A^*}^{\mathbb{R}}$, which equals its real dual cone, is $\mathfrak{r}_{A^{**}}$. Thus the second dual cone of $\mathfrak{r}_A$ is $\mathfrak{r}_{A^{**}}$, and hence $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$.*

Proof. This is proved in [13] in a more general setting, but there is a simpler proof in our case. By Corollary 2.8, every real positive $\varphi : A \to \mathbb{R}$ extends to a real positive real functional on B, and the latter is the real part of a $\mathbb{C}$-linear real positive functional ψ on B. Clearly ψ is positive in the usual sense, and hence ψ is a positive multiple of a state on B. Restricting to A, we see that φ is the real part of a positive multiple of a state on A. Thus

$$\mathfrak{c}_{A^*}^{\mathbb{R}} = \{t\,\mathrm{Re}(\psi) : \psi \in S(A), t \in [0,\infty)\}.$$

In any C^*-algebra B it is well known that $b \geq 0$ iff $\varphi(b) \geq 0$ for all states φ of B. Hence $a \in \mathfrak{r}_A = A \cap \mathfrak{r}_B$ iff $2\,\mathrm{Re}(\varphi(a)) = \varphi(a + a^*) \geq 0$ for all states φ, and so iff $a \in (\mathfrak{c}_{A^*}^{\mathbb{R}})_\circ$. The polar of $\mathfrak{c}_{A^*}^{\mathbb{R}}$ is

$$\{\eta \in A^{**} : \mathrm{Re}(\eta(\psi)) \geq 0 \text{ for all } \psi \in S(A)\} = \mathfrak{r}_{B^{**}} \cap A^{**} = \mathfrak{r}_{A^{**}},$$

since

$$\mathfrak{r}_{B^{**}} = \{\eta \in B^{**} : \mathrm{Re}(\eta(\psi)) \geq 0 \text{ for all } \psi \in S(B)\}.$$

So the real bipolar $(\mathfrak{r}_A)^{\circ\circ}$ is $\mathfrak{r}_{A^{**}}$. By the bipolar theorem, $\mathfrak{r}_A$ is weak* dense in $\mathfrak{r}_{A^{**}}$. ■

We remark that the last several results have some depth; indeed, one can show that they are each essentially equivalent to Read's theorem on approximate identities (and can be used to give a more order-theoretic proof of that result).

We give some consequences to the theory of real states. A *real state* on an approximately unital operator algebra A will be a contractive $\mathbb{R}$-linear $\mathbb{R}$-valued functional on A such that $\varphi(e_t) \to 1$ for some cai (e_t) of A. This is equivalent to $\varphi^{**}(1) = 1$, where φ^{**} is the canonical $\mathbb{R}$-linear extension to A^{**}, and 1 is the identity of A^{**} (here we are using the canonical identification between real second duals and complex second duals of a complex Banach space [32]). Hence $\varphi(e_t) \to 1$ for every cai (e_t) of A.

Since we can identify A^1 with $A + \mathbb{C}1_{A^{**}}$ if we like, by the last paragraph it follows that real states of A extend to real states of A^1, hence by the Hahn–Banach theorem they extend to real states of $C^*(A^1)$. We claim that a real state ψ on a C^*-algebra B is positive on B_+, and is zero on iB_+. To see this, we may assume that B is a von Neumann algebra (by extending the state to its second dual similarly to as in the last paragraph). For any projection $p \in B$, $C^*(1,p) \cong \ell^\infty_2$, and it is an easy exercise to see that real states on ℓ^∞_2 are positive on $(\ell^\infty_2)_+$ and are zero on $i(\ell^\infty_2)_+$. Thus $\psi(p) \geq 0$ and $\psi(ip) = 0$ for any projection p, hence ψ is positive on B_+ and zero on iB_+ by the Krein–Milman theorem.

We deduce:

Corollary 2.11. *Real states on an approximately unital operator algebra A are in $\mathfrak{c}^{\mathbb{R}}_{A^*}$. Indeed, real states are just the real parts of ordinary states on A.*

Proof. Certainly the real part of an ordinary state is a real state. If φ is a real state on A, if $a + a^* \geq 0$, and if $\tilde{\varphi}$ is the real state extension above to $B = C^*(A^1)$, then

$$\varphi(a) = \frac{1}{2}\tilde{\varphi}(a + a^*) + \frac{1}{2}\tilde{\varphi}(-i \cdot i(a - a^*)) = \frac{1}{2}\tilde{\varphi}(a + a^*) \geq 0,$$

since $i(a - a^*) \in B_{\mathrm{sa}} = B_+ - B_+$, and $\tilde{\varphi}(i(B_+ - B_+)) = 0$, as we said above. So $\varphi \in \mathfrak{c}^{\mathbb{R}}_{A^*}$. By [13, Corollary 6.3], φ is the real part of a quasistate of A, and it is easy to see that the latter must be a state. ∎

Corollary 2.12. *Any real state on an approximately unital closed subalgebra A of an approximately unital operator algebra B extends to a real state on B. If A is a HSA in B then this extension is unique.*

Proof. The first part is as in [35, Proposition 3.1.6]. Suppose that A is a HSA in B and that φ_1, φ_2 are real states on B extending a real state on A. By the above we may write $\varphi_i = \mathrm{Re}(\psi_i)$ for ordinary states on B. Since $\varphi_1 = \varphi_2$ on A we have $\psi_1 = \psi_2$ on A. Hence $\psi_1 = \psi_2$ on B by [9, Theorem 2.10]. So $\varphi_1 = \varphi_2$ on B. ∎

2.4. Principal r-ideals. In the predecessor to this paper ([16]), we proved several facts about principal and algebraically finitely generated r-ideals, and these were generalized to Banach algebras in [13] with essentially the same proofs. The main difference is that in [13] one always had the condition that A be approximately unital, whose purpose was simply to ensure that $\mathfrak{r}_A$ makes sense. For operator algebras, $\mathfrak{r}_A$ always makes sense, so that one can delete 'approximately unital' in the statements of 3.21–3.25

in [13]. One may also replace 'idempotent' by 'projection' in those results, since for operator algebras the support $s(x)$ is a projection for $x \in \mathfrak{r}_A$. One may also delete the word 'left' in [13, Corollary 3.25] since a left identity is a two-sided identity if A is approximately unital (since $ee_t = e_t \to e$ for the cai (e_t)). Moreover, the proofs show that all of Theorem 3.2 of [14] is valid for $x \in \mathfrak{r}_A$. Similarly, the proof of [13, Corollary 4.7] gives

COROLLARY 2.13. *Let A be an operator algebra. A closed r-ideal J in A is algebraically finitely generated as a right module over A iff $J = eA$ for a projection $e \in A$. This is also equivalent to J being algebraically finitely generated as a right module over A^1.*

2.5. Roots of accretive elements

LEMMA 2.14. *Suppose that B is a C^*-algebra in its universal representation, so that $B^{**} \subset B(H)$ as a von Neumann algebra containing I_H. Let $x \in \frac{1}{2}\mathfrak{F}_B$ and let $s(x)$ be its support projection, viewed in $B(H)$. Then $x^{1/n} \to s(x)$ in the strong operator topology.*

Proof. If $\zeta \in H$, and $a_n = x^{1/n}$ then $a_n \in \frac{1}{2}\mathfrak{F}_B$ by [14, Proposition 2.3]. Hence $a_n^* a_n \le \mathrm{Re}(a_n)$, and

$$\|(a_n - s(x))\zeta\|^2 = \langle (a_n^* a_n - 2\,\mathrm{Re}(a_n) + s(x))\zeta, \zeta\rangle \le \langle (s(x) - \mathrm{Re}(a_n))\zeta, \zeta\rangle \to 0,$$

since a_n, and hence a_n^* and $\mathrm{Re}(a_n)$, converges weak* to $s(x)$. ∎

LEMMA 2.15. *Let A be an operator algebra, and $x \in A$.*

1. *If the numerical range of x is contained in a sector S_ρ for $\rho < \pi/2$ (see notation above Lemma* 1.1*), then $x/\|\mathrm{Re}(x)\| \in \frac{\sec^2 \rho}{2}\,\mathfrak{F}_A$. So $x \in \mathfrak{c}_A$.*
2. *If $x \in \mathfrak{r}_A$ then $x^\alpha \in \mathfrak{c}_A$ for any $\alpha \in (0,1)$.*

In particular, the elements of A which are sectorial of angle $< \pi/2$ are a dense subcone of $\mathfrak{c}_A$.

Proof. (1) Write $x = a + ib$, for positive a and selfadjoint b in a containing $B(H)$. By the argument in the proof of [14, Lemma 8.1], there exists a selfadjoint $c \in B(H)$ with $b = a^{1/2} c a^{1/2}$ and $\|c\| \le \tan\rho$. Then $x = a^{1/2}(1 + ic)a^{1/2}$, and

$$x^* x = a^{1/2}(1+ic)^* a (1+ic) a^{1/2} \le Ca.$$

By the C^*-identity $\|(1+ic)^* a (1+ic)\|$ equals

$$\|a^{1/2}(1+ic)(1+ic)^* a^{1/2}\| \le \|a\|(1 + \|c\|^2) \le \|a\|(1 + \tan^2\rho) = \|a\|\sec^2\rho.$$

So we can take $C = \|a\|\sec^2\rho$. Saying that $x^* x \le C\,\mathrm{Re}(x)$ is the same as saying that $x \in \frac{C}{2}\mathfrak{F}_A$.

(2) This follows from (1) since in this case the numerical range of x^α is contained in a sector S_ρ with $\rho < \pi/2$.

The final assertion follows from (1), and from the facts from the Introduction that $x = \lim_{t\to 1^-} x^t$ and that x^t is sectorial of angle $< \pi/2$ if $0 < t < 1$. ∎

Remark. The last result is related to the remark before [14, Lemma 8.1].

Of course $\|\mathrm{Im}(x^{1/n})\| \to 0$ as $n \to \infty$, for $x \in \mathfrak{r}_A$ (as is clear e.g. from the above, or from the computation in the centered line on the second page of our paper).

Lemma 2.16. *If $a \in \mathfrak{r}_A$ for an operator algebra A, and v is a partial isometry in any containing C^*-algebra B with $v^*v = s(a)$, then $vav^* \in \mathfrak{r}_B$ and $(vav^*)^r = va^rv^*$ if $r \in (0,1) \cup \mathbb{N}$.*

Proof. This is clear if $r = k \in \mathbb{N}$. It is also clear that $vav^* \in \mathfrak{r}_B$. We will use the Balakrishnan representation above to check that $(vav^*)^r = va^rv^*$ if $r \in (0,1)$ (it can also be deduced from the $\mathfrak{F}_A$ case in [12]). Claim: $(t + vav^*)^{-1}vav^* = v(t + a)^{-1}av^*$. Indeed, since $v^*va = a$ we have

$$(t + vav^*)v(t + a)^{-1}av^* = v(t + a)(t + a)^{-1}av^* = vav^*,$$

proving the claim. Hence for any $\zeta, \eta \in H$ we have

$$\langle (t + vav^*)^{-1}vav^*\zeta, \eta\rangle = \langle v(t + a)^{-1}av^*\zeta, \eta\rangle = \langle (t + a)^{-1}av^*\zeta, v^*\eta\rangle.$$

Hence by the Balakrishnan representation

$$\langle (vav^*)^r\zeta, \eta\rangle = \frac{\sin(r\pi)}{\pi}\int_0^\infty t^{r-1}\langle (t + vav^*)^{-1}vav^*\zeta, \eta\rangle\, dt,$$

which equals

$$\frac{\sin(r\pi)}{\pi}\int_0^\infty t^{r-1}\langle (t + a)^{-1}av^*\zeta, v^*\eta\rangle dt = \langle va^rv^*\zeta, \eta\rangle,$$

as desired. ∎

The last result generalizes [11, Lemma 1.4]. With the last few results in hand, and [13, Lemma 3.6], it appears that all of the results in [11] stated in terms of $\mathfrak{F}_A$ (or $\frac{1}{2}\mathfrak{F}_A$ or $\mathfrak{c}_A$) should generalize without problem to the $\mathfrak{r}_A$ case. We admit that we have not yet carefully checked every part of every result in [11] for this though, but hope to in forthcoming work.

2.6. Concavity, monotonicity, and operator inequalities. The usual operator concavity/convexity results for C^*-algebras seem to fail for the $\mathfrak{r}$-ordering. That is, results of the type in [35, Proposition 1.3.11] and its proof fail. Indeed,, functions like $\mathrm{Re}(z^{1/2}), \mathrm{Re}(z(1 + z)^{-1}), \mathrm{Re}(z^{-1})$ are not operator concave or convex, even for operators $x, y \in \frac{1}{2}\mathfrak{F}_A$. In fact, this fails even in the simplest case $A = \mathbb{C}$: take $x = 1/2$, $y = (1 + i)/2$. Similar remarks hold for 'operator monotonicity' with respect to the $\mathfrak{r}_A$-ordering for these functions.

For the $\mathfrak{r}$-ordering, one way one can often prove operator inequalities, or that something is increasing, is via the functional calculus, as follows.

LEMMA 2.17. *Suppose that A is a unital operator algebra and f, g are functions in the disk algebra with* $\mathrm{Re}(g - f) \geq 0$ *on the closed unit disk. Then $f(1 - x) \preccurlyeq g(1 - x)$ for $x \in \mathfrak{F}_A$.*

Proof. Here e.g. $f(1-x)$ is the 'disk algebra functional calculus', arising from von Neumann's inequality for the contraction $1-x$. The result follows by applying [39, Proposition 3.1, Chapter IV] to $g - f$. ∎

A good illustration of this principle is the proof at the end of [7] that for any $x \in \frac{1}{2}\mathfrak{F}_A$, the sequence $(\mathrm{Re}(x^{1/n}))$ is increasing. The last fact is another example of $\frac{1}{2}\mathfrak{F}_A$ behaving better than $\mathfrak{r}_A$: for contractions $x \in \mathfrak{r}_A$, we do not in general have $(\mathrm{Re}(x^{1/m}))$ increasing with m. The matrix example

$$\begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}$$

(communicated to us by Christian Le Merdy) will demonstrate this. This example also shows that one need not have $\|x^{1/m}\| \leq \|x\|^{1/m}$ for $x \in \mathfrak{r}_A$, so that one can have $x \in \mathfrak{r}_A \cap \mathrm{Ball}(A)$ but $\|x^{1/m}\| > 1$. However, one can show that for any $x \in \mathfrak{r}_A$ there exists a constant $c > 0$ such that $(\mathrm{Re}((x/c)^{1/m}))_{m\geq 2}$ is increasing with m. Indeed, if $c = (2\|\mathrm{Re}(x^{1/2})\|)^2$, then by Lemma 2.15(2) we have $(x/c)^{1/2} \in \frac{1}{2}\mathfrak{F}_A$. Thus $\mathrm{Re}((x/c)^t)$ increases as $t \searrow 0$ (see the proof of [7, Proposition 3.4]), from which the desired assertion follows.

Finally, we clarify a few imprecisions in a couple of the positivity results in [14, 15]. In the second last paragraph of Section 4 of [15], states on a nonunital algebra should probably also be assumed to have norm 1 (this seems a sensible requirement, although the arguments there do not need this and the results as stated are correct). In the statement of [14, Proposition 4.3] we should have explicitly added the hypothesis that A is approximately unital since the proof needs this (otherwise $E_{11}M_2$ is a counterexample). There are some small typos in the proof of [14, Theorem 2.12] but the reader should have no problem correcting these, and the result itself is correct.

3. Strictly real positive elements. We will study here a variant of the notion of a 'strictly real positive element' introduced after Lemma 2.10 in [14]. In the present paper, in contrast to the latter, we will say an element x in A is *strictly real positive* if $\mathrm{Re}(\varphi(x)) > 0$ for all states on A^1 whose restriction to A is nonzero. Such x are in $\mathfrak{r}_A$. This includes the $x \in A$ with

$\mathrm{Re}(x)$ strictly positive (in the usual C^*-algebra sense) in some C^*-algebra generated by A. If A is approximately unital, then these conditions are in fact equivalent, as the next result shows, so that in this case we are not introducing a notion different to that of [14]. For algebras that are not approximately unital it may possibly turn out in the future that for some purposes our new definition above is too general, but for now we investigate the 'broadest class'.

LEMMA 3.1. *Let A be an approximately unital operator algebra, which generates a C^*-algebra $C^*(A)$. An element $x \in A$ is strictly real positive in the sense above iff* $\mathrm{Re}(x)$ *is strictly positive in $C^*(A)$.*

Proof. The one direction follows because any state on A^1 whose restriction to A is nonzero extends to a state on $C^*(A)^1$ which is nonzero on $C^*(A)$. The restriction to $C^*(A)$ of the latter state is a positive multiple of a state.

For the other direction recall that we showed in the introduction that any state on $C^*(A)$ gives rise to a state on A^1. Since any cai of A is a cai of $C^*(A)$, the latter state cannot vanish on A. ∎

REMARK. Note that if $\mathrm{Re}(x) \geq \epsilon 1$ in $C^*(A)^1$, then there exists a constant $C > 0$ with $\mathrm{Re}(x) \geq \epsilon 1 \geq C x^* x$, and it follows that $x \in \mathbb{R}_+ \mathfrak{F}_A$. Thus if A is unital then every strictly real positive in A is in $\mathbb{R}_+ \mathfrak{F}_A$. However, this is false if A is approximately unital (it is even easily seen to be false in the C^*-algebra $A = c_0$). Conversely, note that if A is an approximately unital operator algebra with no r-ideals and no identity, then every nonzero element of $\mathbb{R}_+ \mathfrak{F}_A$ is strictly real positive by [14, Theorem 4.1].

We also remark that it is tempting to define an element $x \in A$ to be strictly real positive if $\mathrm{Re}(x)$ strictly positive in some C^*-algebra generated by A. However, this definition can depend on the particular generated C^*-algebra, unless one only uses states on the latter that are not allowed to vanish on A (in which case it is equivalent to the other definition). As an example of this, consider the algebra of 2×2 matrices supported on the first row, and the various C^*-algebras it can generate.

We next discuss how some results in [14] generalize, particularly those related to strict real positivity if we use the definition at the start of the present section. We recall that in [14], many 'positivity' results were established for elements in $\mathfrak{F}_A$ or $\frac{1}{2}\mathfrak{F}_A$, and by extension for the proper cone $\mathfrak{c}_A = \mathbb{R}_+ \mathfrak{F}_A$. In [15, Section 3] we pointed out several of these facts that generalized to the larger cone $\mathfrak{r}_A$, and indicated that some of this would be discussed in more detail in [7]. In [15, Section 4] we pointed out that the hypothesis in many of these results that A be approximately unital could be simultaneously relaxed. In the next few paragraphs we give more details that indicate the similarities and differences between these cones, particularly focusing on the results involving strictly real positive elements. The following list should be added to the list in [15, Section 3], and some complementary details are discussed in [7]. Since there are a large number of results being referenced we will not state the results in full, but the interested reader will easily be able to do so from what we do say. Similarly, we will not prove the results that generalize, since in most cases it is obvious that the original proof will work in the new setting. If there needs to be a slight variant of the argument then we indicate below what the small changes are. We also state what results do not generalize, and will indicate counterexamples in these cases.

In [14, Lemma 2.9] the ($\Leftarrow$) direction is correct for $x \in \mathfrak{r}_A$ with the same proof. Also one need not assume there that A is approximately unital, as we said towards the end of Section 4 in [15]. The other direction is not true in general, but there is a partial result, Lemma 3.3 below. For example, a counterexample in the $\mathfrak{r}_A$ case both to one implication of [14, Lemma 2.9], and to some implications in [14, Lemma 2.10], is given by $x = (i, -i) \in \ell^\infty_2$. Here $x \in \mathfrak{r}_A$ but $x \notin \mathfrak{F}_A$, and $s(x) = (1, 1)$, but $\mathrm{Re}(x)$ is not strictly positive, and $\varphi(x) = 0$ for the state $\varphi((a, b)) = \frac{1}{2}(a+b)$. In [14, Lemma 2.10], if $x \in \mathfrak{r}_A$ then (i), (ii) and (iii) are still equivalent; and (v) implies (iv) implies (iii). This uses the $\mathfrak{r}_A$ version above of the ($\Leftarrow$) direction of [14, Lemma 2.9], and [15, Theorem 3.2] (which gives $s(x) = s(\mathfrak{F}(x))$). However, none of the other implications in that lemma are correct in the $\mathfrak{r}_A$ case, even in ℓ^∞_2, as we said above.

Proposition 2.11 and Theorem 2.19 of [14] are correct in their $\mathfrak{r}_A$ variant, which should be phrased in terms of strictly real positive elements in $\mathfrak{r}_A$ as defined above at the start of the present section. Indeed, this variant of Proposition 2.11 is true even for nonunital algebras if in the proof we replace $C^*(A)$ by A^1. Theorem 2.19 of [14] may be seen using the parts of [14, Lemma 2.10] which are true for $\mathfrak{r}_A$ in place of $\mathfrak{F}_A$, and [15, Theorem 3.2] (which gives $s(x) = s(\mathfrak{F}(x))$). Thus we have:

Theorem 3.2. *Let A be an approximately unital operator algebra. The following are equivalent:*

(i) *A has a countable cai.*
(ii) *A has a strictly real positive element.*
(iii) *There is an element x in $\mathfrak{r}_A$ with $s(x) = 1_{A^{**}}$.*

Indeed, if $x \in \mathfrak{r}_A$ is strictly real positive then $s(x) = 1_{A^{**}}$; and the converse holds if $x \in \mathfrak{c}_A$.

Lemma 2.14 of [14] is clearly false in the $\mathfrak{r}_A$ case even in $\mathbb{C}$, however it is true with essentially the same proof if the elements x_k there are strictly real positive elements, or more generally if they are in $\mathfrak{r}_A$ and their numerical ranges in A^1 intersect the imaginary axis only possibly at 0. Also, this does not affect the correctness of the important results that follow it in [14, Section 2]. Indeed, as stated in [15], all descriptions of r-ideals and ℓ-ideals and HSA's from [14] are valid with $\mathfrak{r}_A$ in place of $\mathfrak{F}_A$, sometimes by using [15, Corollaries 3.4 and 3.5]). We remark that Proposition 2.22 of [14] is clearly false with $\mathfrak{F}_A$ replaced by $\mathfrak{r}_A$, even in $\mathbb{C}$.

Similarly, in [14, Theorem 4.1], (c) implies (a) and (b) there with $\mathfrak{r}_A$ in place of $\mathfrak{F}_A$. However, the Volterra algebra [14, Example 4.3] is an example where (a) of [14, Theorem 4.1] holds in the $\mathfrak{r}_A$ case but not (c) (note that the Volterra operator V is in $\mathfrak{r}_A$, but V is not strictly real positive in A). The results in Section 3 of [14] were discussed for the $\mathfrak{r}_A$ case in Subsection 2.4 and [13]. It follows as in [14] that if x is a strictly real positive element (in our new sense above) in a nonunital approximately unital operator algebra A, then xA is never closed. For if xA is closed then by the $\mathfrak{r}_A$ version of [14, Lemma 2.10] discussed above, we have $xA = A$. Now apply Corollary 2.13 to see that A has a left identity (which as we said in Subsection 2.4 forces it to have an identity).

Lemma 3.3. *In an operator algebra A, suppose that $x \in \mathfrak{r}_A$ and either x is strictly real positive, or the numerical range $W(x)$ of x in A^1 is contained in a sector S_ψ of angle $\psi < \pi/2$ (see notation above Lemma* 1.1). *If φ is a state on A or more generally on A^1, then $\varphi(s(x)) = 0$ iff $\varphi(x) = 0$.*

Proof. The one direction is as in [14, Lemma 2.9] as mentioned above. The strictly real positive case of the other direction is obvious (but nonvacuous in the A^1 case). In the remaining case, write $\varphi = \langle \pi(\cdot)\xi, \xi\rangle$ for a unital $*$-representation π of $C^*(A^1)$ on a Hilbert space H, and a unit vector $\xi \in H$. Then $W(\pi(x))$ is contained in a sector of the same angle. By Lemma 5.3 in Chapter IV of [39] we have $\|\pi(x)\xi\|^2 = \varphi(x^*x) = 0$. As e.g. in the proof of [14, Lemma 2.9], this gives $\varphi(s(x)) = 0$. ∎

Corollary 3.4. *Let $x \in \mathfrak{r}_A$ for an operator algebra A. If $\varphi(x^{1/n}) = 0$ for some $n \in \mathbb{N}$, $n \geq 2$, and state φ on A, then $\varphi(s(x)) = 0$ and $\varphi(x^{1/m}) = 0$ for all $m \in \mathbb{N}$. Thus if $\varphi(s(x)) \neq 0$ for a state φ on A, then* $\mathrm{Re}(\varphi(x^{1/n})) > 0$ *for all $n \in \mathbb{N}$, $n \geq 2$.*

Proof. It is clear that $s(x) = s(x^{1/m})$ for all $m \in \mathbb{N}$, by using for example the fact from [15, Section 3] that $x^{1/n} \to s(x)$ weak*. Since the numerical range of $x^{1/n}$ in A^1 is contained in a sector centered on the positive real axis of angle $< \pi$, $\varphi(s(x)) = \varphi(s(x^{1/n})) = 0$ by Lemma 3.3. As we said above, this implies that $\varphi(x) = 0$, and the same argument applies with x replaced by $x^{1/m}$ to give $\varphi(x^{1/m}) = 0$.

The last statement follows from this, since $\mathrm{Re}(\varphi(x^{1/n})) > 0$ is equivalent to $\varphi(x^{1/n}) \neq 0$ if $n \geq 2$. ∎

Remark. Examining the proofs of the last few results shows that they are valid if states on A are replaced by nonzero functionals that extend to states on A^1, or equivalently extend to a C^*-algebra generated by A^1.

Corollary 3.5. *In an operator algebra A, if $x \in \mathfrak{r}_A$ and x is strictly real positive, then $x^{1/n}$ is strictly real positive for all $n \in \mathbb{N}$.*

Proof. If $x^{1/n}$ is not strictly real positive for some $n \geq 2$, then $\varphi(x^{1/n}) = 0$ for some state φ of A^1 which is nonzero on A. Such a state extends to a state on $C^*(A^1)$. By the last Remark, $\varphi(x) = 0$ by Corollary 3.4, a contradiction. ∎

We recall that for a C^*-algebra B, an open projection $p \in B^{**}$ is *σ-compact* if it is the supremum (or weak* limit) of an increasing sequence in B_+ [36]. It is well known from C^*-algebra theory that this is equivalent to saying that p is the support projection of a closed right ideal in B which has a countable left cai; and also equivalent to saying that there exists a strictly positive element in the hereditary subalgebra defined by p.

Lemma 3.6. *If A is a closed subalgebra of a C^*-algebra B, and if p is an open projection in A^{**}, then the following are equivalent:*

(i) *p is the support projection of a closed right ideal in A with a countable left cai.*
(ii) *p is σ-compact in B^{**} in the sense above.*
(iii) *p is the support projection of a closed right ideal in A of the form $\overline{xA}$ for some $x \in \mathfrak{r}_A$. That is, $p = s(x)$ for some $x \in \mathfrak{r}_A$.*
(iv) *There is a sequence $x_n \in \mathfrak{r}_A$ with $x_n = px_n \to p$ weak*.*
(v) *The hereditary subalgebra D of A associated with p contains an element x which is strictly real positive with respect to D.*

If these hold then the sequence (x_n) in (iv) *can be chosen to be increasing with respect to $\preccurlyeq$, and they, and the element x in* (iii) *and* (v), *can be chosen to be in $\frac{1}{2}\mathfrak{F}_A$ and nearly positive. Of course $s(x) = p$ if x is as in* (v).

Proof. We know from the theory in [14, 15] that (i) and (iii) are equivalent, and the element x in (iii) can be chosen to be in $\frac{1}{2}\mathfrak{F}_A$ and nearly positive. Indeed, one direction is similar to the argument in the paragraph after Corollary 2.8. That these imply (iv) is similar, clearly $x_n = x^{1/n}$ has the desired properties for n large enough.

(iv)⇒(iii). If $x_n \in \mathfrak{r}_A$ with $x_n = px_n \to p$ weak*, then p is the support projection of the closed right ideal $J = \{a \in A : pa = a\}$. Indeed, it is easy to see that $J^{\perp\perp} = pA^{**}$. Note that $\overline{\sum_k x_n A}$ is a left ideal in A and J, but actually equals J since its weak* closure contains p and hence contains $pA^{**} = J^{\perp\perp}$. By [14, Proposition 2.14] and [15, Corollary 3.5], $\overline{\sum_k x_n A} = \overline{xA}$ for some $x \in \mathfrak{r}_A$.

(iii)$\Rightarrow$(v). Suppose that D is the hereditary subalgebra defined by p. If $x \in \frac{1}{2}\mathfrak{F}_A$ is as in (iii) then $x \in D = \overline{xAx}$, and

$$\overline{xD} \subset D \subset \overline{x^{1/2}x^{1/2}Ax} \subset \overline{x^{1/2}D} \subset \overline{xAD} \subset \overline{xD}.$$

So $D = \overline{xD}$, and by Theorem 3.2, x is a strictly real positive element of D.

(v)$\Rightarrow$(iii). If $x \in \mathfrak{r}_D \subset \mathfrak{r}_A$ is a strictly real positive element of D then $p = w^* \lim_n x^{1/n} = s(x)$ by Theorem 3.2.

(i)$\Leftrightarrow$(ii). Let $I = pB^{**} \cap B$ and $J = pA^{**} \cap A$. As we said, (ii) is equivalent to I having a countable left cai. From [9, Section 2] we have $I = JB$. So if J has a countable left cai then so does I. Similarly, any left cai for J is a left cai for I. If I has a countable left cai (f_n), choose elements e_n from a left cai for J such that $\|e_n f_n - f_n\| < 2^{-n}$. Then since

$$e_n a = e_n(a - f_n a) + (e_n f_n - f_n)a + f_n a, \qquad a \in A,$$

it is clear that J has a countable left cai. ∎

A similar result holds for left ideals or HSA's.

If A is an operator algebra then an open projection $p \in A^{**}$ will be said to be *σ-compact* with respect to A if it satisfies the equivalent conditions in the previous result. These projections, and the above lemma, will be used in our 'strict Urysohn lemma' in Section 4. Note that if A is separable then every open projection in A^{**} is σ-compact with respect to A, by [14, Corollary 2.17].

4. Positivity in the Urysohn lemma and peak interpolation. In our previous work [9, 14, 12, 15] we had two main settings for noncommutative Urysohn lemmata for a subalgebra A of a C^*-algebra B. In both settings we have a compact projection $q \in A^{**}$, dominated by an open projection u in B^{**}, and we seek to find $a \in \mathrm{Ball}(A)$ with $aq = qa = q$, and both $au^{\perp}$ and $u^{\perp}a$ either small or zero. In the first setting $u \in A^{**}$ too, whereas this is not required in the second setting. We now ask if in both settings one may also have $a \in \frac{1}{2}\mathfrak{F}_A$ and nearly positive (hence 'positive' in our new sense, and as close as we like to a positive operator in the usual sense). In the first setting, all works perfectly:

THEOREM 4.1. *Let A be an operator algebra* (*not necessarily approximately unital*), *and let $q \in A^{**}$ be a compact projection which is dominated by an open projection $u \in A^{**}$. Then there exists nearly positive $a \in \frac{1}{2}\mathfrak{F}_A$ with $aq = qa = q$, and $au = ua = a$.*

Proof. The proofs of [12, Theorem 2.6] and [15, Theorem 6.6 (2)] show that this all can be done with $a \in \frac{1}{2}\mathfrak{F}_A$. Then $a^{1/n}q = qa^{1/n} = q$, as is clear for example using the power series form $a^{1/n} = \sum_{k=0}^{\infty} \binom{1/n}{k}(-1)^k(1-a)^k$ from [14, Section 2], where it is also shown that $a^{1/n} \in \frac{1}{2}\mathfrak{F}_A$. Similarly

$a^{1/n}u = ua^{1/n} = a^{1/n}$, since u is the identity multiplier on $\mathrm{oa}(a)$, and $\mathrm{oa}(a)$ contains these roots [14, Section 2]. That the numerical range of $a^{1/n}$ lies in a cigar centered on the line segment $[0, 1]$ in the x-axis, of height $< \epsilon$, is as in the proof of [14, Theorem 2.4]. ∎

We now turn to the second setting (see e.g. [15, Theorem 6.6 (1)]), where the dominating open projection u is not required to be in $A^{\perp\perp}$. Of course if A has no identity or cai then one cannot expect the 'interpolating' element a to be in $\frac{1}{2}\mathfrak{F}_A$ or $\mathfrak{r}_A$. This may be seen clearly when A is the functions in the disk algebra vanishing at 0. Here $\frac{1}{2}\mathfrak{F}_A$ and $\mathfrak{r}_A$ are (0). Indeed, by the maximum modulus theorem for harmonic functions there are no nonconstant functions in this algebra which have nonnegative real part. The remaining question is the approximately unital case 'with positivity'. We solve this next, also solving the questions posed at the end of [12].

THEOREM 4.2. *Let A be an approximately unital subalgebra of a C^*-algebra B, and let $q \in A^{\perp\perp}$ be a compact projection.*

(1) *If q is dominated by an open projection $u \in B^{**}$ then for any $\epsilon > 0$, there exists an $a \in \frac{1}{2}\mathfrak{F}_A$ with $aq = qa = q$, and $\|a(1-u)\| < \epsilon$ and $\|(1-u)a\| < \epsilon$. Indeed, this can be done with a in addition nearly positive (thus the numerical range (and spectrum) of a within a horizontal cigar centered on the line segment $[0, 1]$ in the x-axis, of height $< \epsilon$).*
(2) *q is a weak* limit of a net (y_t) of nearly positive elements in $\frac{1}{2}\mathfrak{F}_A$ with $y_t q = q y_t = q$.*

Proof. (2) First assume that $q = u(x)$ (this was defined in the Introduction) for some $x \in \frac{1}{2}\mathfrak{F}_A$. We may replace A by the commutative algebra $\mathrm{oa}(x)$, and then q is a minimal projection, since $qp(x) \in \mathbb{C}q$ for any polynomial p. Now q is closed and compact in $(A^1)^{**}$, so by the unital case of (2), which follows from [14, Theorem 2.24] and the closing remarks to [12], there is a net $(z_t) \in \frac{1}{2}\mathfrak{F}_{A^1}$ with $z_t q = q z_t = q$ and $z_t \to q$ weak*. Let $y_t = z_t^{1/2} x^{1/2}$. By [7, Lemma 4.2(3)], we have $y_t \in \frac{1}{2}\mathfrak{F}_{A^1} \cap A = \frac{1}{2}\mathfrak{F}_A$. Also, $x^{1/2} q = q x^{1/2} = q$ by considerations used in the last proof, and similarly $z_t^{1/2} q = q z_t^{1/2} = q$. Thus $y_t^{1/2} q = q y_t^{1/2} = q$. If A is represented nondegenerately on a Hilbert space H, and we identify 1_{A^1} with I_H, then for any $\zeta \in H$, by a result at the end of the Introduction we have

$$\|(y_t - q)\zeta\| = \|(z_t^{1/2} - q)x^{1/2}\zeta\| \leq K\|(z_t - q)x^{1/2}\zeta\|^{1/2} \to 0.$$

Thus $y_t \to q$ strongly and hence weak*.

Next, for an arbitrary compact projection $q \in A^{\perp\perp}$, by [12, Theorem 3.4] there exists a net $x_s \in \frac{1}{2}\mathfrak{F}_A$ with $u(x_s) \searrow q$. By the last paragraph there exist nets $y_t^s \in \frac{1}{2}\mathfrak{F}_A$ with $y_t^s u(x_s) = u(x_s) y_t^s = u(x_s)$, and $y_t^s \to u(x_s)$ weak*.

Then

$$y_t^s q = y_t^s u(x_s) q = u(x_s) q = q$$

for all t, s. It is clear that the y_t^s can be arranged into a net weak* convergent to q.

(1) If A is unital then the first assertion of (1) is [14, Theorem 2.24]. In the approximately unital case, by the ideas in the closing remarks to [12], the first assertion of (1) should be equivalent to (2). Indeed,, by substituting such a net (y_t) into the proof of [12, Theorem 2.1] one obtains the first assertion of (1).

Finally, we obtain the 'cigar' assertion. For (y_t) as in (2), similarly to the last paragraph we substitute the net $(y_t^{1/m})$ into the proof of [12, Theorem 2.1]. Here m is a fixed integer so large that the numerical range of $y_t^{1/m}$ lies within the appropriate horizontal cigar. As in the proof of the previous theorem, $y_t^{1/m} q = q y_t^{1/m} = q$ and $y_t^{1/m} \to q$ weak* with t since if $\zeta \in H$ again then

$$\|(y_t^{1/m} - q)\zeta\| \leq \|(y_t - q)\zeta\|^{1/m} \to 0$$

by the inequality at the end of the Introduction. ∎

Remark. The recent paper [20] contains a special kind of 'Urysohn lemma with positivity' for function algebras. It seems that our Urysohn lemma applied to a function algebra has much weaker (fewer) hypotheses, and has stronger conclusions except that our interpolating element has range in the usual thin cigar in the right half-plane which we like to use, and this is contained in their Stolz region which contains 0 as an interior point, except for a tiny region just to the left of 1. Hopefully our results could be helpful in such applications.

We now turn to our analogue of the 'strict Urysohn lemma'. We recall that the classical form of the strict Urysohn lemma in topology finds a positive continuous function which is 0 and 1 on the two given closed sets, and which is strictly between 0 and 1 outside of these two sets. The latter is essentially equivalent to saying that there is a positive contraction f in the algebra such that the given closed sets are peak sets for f and $1 - f$.

With this in mind we state some preliminaries related to peak projections, the noncommutative generalization of peak sets. There are many equivalent definitions of peak projections (see e.g. [28, 9, 12, 15]), but basically they are the closed projections q for which there is a contraction x with $xq = qx = q$ (so x 'equals one on' q), and $|x| < 1$ in some sense (which is made precise in the above references) on $q^\perp$; we write $q = u(x)$. More generally, if B is a C^*-algebra and $x \in \mathrm{Ball}(B)$ one can define $u(x) = w^* \lim_n x(x^*x)^n$, which always exists in B^{**} and is a partial isometry (see e.g. [24] and [12, Lemma 3.1]). When this is a nonzero projection q it is a peak projection and equals $w^* \lim_n x^n$, and we say that x *peaks at* q; if in addition A is a closed subalgebra of B with $x \in A$ and $q \in A^{\perp\perp}$ then we say that q is a peak projection for (or relative to) A. This is the case for example for any element of $\frac{1}{2}\mathfrak{F}_A$ of norm 1 (or equivalently, with 1 in its spectrum, or attaining value 1 on some state; see [12, Corollary 3.3]), for any closed subalgebra A of B, and here the peak projection $u(x)$ is in A^{**}. However, it need not be the case for any norm 1 real positive element, even in a unital C^*-algebra. For example, if V is the Volterra operator, which is accretive, and $x = \frac{1}{\|V+\epsilon I\|}(V + \epsilon I)$, then one can show that $w^* \lim_n x^n = 0$.

The following is implicit in [12, Lemma 3.1]:

LEMMA 4.3. *Suppose that B is a C^*-algebra, $x \in \mathrm{Ball}(B)$ and $q \in B^{**}$ is a closed projection with $qx = q$. Then x peaks at q (that is, q is a peak projection and equals $u(x)$) iff $\varphi(x^*x) < 1$ for every state φ of B with $\varphi(q) = 0$.*

Proof. ($\Leftarrow$) This follows from [12, Lemma 3.1].

($\Rightarrow$) If $q = u(x)$ then the last assertions of [12, Lemma 3.1] show that (3) there holds. If φ is a state of B with $\varphi(q) = 0$, then $\varphi(1-q) = 1$ and by Cauchy–Schwarz $\varphi(x^*xq) = 0$, So $\varphi(x^*x) < 1$ by (3) there. ∎

REMARK. In place of using states with $\varphi(q) = 0$ in the lemma and its application below, one can use minimal or compact projections dominated by $1 - q$, as in the proof of [12, Theorem 3.4(2)].

LEMMA 4.4. *Suppose that A is an approximately unital operator algebra, with $e = 1_{A^{**}}$ as usual, that $q \in A^{**}$ is compact, and that $p = q^\perp = e - q$ is σ-compact in A^{**}. Then q is a peak projection for A, indeed $q = u(x)$ for some nearly positive $x \in \frac{1}{2}\mathfrak{F}_A$.*

Proof. It is only necessary to find such $x \in \frac{1}{2}\mathfrak{F}_A$; the claim about near positivity will follow from [12, Corollary 3.3]. If A is unital then by [14, Proposition 2.22] we have $q = s(a)^\perp = u(1-a)$, and we are done. If A is nonunital, by the above applied in A^1 we have $1 - s(a) = u(b)$ where $b = 1 - a \in \frac{1}{2}\mathfrak{F}_{A^1}$. Since q is compact there exists $r \in \mathrm{Ball}(A)$ with $q = rq$. We follow the idea in the proof of [12, Theorem 3.4(3)]. Let $d = rb \in \mathrm{Ball}(A)$. Then

$$dq = rbq = rb(1-p)q = r(1-p)q = rq = q.$$

If $\varphi \in S(B)$ with $\varphi(q) = 0$, then φ extends to a state $\psi \in S(B^1)$ with $\psi(q) = 0$. By Lemma 4.3 applied in B^1 we have $\psi(b^*b) < 1$, so that

$$\varphi(d^*d) = \psi(d^*d) < \psi(b^*b) < 1.$$

Thus $q = u(d)$ is a peak projection for A by [12, Corollary 3.3]. By [12, Theorem 3.4(3)], $q = u(x)$ for some $x \in \frac{1}{2}\mathfrak{F}_A$. ∎

Corollary 4.5. *Suppose that A is a* (*not necessarily approximately unital*) *operator algebra, and B is a C^*-algebra containing A. If a peak projection for B lies in $A^{\perp\perp}$ then it is also a peak projection for A.*

Proof. Suppose that $q = u(b) \in A^{\perp\perp} \subset (A^1)^{\perp\perp}$ for some $b \in \frac{1}{2}\mathfrak{F}_B$. Then by [14, Proposition 2.22], $s(1-b) = 1-q$ is a σ-compact projection in $(A^1)^{\perp\perp}$. So by Lemma 3.6 we have $1-q = s(a)$ for some $a \in \frac{1}{2}\mathfrak{F}_{A^1}$. By [14, Proposition 2.22] again, $q = u(1-a)$. By [15, Proposition 6.4], q is a peak projection for A. ∎

Theorem 4.6 (A strict noncommutative Urysohn lemma for operator algebras). *Suppose that A is any* (*possibly not approximately unital*) *operator algebra and that q and p are respectively compact and open projections in A^{**} with $q \leq p$, and $p-q$ σ-compact* (*note that the latter is automatic if A is separable*). *Then there exists $x \in \frac{1}{2}\mathfrak{F}_A$ such that $xq = qx = q$, $xp = px = x$, and x peaks at q* (*that is, $u(x) = q$*) *and $s(x) = p$, and $1-x$ peaks at $1-p$ with respect to A^1* (*that is, $u(1-x) = 1-p$*). *The latter identities imply that x is real strictly positive in the hereditary subalgebra C associated with p, and $1-x$ is real strictly positive in the hereditary subalgebra in A^1 associated with $1-q$. Also, $s(x(1-x)) = p-q$, so that $x(1-x)$ is real strictly positive in the hereditary subalgebra in A associated with $p-q$. We can also have x 'almost positive', in the sense that if $\epsilon > 0$ is given one can choose x as above but also satisfying $\mathrm{Re}(x) \geq 0$ and $\|x - \mathrm{Re}(x)\| < \epsilon$.*

Proof. That $p-q$ is automatically σ-compact if A is separable follows from the fact at the end of Section 3. Consider the hereditary subalgebra C associated with p. It is clear e.g. from Lemma 3.6 that $p-q$ is a σ-compact projection with respect to C. Applying Lemma 4.4 in C, we can choose $b \in \frac{1}{2}\mathfrak{F}_C \subset \frac{1}{2}\mathfrak{F}_A$ with $u(b) = q$. By the last Urysohn lemma above, we can choose $r \in \frac{1}{2}\mathfrak{F}_A$ with $rp = pr = r$ and $rq = qr = q$. The argument in the proof of [12, Theorem 3.4(3)] shows that the closed algebra D generated by $x = rbr$ and b is approximately unital, and that there is an element $f_2 \in D \cap \frac{1}{2}\mathfrak{F}_A$ with $u(f_2) = q$. Note that $f_2 p = pf_2 = f_2$. By taking roots we can assume that f_2 is nearly positive.

Similarly, but working in A^1, one sees that there is a nearly positive $f_1 \in \frac{1}{2}\mathfrak{F}_{A^1}$ with $f_1 q = qf_1 = 0$ and $u(f_1) = 1-p$. We have $f_1(1-p) = 1-p$, which implies that $(1-f_1)p = 1-f_1$. Let $x = \frac{1}{2}(f_2 + (1-f_1)) \in \frac{1}{2}\mathfrak{F}_{A^1}$. Since f_1, f_2 are nearly positive, it is easy to see that x is almost positive in the sense above, by a variant of the computation involving $\|\mathrm{Im}(x)\|$ in one of the early paragraphs of our paper. We have $1-x = \frac{1}{2}((1-f_2)+f_1)$. Within $(A^1)^{**}$, by [12, Proposition 1.1] (and the fact that a tripotent dominated by a projection in the natural ordering on tripotents is a projection) we have

$$u(x) = u(f_2) \wedge u(1-f_1) = u(f_2) = q,$$

since $(1-f_1)q = q$ which implies $u(1-f_1) \geq q$. Similarly

$$u(1-x) = u(1-f_2) \wedge u(f_1) = u(f_1) = 1-p,$$

since $(1-f_2)(1-p) = 1-p$ and so $u(1-f_2) \geq 1-p$.

Since $xp = x$ and $p \in A^{\perp\perp}$, and $A^{\perp\perp}$ is an ideal in $(A^1)^{**}$, we see that $x \in A^{\perp\perp} \cap A^1 = A$.

Note that $\|1-4(x-x^2)\| = \|(1-2x)^2\| \leq 1$, so $x(1-x) \in \frac{1}{4}\mathfrak{F}_A$. Then by Lemma 2.14, $s(x(1-x))$ may be regarded as the strong limit of $(x(1-x))^{1/n} = x^{1/n}(1-x)^{1/n}$ (see e.g. [7] for the last identity), which is $s(x)s(1-x) = p(1-q) = p-q$. The 'strictly real positive' assertions follow from Lemma 3.6. ∎

Remarks. 1) One may replace the hypothesis in Theorem 4.6 that $p-q$ be σ-compact by the premise that both q and $1-p$ are peak projections (in A and A^1). Indeed, if $q = u(w), 1-p = u(z)$, then by [12, Corollary 3.5] $1+q-p = u(w)+u(z) = u(k)$ say. Hence by [14, Proposition 2.22] and Lemma 3.6 we conclude that $p-q = 1-u(k) = s(k)$ is σ-compact.

2) Under a commuting hypothesis we offer a quicker proof inspired by the proof of [36, Theorem 2]: choose $b \in \frac{1}{2}\mathfrak{F}_A$ with $s(b) = p-q$. Then if r is as in the last proof, and $br = rb$, set $x = (1-r)b + (1-b)r$. Then $1-2x = (1-2b)(1-2r)$, a contraction, so that $x \in \frac{1}{2}\mathfrak{F}_A$, and it is easy to see that $xq = q$ and $xp = x$.

We give an application of our strict noncommutative Urysohn lemma to the lifting of projections, a variant of [36, Corollary 4]. First we will need a sharpening of [14, Proposition 6.2]. Recall that if A is an operator algebra containing a closed approximately unital two-sided ideal J with support projection p, then p is central in $(A^1)^{**}$ since J is a two-sided ideal. We may view $A/J \subset A^{**}(1-p)$ via the map $a + J \mapsto a(1-p)$, in view of the identifications

$$(A/J)^{**} \cong A^{**}/J^{\perp\perp} \cong A^{**}/A^{**}p \cong A^{**}(1-p) \subset A^{**}.$$

Lemma 4.7. *Let A be an operator algebra containing a closed approximately unital two-sided ideal J with support projection p, and suppose that D is a HSA in A/J. Regarding $(A/J)^{**} \cong A^{**}(1-p)$ as above, let r be the projection in $A^{**}(1-p)$ corresponding to the support projection of D. Then the preimage of D in A under the quotient map is a HSA in A with support projection $p+r$.*

Proof. By the proof of [14, Proposition 6.2 and Corollary 6.3], the preimage C of D in A is a HSA in A, and $C/J \cong D$. Thus $C^{**} \cong J^{\perp\perp} \oplus^\infty D^{**}$, and we can view the isomorphism $C^{**} \to J^{\perp\perp} \oplus^\infty D^{**}$ here as the restriction of the completely isometric map $\eta \mapsto (\eta p, \eta(1-p))$ setting up the isomorphism $A^{**} \cong J^{\perp\perp} \oplus^\infty A^{**}(1-p)$. If $\eta \in A^{**}$ with $\eta(1-p) \in rA^{**}r \cong D^{**}$, then

$\eta \in \eta p + rA^{**}r \subset (p+r)A^{**}(p+r)$. Hence

$$C^{\perp\perp} = \{\eta \in A^{**} : \eta(1-p) \in rA^{**}r\} = (p+r)A^{**}(p+r).$$

Thus $p+r$ is the support projection of C, so is open. ∎

COROLLARY 4.8 (cf. [36, Corollary 4]). *Let A be an operator algebra containing a closed two-sided ideal J with a countable cai, or equivalently, with a σ-compact (as defined after Lemma 3.6) support projection p. Also, suppose that q is a projection in A/J. Then there exists an almost positive (in the sense of the last theorem) $x \in \frac{1}{2}\mathfrak{F}_A$ such that $x + J = q$. Also, the peak $u(x)$ for x equals the canonical copy of q in $A^{**}(1-p)$.*

Proof. That J has a countable cai iff it has a σ-compact support projection follows from e.g. Lemma 3.6. By [14, Proposition 2.22], q has a lift $y \in \frac{1}{2}\mathfrak{F}_A$, so that the copy of q in $A^{**}(1-p)$ is $r = y(1-p)$. Thus r is a projection in A^{**}. Also, $r = (y(1-p))^n = y^n(1-p) \to u(y)(1-p)$ weak*. This implies that $r = u(y)(1-p) = u(y) \wedge (1-p)$ is a closed projection in A^1, hence is a compact projection in A^{**}. Clearly $r = yr$. By Lemma 4.7 the projection $p+r$ is open in A^{**}, and it dominates r. We apply Theorem 4.6 to see that there exists an almost positive $x \in \frac{1}{2}\mathfrak{F}_A$ such that $u(x) = r$, and $xr = rx = r$ and $x(p+r) = (p+r)x = x$. Thus $y(1-p) = r = xr = x(1-p)$, and so $x + J = q$. ∎

REMARK. In the last result, by Theorem 4.6 we can have x real strictly positive in the HSA associated with $p+r$; and also $s(x(1-x)) = p$, so that $x - x^2$ is real strictly positive in J.

We now turn to noncommutative peak interpolation. The following is an improvement of [8, Lemma 2.1].

PROPOSITION 4.9. *Suppose that A is an approximately unital operator algebra, and B is a C^*-algebra generated by A. If $c \in B_+$ with $\|c\| < 1$ then there exists an $a \in \frac{1}{2}\mathfrak{F}_A$ with $|1-a|^2 \le 1 - c$. Indeed, such an a can be chosen to also be nearly positive.*

Proof. By Theorem 2.1(2′), there exists a nearly positive $a \in \frac{1}{2}\mathfrak{F}_A$ with

$$c \le \mathrm{Re}(a) \le 2\,\mathrm{Re}(a) - a^*a$$

since $a^*a \le \mathrm{Re}(a)$ if $a \in \frac{1}{2}\mathfrak{F}_A$. Thus $|1-a|^2 \le 1-c$. ∎

In the last result one cannot hope to replace the hypothesis $\|c\| < 1$ by $\|c\| \le 1$, as can be seen with the example in Remark 1 after Theorem 2.1.

Proposition 4.9, like several other results in this paper, is equivalent to Read's theorem from [37]. Indeed, if e is an identity of norm 1 for A^{**}, and if we choose $a_t \in \frac{1}{2}\mathfrak{F}_A$ with $|e - a_t|^2 \le e - e_t$, where (e_t) is any positive cai in $\mathcal{U}_B$, then

$$|\langle (e-a_t)\zeta, \eta\rangle|^2 \le \|(e-a_t)\zeta\|^2 = \langle |e-a_t|^2\zeta, \zeta\rangle \le \langle (e-e_t)\zeta, \zeta\rangle \to 0$$

for all $\zeta \in H$. Thus e is a weak* limit of a net in $\frac{1}{2}\mathfrak{F}_A$, and hence by the usual argument there exists a cai in $\frac{1}{2}\mathfrak{F}_A$.

As in [8, Lemma 2.1], Proposition 4.9 can be interpreted as a noncommutative peak interpolation result. Namely, if the projection $q = 1_{A^1} - e$ is dominated by $d = 1 - c$ then there exists an element $g = 1 - a \in A^1$ with $gq = qg = q$, and $g^*g \leq d$. The new point is that a is in $\frac{1}{2}\mathfrak{F}_A$ and nearly positive.

This leads one to ask whether the other noncommutative peak interpolation results we have obtained in earlier papers can also be done with the interpolating element in $\frac{1}{2}\mathfrak{F}_A$, or more generally with the interpolating element having prescribed numerical range. We will discuss this below. As discussed at the end of [13], lifting elements without increasing the norm, while keeping the numerical range in a fixed compact convex subset E of the plane, may be regarded as a kind of Tietze extension theorem. (In the usual Tietze theorem $E = [-1, 1]$. It should be pointed out that in the usual Tietze theorem one can lift elements from the multiplier algebra, whereas here we are being more modest.) We refer the reader to [19, Section 3] for a discussion of some other kinds of Tietze theorems for C^*-algebras.

The following two theorems may be regarded as peak interpolation theorems 'with positivity'. They are sharpenings of [15, Theorem 5.1] (see also Corollary 2.2 in that reference).

THEOREM 4.10. *Suppose that A is an operator algebra* (*not necessarily approximately unital*), *and q is a closed projection in $(A^1)^{**}$. Suppose that $b \in A$ with $bq = qb$ and $\|bq\| \leq 1$, and $\|(1-2b)q\| \leq 1$. Then there exists an element $g \in \frac{1}{2}\mathfrak{F}_A \subset \mathrm{Ball}(A)$ with $gq = qg = bq$.*

Proof. We modify the proof of [15, Theorem 5.1]. In that proof a closed subalgebra C of A^1 is constructed which contains b and 1_{A^1}, such that q is in the center of $C^{\perp\perp} \cong C^{**}$. So $q^\perp$ supports a closed two-sided ideal J in C. Then we set $I = C \cap A$, an ideal in C containing b. Finally, an M-ideal D in I was constructed there; this will be an approximately unital ideal in I. Using the language of the proof of [15, Theorem 5.1], since $P(I^\perp) \subset I^\perp$ it follows that $I^{\perp\perp}$ is invariant under P^*. By [27, Proposition I.1.16] we see that $I + \tilde{D}$ is closed, hence it follows similarly to the centered equation in the proof of [14, Proposition 7.3], and the two lines above it, that

$$D^{\perp\perp} = (I \cap \tilde{D})^{\perp\perp} = I^{\perp\perp} \cap \tilde{D}^{\perp\perp} = (1-q)C^{**} \cap I^{\perp\perp} = (1-q)I^{\perp\perp}.$$

Thus the M-projection from I^{**} onto $D^{\perp\perp}$ is multiplication by $1-q$, which is also the restriction of P^* to $I^{\perp\perp}$. Now I/D is an operator algebra; indeed, it may be viewed, via the map $x + D \mapsto qx$, as a subalgebra of

$$(I/D)^{**} \cong I^{**}/D^{\perp\perp} \cong qI^{\perp\perp} \subset qC^{**} \subset q(A^1)^{**}q.$$

Indeed, it is not hard to see that I/D may be regarded as an ideal in the unital subalgebra C/J of qC^{**}, where J was defined above.

If $\|(1-2b)q\| \leq 1$ then $bq \in \frac{1}{2}\mathfrak{F}_{q(A^1)^{**}q}$, so that $b+D \in \frac{1}{2}\mathfrak{F}_{I/D}$. Hence by [14, Proposition 6.1] there exists $g \in \frac{1}{2}\mathfrak{F}_I \subset \frac{1}{2}\mathfrak{F}_A$ with $g+D = b+D$. We have $gq = qg = bq$. ∎

We will need a simple corollary of Meyer's theorem mentioned in the introduction:

LEMMA 4.11. *Suppose that A and B are closed subalgebras of unital operator algebras C and D respectively, with $1_C \notin A$ and $1_B \notin D$, and $q : A \to B$ is a complete quotient map and homomorphism. Then the unique unital extension of q to a unital map from $A+\mathbb{C}1_C$ to $B+\mathbb{C}1_D$ is a complete quotient map.*

Proof. Let $J = \operatorname{Ker} q$, let $\tilde{q} : A/J \to B$ be the induced complete isometry, and let $\theta : A + \mathbb{C}1_C \to B + \mathbb{C}1_D$ be the unique unital extension of q. This gives a one-to-one homomorphism $\tilde{\theta} : (A+\mathbb{C}1_C)/J \to B+\mathbb{C}1_D$ which equals $\tilde{q}$ on A/J. If B, and hence A/J, is not unital then $\tilde{\theta}$ is a completely isometric isomorphism by Meyer's result mentioned in the Introduction (since both $(A+\mathbb{C}1_C)/J$ and $B+\mathbb{C}1_D$ are 'unitizations' of $A/J \cong B$). Similarly, if B is unital, then $\tilde{\theta}$ is a completely isometric isomorphism by the (almost trivial) uniqueness of the unitization of an already unital operator algebra. So in either case we may deduce that $\tilde{\theta}$ is a complete isometry and θ is a complete quotient map. ∎

The following is a noncommutative peak interpolation theorem which is also, as discussed in the paragraph before Theorem 4.10, a kind of 'Tietze theorem'. It also yields a peak interpolation theorem 'with positivity': if one insists that the set E appearing here lies in the right half-plane, or in the usual 'cigar' centered on $[0, 1]$, then the interpolation or extension is preserving 'positivity' in our new sense.

THEOREM 4.12 (A noncommutative Tietze theorem). *Suppose that A is an operator algebra (not necessarily approximately unital), and q is a closed projection in $(A^1)^{**}$. Suppose that $b \in A$ with $bq = qb$ and $\|bq\| \leq 1$, and the numerical range of bq (in e.g. $q(A^1)^{**}q$) is contained in a compact convex set E in the plane. Also suppose, by fattening it slightly if necessary, that E is not a line segment. If both A is nonunital and $q \in A^{\perp\perp}$, then we will also insist that $0 \in E$. Then there exists $g \in \operatorname{Ball}(A)$ with $gq = qg = bq$ such that the numerical range of b with respect to A^1 is contained in E.*

Proof. This is the same as the proof of the last theorem except that the last paragraph should be replaced by the following. Suppose that the numerical range $W_{qC^{**}}(bq)$ lies in the convex set E described. If $1_{A^1} \in I$ (which is the case for example if A is unital) then I/D viewed in qC^{**} as above has identity q. Then the numerical range of $b+D$ in I/D is a subset of E. By [21, Theorem 3.1] and the Claim at the end of [13], there exists a contractive lift $g \in I \subset C$ with numerical range with respect to C, and hence with respect to A^1, contained in E. We have $gq = qg = bq$ since $g+D = b+D$. This proves the result. Thus henceforth we can assume that $1_C = 1_{A^1} \notin I$ and A is nonunital.

Next suppose that the copy qI of I/D in qC^{**} above does not contain q. This will be the case for example if $q \notin A^{\perp\perp}$ (for if $q = qx$ for some $x \in I$ then $q \in qA \subset A^{\perp\perp}$, since the latter is an ideal in $(A^1)^{**}$). By Lemma 4.11 we can extend the quotient map $I \to I/D$ to a complete quotient map $\theta : I + \mathbb{C}1_C \to I/D + \mathbb{C}q$ (the latter viewed as above in qC^{**}). By [21, Theorem 3.1] and the Claim at the end of [13], there exists a contractive lift $g \in I + \mathbb{C}1_C$ with numerical range with respect to C, and hence with respect to A^1, contained in E. If $g = x + \lambda 1_C$ with $x \in I$ then $b + D = g + D = \lambda q + x + D \in I/D$, which forces $\lambda = 0$. So $g \in I \subset A$, and $gq = qg = bq$ again as above. Finally, suppose that I/D contains q, and $0 \in E$, so that $E = [0,1]E$. Here q is the identity of C/J (viewed as above in qC^{**}). Since I/D is an ideal, we have $I/D = C/J$. Consider $I \oplus c_0$ and its ideal $D \oplus (0)$. The quotient here is $(I/D) \oplus c_0$, which may be viewed as a subalgebra of $qC^{**} \oplus c$. The numerical range of an element $(x, 0)$ in a direct sum $A_1 \oplus^\infty A_2$ of unital Banach algebras is easily seen to be $[0,1]W_{A_1}(x)$. Hence the numerical range of $(b+I, 0)$ in $(I/D) \oplus c$ is contained in $[0,1]E = E$. By Lemma 4.11 the canonical complete quotient map $I \oplus c_0 \to (I/D) \oplus c_0$ extends to a unital complete quotient map $(I \oplus c_0) + \mathbb{C}(1_C, \vec{1}) \to (I/D) \oplus c$. By [21, Theorem 3.1] and the Claim at the end of [13], there exists a contractive lift $(g, 0) \in (I \oplus c_0) + \mathbb{C}(1_C, \vec{1})$ whose numerical range in the latter space, and hence in $C \oplus c$, is contained in E. By the Banach algebra sum fact a few lines earlier, we deduce that $W_C(g) \subset E$, and hence $W_{A^1}(g) \subset E$. Clearly $g \in I \subset A$, and $gq = qg = bq$ as before. ∎

Remark. By considering examples such as $C_0((0,1])/C_0((0,1)) \cong \mathbb{C}$ one sees the necessity of the condition $0 \in E$ if A is nonunital and $q \in A^{\perp\perp}$. As in [21], by considering the quotient of the disk algebra by an approximately unital codimension 2 ideal, one sees the necessity of the condition that E not be a line segment. We remark that the case where E is a point is technically covered by the theorem: in this case bq is a scalar multiple of q. The case where the scalar is zero, or where A is unital, is trivial. Otherwise $q \in A^{\perp\perp}$, where the hypotheses force $[0,1]E = E$.

Our best noncommutative peak interpolation result [8, Theorem 3.4] (and its variant [15, Corollary 5.4]) should also have 'positive/Tietze versions' analogous to the two cases considered in Theorems 4.10 and 4.12 above. However, there is an obstacle to using the approach for the latter results to improve [8, Theorem 3.4] say. Namely the quotient one now has to deal with is $(If)/(Df)$ as opposed to I/D. (We remark that unfortunately in the proof of [8, Theorem 3.4] we forgot to repeat that $f = d^{-1/2}$, as was the case in the earlier proof from that paper that it is mimicking.) This is not an operator algebra quotient, and so we are not sure at this point how to deal with it. We remark that the Tietze variant here initially seems promising, since the key tool above used in that case is the numerical range lifting result from [21], and this is stated in that paper in utmost generality. However, we were not able to follow the proof of the latter in this generality, although as we said at the end of [13] we were able to verify it in the less general setting needed in the last proof, and in [13].

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