STUDIA MATHEMATICA 165 (3) (2004)

# Minimal displacement in Hilbert spaces

by

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**Abstract.** We give a lower bound for the minimal displacement characteristic in Hilbert spaces.

**1. Introduction and notation.** In this paper we study the minimal displacement problem. Roughly speaking, this problem is connected with a quantitative measure of lack of fixed points of a mapping. More precisely, if $X$ is an infinite-dimensional real Banach space, then, given a bounded, closed, convex and nonempty set $C$ in $X$ and a mapping $T : C \to C$, the *minimal displacement problem* is to find the quantity

$$\eta(T) = \inf\{\|x - Tx\| : x \in C\},$$

called the *mimimal displacement* of $T$.

For Lipschitzian mappings in Banach spaces the study of the mimimal displacement problem started in 1973 in a paper of Goebel ([7]). However only in 1983 Benyamini and Sternfeld ([2]), following the work of Nowak ([15]), proved that in every infinite-dimensional Banach space there exists a fixed point-free Lipschitzian mapping from the unit closed ball into itself. More generally, Lin and Sternfeld proved the following:

THEOREM 1 ([14]). *For any nonempty, noncompact, bounded, closed and convex subset $C$ of an infinite-dimensional Banach space there exists a Lipschitzian mapping $T : C \to C$ for which $\eta(T) > 0$.*

So if we denote by $\mathcal{L}(k)$ the family of all $k$-Lipschitzian mappings from the closed unit ball $B(X)$ into itself, the above mentioned result naturally leads to the definition of the function

$$\psi_X(k) = \sup_{T\in\mathcal{L}(k)} \eta(T),$$

called the *minimal displacement characteristic* of $X$, and this function is the main object of study relating to the minimal displacement problem.

2000 *Mathematics Subject Classification*: Primary 47H10.

*Key words and phrases*: Lipschitz maps, minimal displacement.

The essential properties of the function $\psi_X(k)$ can be found in the book by Goebel and Kirk ([10]). More recent results can be found in [1] and [3]–[6], [8] and [9].

The aim of this paper is to give a lower bound for the minimal displacement characteristic of $X$ when $X$ is a real infinite-dimensional Hilbert space.

**2. Lower bound.** In this section we prove the following result:

$$\psi_H(k) \geq 1 - \frac{2\sqrt{\sqrt{2}(k+1)}}{k},$$

where $H$ is an infinite-dimensional real Hilbert space.

To obtain this lower bound we use the Hilbert space $L^2[0,1]$ and the following mapping: for $k > 1$ and $f \in B(L^2[0,1])$ define $T_1 : B(L^2[0,1]) \to B(L^2[0,1])$ by

$$(T_1 f)(t) = \begin{cases} 1 + k|f(t)| & \text{if } 0 \leq t \leq t(f), \\ 0 & \text{if } t(f) < t \leq 1, \end{cases}$$

where $t(f)$ is the only solution in $[0,1]$ of the equation

$$\int_0^t (1 + k|f(s)|)^2 \, ds = 1.$$

This mapping is studied in [4], where it is proved that

$$\|f - T_1 f\| \geq 1 - 1/k. \tag{1}$$

We show that this map has a particular Hölder property.

Proposition 1. *For every $f, g \in B(L^2[0,1])$,*

$$\|T_1 f - T_1 g\|^2 \leq 2k\|f - g\|.$$

*Proof.* Let $f, g \in B(L^2[0,1])$ and suppose $t(f) \leq t(g)$. Then

$$\begin{aligned} \|T_1 f - T_1 g\|^2 &= \int_0^{t(f)} (k|f| - k|g|)^2 + \int_{t(f)}^{t(g)} (1 + k|g|)^2 \\ &= \int_0^{t(f)} (k|f| - k|g|)^2 + 1 - \int_0^{t(f)} (1 + k|g|)^2 \\ &= \int_0^{t(f)} (k|f| - k|g|)^2 + \int_0^{t(f)} (1 + k|f|)^2 - \int_0^{t(f)} (1 + k|g|)^2 \end{aligned}$$

$$= \int_0^{t(f)} (k^2|f|^2 + k^2|g|^2 - 2k^2|fg| + 1 + 2k|f| + k^2|f|^2 - 1 - 2k|g| - k^2|g|^2)$$

$$= \int_0^{t(f)} (2k^2|f|^2 - 2k^2|f|\,|g| + 2k(|f| - |g|))$$

$$= \int_0^{t(f)} 2k(|f| - |g|)(1 + k|f|) \le 2k \int_0^{t(f)} |f - g|(1 + k|f|)$$

$$\le 2k\Big(\int_0^{t(f)} |f - g|^2\Big)^{1/2}\Big(\int_0^{t(f)} (1 + k|f|)^2\Big)^{1/2} \le 2k\|f - g\|. \ \blacksquare$$

REMARK. In [4] it is proved that

$$\|T_1 f - T_1 g\|^2 \le k^2\|f - g\|^2 + 2k(k + 1)\|f - g\|.$$

The map we have described is obviously not Lipschitzian so we shall use the technique of [13] (see also [4], [10]), that is, first we restrict the map to a particular subset $\widetilde{W}$ of $B(L^2[0, 1])$ on which the restriction is a Lipschitzian map and then we extend this restriction to the space $H$ using the Kirszbraun extension theorem ([11]). However, to obtain a better bound, we shall be more careful in the choice of $\widetilde{W}$ than in [13] .

In fact we will use the following theorem:

THEOREM 2 ([12]). *Let $\xi$ be an infinite cardinal number for which $\xi^{\aleph_0} = \xi$. Then $l_2(\xi)$ contains a $\sqrt{2}$-dispersed proximinal set $\widetilde{W}$ such that* $\inf\{\|x - w\| : w \in \widetilde{W}\} \le 1$ *for all $x \in l_2(\xi)$.*

We recall that a subset $W$ of $X$, with at least two points, is *$\delta$-dispersed* if $\|x - y\| > \delta$ for each pair $x, y$ of distinct points of $W$. A subset $W$ of $X$ is *proximinal* if for each $x \in X$ there exists an element $w(x) \in W$ such that $\|x - w(x)\| = \operatorname{dist}(x, W)$.

Choose $\varepsilon > 0$ and consider the set $W = \varepsilon\widetilde{W} \cap B(l_2(\xi))$ in the Hilbert space $l_2(\xi)$. Obviously if $x, y \in W$ we have

$$\|x - y\| \ge \varepsilon\sqrt{2}$$

and for every $x \in B(l_2(\xi))$ there exists a $z \in W$ such that

$$\|x - z\| \le \varepsilon.$$

We embed $L^2[0, 1]$ in $l_2(\xi)$ as a closed subspace and we denote by $P$ the orthogonal projection onto it. If $T_2 = T_1 P$ then

$$\|T_2 x - T_2 y\|^2 \le 2k\|Px - Py\| \le 2k\|x - y\|$$

and, by (1),

$$\|x - T_2x\|^2 = \|x - Px + Px - T_1Px\|^2 = \|x - Px\|^2 + \|Px - T_1Px\|^2$$
$$\geq \|x - Px\|^2 + (1 - 1/k)^2 \geq (1 - 1/k)^2.$$

Now let $T_3$ be the restriction of $T_2$ to $W$. Then $T_3$ is a Lipschitz mapping with constant $\sqrt{\sqrt{2}\,k/\varepsilon}$. In fact, if $x, y \in W$, we have

$$\|T_3x - T_3y\| \leq \sqrt{2\,k\|x - y\|} \leq \sqrt{\frac{2k\|x - y\|^2}{\sqrt{2}\,\varepsilon}} = \sqrt{\frac{\sqrt{2}\,k}{\varepsilon}}\,\|x - y\|.$$

Using the Kirszbraun theorem we extend $T_3$ to all $l_2(\xi)$ keeping the same Lipschitz constant and we call this extension $T_4$. Notice that $T_4$ takes values in $\overline{\mathrm{co}}(T_3(B(l_2(\xi)))) \subset B(L^2[0,1])$. Finally, denote by $T$ the restriction of $T_4$ to $B(L^2[0,1])$. Obviously $T$ is a Lipschitzian mapping from $B(L^2[0,1])$ to $B(L^2[0,1])$ with Lipschitz constant $\sqrt{\sqrt{2}\,k/\varepsilon}$.

Now let $x \in B(L^2[0,1])$ and take $z \in W$ such that $\|x - y\| \leq \varepsilon$. We have

$$\|Tx - x\| = \|z - (z - x) - T_4z - (T_4z - Tx)\|$$
$$\geq \|z - T_4z\| - \|x - z\| - \|T_4z - Tx\|$$
$$\geq (1 - 1/k) - \varepsilon - \varepsilon\sqrt{\sqrt{2}\,k/\varepsilon}.$$

So

$$\psi_{L^2}(\sqrt{\sqrt{2}\,k/\varepsilon}) \geq (1 - 1/k) - \varepsilon - \varepsilon\sqrt{\sqrt{2}\,k/\varepsilon}$$

and from this inequality we obtain

$$\psi_{L^2}(k) \geq 1 - \frac{\sqrt{2}}{\varepsilon k^2} - \varepsilon(1 + k).$$

Elementary computations show that the optimal choice is $\varepsilon = \sqrt[4]{2}/k\sqrt{k+1}$. So we obtain

$$\psi_H(k) \geq 1 - \frac{2\sqrt{\sqrt{2}(k+1)}}{k}. \tag{2}$$

Remark. In [4] it is proved that

$$\psi_H(k) \geq 1 - \frac{2 + \varepsilon}{\sqrt{1 + \varepsilon(\varepsilon + 2)k^2} - 1} - \varepsilon(k + 1).$$

This formula seems to be more difficult to handle when you try to find explicitly the optimal value of $\varepsilon$. As the author of [4] notices, the value of $\psi(50)$ is greater than 0.25 (taking $\varepsilon = 0.005$). Formula (2) gives a value greater than 0.66.

Remark. Also formula (2) allows one to find a lower bound of $\psi'(1)$. In fact since the function $\psi_X$ is concave with respect to 1 (see [10, p. 215]), if

$r(k) = m(k-1)$ is the tangent line to the function $f(k) = 1-2\sqrt{\sqrt{2}(k+1)}/k$ we obtain $\psi'(1) \geq m$. Numerical methods show that $\psi'(1) \geq 0.026$. To obtain a good lower bound for $\psi'(1)$ is particularly important since this value is directly related to the Lipschitz constant of the retractions of the unit ball onto the unit sphere (see [9]).

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*Received September 20, 2002*
*Revised version May 17, 2004* (5038)