Minimal displacement in Hilbert spaces

by

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Abstract. We give a lower bound for the minimal displacement characteristic in Hilbert spaces.

1. Introduction and notation. In this paper we study the minimal displacement problem. Roughly speaking, this problem is connected with a quantitative measure of lack of fixed points of a mapping. More precisely, if $X$ is an infinite-dimensional real Banach space, then, given a bounded, closed, convex and nonempty set $C$ in $X$ and a mapping $T : C \to C$, the minimal displacement problem is to find the quantity

$$\eta(T) = \inf \{ \| x - Tx \| : x \in C \},$$

called the minimal displacement of $T$.

For Lipschitzian mappings in Banach spaces the study of the minimal displacement problem started in 1973 in a paper of Goebel ([7]). However only in 1983 Benyamini and Sternfeld ([2]), following the work of Nowak ([15]), proved that in every infinite-dimensional Banach space there exists a fixed point-free Lipschitzian mapping from the unit closed ball into itself. More generally, Lin and Sternfeld proved the following:

Theorem 1 ([14]). For any nonempty, noncompact, bounded, closed and convex subset $C$ of an infinite-dimensional Banach space there exists a Lipschitzian mapping $T : C \to C$ for which $\eta(T) > 0$.

So if we denote by $\mathcal{L}(k)$ the family of all $k$-Lipschitzian mappings from the closed unit ball $B(X)$ into itself, the above mentioned result naturally leads to the definition of the function

$$\psi_X(k) = \sup_{T \in \mathcal{L}(k)} \eta(T),$$

called the minimal displacement characteristic of $X$, and this function is the main object of study relating to the minimal displacement problem.

2000 Mathematics Subject Classification: Primary 47H10.

Key words and phrases: Lipschitz maps, minimal displacement.
The essential properties of the function \( \psi_X(k) \) can be found in the book by Goebel and Kirk ([10]). More recent results can be found in [1] and [3]–[6], [8] and [9].

The aim of this paper is to give a lower bound for the minimal displacement characteristic of \( X \) when \( X \) is a real infinite-dimensional Hilbert space.

2. Lower bound. In this section we prove the following result:

\[
\psi_H(k) \geq 1 - \frac{2\sqrt{2}(k + 1)}{k},
\]

where \( H \) is an infinite-dimensional real Hilbert space.

To obtain this lower bound we use the Hilbert space \( L^2[0, 1] \) and the following mapping: for \( k > 1 \) and \( f \in B(L^2[0, 1]) \) define \( T_1 : B(L^2[0, 1]) \rightarrow B(L^2[0, 1]) \) by

\[
(T_1 f)(t) = \begin{cases} 
1 + k|f(t)| & \text{if } 0 \leq t \leq t(f), \\
0 & \text{if } t(f) < t \leq 1,
\end{cases}
\]

where \( t(f) \) is the only solution in \([0, 1]\) of the equation

\[
\int_0^t (1 + k|f(s)|)^2 \, ds = 1.
\]

This mapping is studied in [4], where it is proved that

\[
(1) \quad \|f - T_1 f\| \geq 1 - 1/k.
\]

We show that this map has a particular Hölder property.

PROPOSITION 1. For every \( f, g \in B(L^2[0, 1]) \),

\[
\|T_1 f - T_1 g\|^2 \leq 2k\|f - g\|.
\]

Proof. Let \( f, g \in B(L^2[0, 1]) \) and suppose \( t(f) \leq t(g) \). Then

\[
\begin{align*}
\|T_1 f - T_1 g\|^2 &= \int_0^{t(f)} (k|f| - k|g|)^2 + \int_{t(f)}^{t(g)} (1 + k|g|)^2 \\
&= \int_0^{t(f)} (k|f| - k|g|)^2 + 1 - \int_0^{t(f)} (1 + k|g|)^2 \\
&= \int_0^{t(f)} (k|f| - k|g|)^2 + \int_0^{t(f)} (1 + k|f|)^2 - \int_0^{t(f)} (1 + k|g|)^2
\end{align*}
\]
\[
\begin{align*}
&= \int_0^1 (k^2|f|^2 + k^2|g|^2 - 2k^2fg + 1 + 2k|f| + k^2|f|^2 - 1 - 2k|g| - k^2|g|^2) \\
&= \int_0^1 (2k^2|f|^2 - 2k^2|f||g| + 2k(|f| - |g|)) \\
&= \int_0^1 2k(|f| - |g|)(1 + k|f|) \leq 2k \int_0^1 |f - g|(1 + k|f|) \\
&\leq 2k \left( \int_0^1 |f - g|^2 \right)^{1/2} \left( \int_0^1 (1 + k|f|)^2 \right)^{1/2} \leq 2k\|f - g\|. ~\blacksquare
\end{align*}
\]

**Remark.** In [4] it is proved that
\[
\|T_1 f - T_1 g\|^2 \leq k^2\|f - g\|^2 + 2k(k + 1)\|f - g\|.
\]

The map we have described is obviously not Lipschitzian so we shall use the technique of [13] (see also [4], [10]), that is, first we restrict the map to a particular subset \(\tilde{W}\) of \(B(L^2[0,1])\) on which the restriction is a Lipschitzian map and then we extend this restriction to the space \(H\) using the Kirszbraun extension theorem ([11]). However, to obtain a better bound, we shall be more careful in the choice of \(\tilde{W}\) than in [13].

In fact we will use the following theorem:

**Theorem 2 ([12]).** Let \(\xi\) be an infinite cardinal number for which \(\xi^{\aleph_0} = \xi\).
Then \(l_2(\xi)\) contains a \(\sqrt{2}\)-dispersed proximinal set \(\tilde{W}\) such that \(\inf\{\|x - w\| : w \in \tilde{W}\} \leq 1\) for all \(x \in l_2(\xi)\).

We recall that a subset \(W\) of \(X\), with at least two points, is \(\delta\)-dispersed if \(\|x - y\| > \delta\) for each pair \(x, y\) of distinct points of \(W\). A subset \(W\) of \(X\) is proximinal if for each \(x \in X\) there exists an element \(w(x) \in W\) such that \(\|x - w(x)\| = \text{dist}(x,W)\).

Choose \(\varepsilon > 0\) and consider the set \(W = \varepsilon\tilde{W} \cap B(l_2(\xi))\) in the Hilbert space \(l_2(\xi)\). Obviously if \(x, y \in W\) we have
\[
\|x - y\| \geq \varepsilon \sqrt{2}
\]
and for every \(x \in B(l_2(\xi))\) there exists a \(z \in W\) such that
\[
\|x - z\| \leq \varepsilon.
\]

We embed \(L^2[0,1]\) in \(l_2(\xi)\) as a closed subspace and we denote by \(P\) the orthogonal projection onto it. If \(T_2 = T_1 P\) then
\[
\|T_2 x - T_2 y\|^2 \leq 2k\|P x - P y\| \leq 2k\|x - y\|
\]
and, by (1),
\[ \|x - T_2 x\|^2 = \|x - P x + P x - T_1 P x\|^2 = \|x - P x\|^2 + \|P x - T_1 P x\|^2 \]
\[ \geq \|x - P x\|^2 + (1 - 1/k)^2 \geq (1 - 1/k)^2. \]

Now let \( T_3 \) be the restriction of \( T_2 \) to \( W \). Then \( T_3 \) is a Lipschitz mapping with constant \( \sqrt{\frac{2}{k}}/\varepsilon \). In fact, if \( x, y \in W \), we have
\[
\|T_3 x - T_3 y\| \leq \sqrt{2} k \|x - y\| \leq \frac{\sqrt{2}}{\varepsilon} \|x - y\|.
\]

Using the Kirszbraun theorem we extend \( T_3 \) to all \( l_2(\xi) \) keeping the same Lipschitz constant and we call this extension \( T_4 \). Notice that \( T_4 \) takes values in \( \mathbb{C}(T_3(B(l_2(\xi)))) \subset B(L^2[0,1]) \). Finally, denote by \( T \) the restriction of \( T_4 \) to \( B(L^2[0,1]) \). Obviously \( T \) is a Lipschitzian mapping from \( B(L^2[0,1]) \) to \( B(L^2[0,1]) \) with Lipschitz constant \( \sqrt{2} k/\varepsilon \).

Now let \( x \in B(L^2[0,1]) \) and take \( z \in W \) such that \( \|x - y\| \leq \varepsilon \). We have
\[
\|T x - x\| = \|z - (z - x) - T_4 z - (T_4 z - T x)\| \geq \|z - T_4 z\| - \|x - z\| - \|T_4 z - T x\| \geq (1 - 1/k) - \varepsilon - \varepsilon \sqrt{2} k / \varepsilon.
\]

So
\[
\psi_{L^2}(\sqrt{2} k/\varepsilon) \geq (1 - 1/k) - \varepsilon - \varepsilon \sqrt{2} k / \varepsilon
\]

and from this inequality we obtain
\[
\psi_{L^2}(k) \geq 1 - \frac{\sqrt{2}}{\varepsilon k^2} - \varepsilon(1 + k).
\]

Elementary computations show that the optimal choice is \( \varepsilon = \sqrt{2} k \sqrt{k + 1} \).

So we obtain
\[
(2) \quad \psi_H(k) \geq 1 - \frac{2 \sqrt{2} (k + 1)}{k}.
\]

**Remark.** In [4] it is proved that
\[
\psi_H(k) \geq 1 - \frac{2 + \varepsilon}{\sqrt{1 + \varepsilon(\varepsilon + 2) k^2}} - \varepsilon(k + 1).
\]

This formula seems to be more difficult to handle when you try to find explicitly the optimal value of \( \varepsilon \). As the author of [4] notices, the value of \( \psi(50) \) is greater than 0.25 (taking \( \varepsilon = 0.005 \)). Formula (2) gives a value greater than 0.66.

**Remark.** Also formula (2) allows one to find a lower bound of \( \psi'(1) \). In fact since the function \( \psi_X \) is concave with respect to 1 (see [10, p. 215]), if
$r(k) = m(k-1)$ is the tangent line to the function $f(k) = 1 - 2\sqrt{\frac{2(k+1)}{k}}$ we obtain $\psi'(1) \geq m$. Numerical methods show that $\psi'(1) \geq 0.026$. To obtain a good lower bound for $\psi'(1)$ is particularly important since this value is directly related to the Lipschitz constant of the retractions of the unit ball onto the unit sphere (see [9]).

References


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Received September 20, 2002
Revised version May 17, 2004 (5038)