

The inclusion theorem for multiple summing operators

by

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Abstract. We prove that, for $1 \leq p \leq q < 2$, each multiple p -summing multilinear operator between Banach spaces is also q -summing. We also give an improvement of this result for an image space of cotype 2. As a consequence, we obtain a characterization of Hilbert–Schmidt multilinear operators similar to the linear one given by A. Pełczyński in 1967. We also give a multilinear generalization of Grothendieck’s Theorem for GT spaces.

1. Introduction. Motivated by the importance of the theory of absolutely summing linear operators, there have been some attempts to generalize this concept and the related results and tools to the multilinear setting (see [1], [7], [21] or [23]). However, the results obtained were not completely satisfactory. Recently, M. C. Matos [11] and, independently, F. Bombal, I. Villanueva and the author [2], [17], [18] have defined and studied the class of multiple summing operators (although the origin of this class goes back to [23]). It is shown in these works that this class behaves better in many ways than the previous definitions of p -summing multilinear operators. In fact, this class has most of the main properties of its linear analogue. For example, we have

THEOREM 1.1. (i) *Every multilinear operator from \mathcal{L}_∞ -spaces into an \mathcal{L}_1 -space is multiple 2-summing [2].*

(ii) *Every multilinear operator from \mathcal{L}_1 -spaces into a Hilbert space is multiple p -summing for $1 \leq p \leq 2$ [17].*

(iii) *The composition of linear 2-summing operators with a multiple 2-summing multilinear operator is integral and even nuclear [16], [18].*

(iv) *X_1, \dots, X_n are \mathcal{L}_∞ -spaces if and only if, for every Banach space Y , the class of multiple 1-summing multilinear operators $T : X_1 \times \dots \times X_n \rightarrow Y$ coincides with the class of integral operators [18].*

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We also have close relations with the Hilbert–Schmidt multilinear operators. In fact, it is proved in [11] that

THEOREM 1.2. *The class of Hilbert–Schmidt multilinear operators and the class of multiple 2-summing operators are identically isometric.*

THEOREM 1.3. *For a multilinear operator $T : X_1 \times \cdots \times X_n \rightarrow Y$, the following are equivalent:*

- (i) *T is Hilbert–Schmidt.*
- (ii) *T is multiple p -summing for every $p \in [2, \infty)$.*
- (iii) *T is multiple p -summing for some $p \in [2, \infty)$.*

THEOREM 1.4. *Every Hilbert–Schmidt multilinear operator is multiple p -summing for every $1 \leq p < 2$.*

The class of Hilbert–Schmidt multilinear operators was introduced in [8], and studied and used in, for example, [3], [4], [9] or [21] (see Section 5 for more details).

However, as shown in [18], for every $q > 2$ there exists a bilinear form that is multiple p -summing for every $1 \leq p \leq 2$ but not q -summing. Therefore, and this is one of the main defects of this class, we cannot expect an Inclusion Theorem similar to [6, Theorem 2.8]. The aim of this paper is to solve this problem partially by showing (see Theorem 3.4) that if $1 \leq p \leq q < 2$, then every multiple p -summing operator is also multiple q -summing. In Theorem 3.10 we improve this result for the case of an image space of cotype 2.

As a first application, we give in Theorem 4.2 a multilinear version of the classical characterization of Hilbert–Schmidt linear operators given by A. Pełczyński [15] (see also [19]). Thus, we complete the relations given in Theorems 1.2–1.4.

Moreover, we prove in Theorem 4.5 that, for GT spaces, every multilinear operator into a Hilbert space is 1-summing. Though this result can be obtained by using the approach of [2], this new proof gives us an optimal constant, improving the proofs in [2] and [17]. The Inclusion Theorem allows us to extend this result to $1 \leq p \leq 2$ and so we conclude that Theorem 1.1(ii) is also valid for GT spaces.

2. Notation and definitions. The notation and terminology used along the paper is standard in Banach space theory, as for instance in [5] or [6]. These books are also our main references for basic facts, definitions and unexplained notation. However, before going any further, we shall establish some terminology. All along this paper all the operators are supposed to be continuous. Given Banach spaces X, Y , $\mathcal{L}(X, Y)$ will denote the Banach space of linear (and continuous) operators between them and $X \otimes_\varepsilon Y$ will be their injective tensor product. X^* will be the dual of X and B_X its unit ball.

We will denote by $Y \xrightarrow{1} X$ an isometric inclusion and $X \stackrel{1}{=} Y$ will denote that X and Y are isometrically isomorphic. Given linear operators $u : X \rightarrow Y$ and $v : Z \rightarrow W$, $u \otimes v : X \otimes Z \rightarrow T \otimes W$ will be the linear operator given by $u \otimes v(x \otimes z) = u(x) \otimes v(z)$. As usual, u^* will denote the adjoint operator of u ; and \mathbb{K} will denote the basic field, which can be both \mathbb{R} and \mathbb{C} .

For a finite sequence $(x_i)_{i=1}^m \subset X$ and $1 \leq p < \infty$, we will write $\|(x_i)_{i=1}^m\|_p^\omega$ for

$$\sup \left\{ \left(\sum_{i=1}^m |x^*(x_i)|^p \right)^{1/p} : x^* \in B_{X^*} \right\}.$$

A linear operator $T : X \rightarrow Y$ is said to be *p-summing* if there exists a constant $K > 0$ such that $(\sum_{i=1}^m \|T(x_i)\|^p)^{1/p} \leq K \|(x_i)_{i=1}^m\|_p^\omega$ for each finite sequence $(x_i)_{i=1}^m \subset X$. We can extend this definition to the multilinear case:

DEFINITION 2.1. Let $1 \leq p < \infty$. A multilinear operator $T : X_1 \times \dots \times X_n \rightarrow Y$ is *multiple p-summing* if there exists a constant $K > 0$ such that, for every choice of sequences $(x_{i_j}^j)_{i_j=1}^{m_j} \subset X_j$,

$$(1) \quad \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^p \right)^{1/p} \leq K \prod_{j=1}^n \|(x_{i_j}^j)_{i_j=1}^{m_j}\|_p^\omega.$$

In that case, we define the *multiple p-summing norm* of T by

$$\pi_p(T) = \min\{K : K \text{ satisfies (1)}\}.$$

The class $\Pi_p^n(X_1, \dots, X_n; Y)$ of multiple p -summing n -linear operators is a Banach space with the norm π_p .

Another class related to p -summing operators is the class of 2-dominated operators (see [5] or [6]). A linear operator $u : X \rightarrow Y$ is said to be *2-dominated* if there exist a Hilbert space H and linear operators $v : X \rightarrow H$ and $w : H \rightarrow Y$ such that $u = wv$ and v and w^* are 2-summing. We define $\Delta_2(u) = \inf\{\pi_2(v)\pi_2(w^*)\}$, where the infimum is taken over all factorizations as above.

It is not difficult to see that $u : X \rightarrow Y$ is 2-dominated if and only if u^* is. In this case $\Delta_2(u) = \Delta_2(u^*)$.

Grothendieck's Theorem says that there exists a universal constant $K > 0$ such that every linear operator $u : \ell_1 \rightarrow \ell_2$ is 1-summing with $\pi_1(u) \leq K\|u\|$. We will denote by K_G the least constant for which Grothendieck's Theorem is valid.

Following [22], we say that a Banach space X is a *GT space*, or that X satisfies *Grothendieck's Theorem*, if there exists $K > 0$ such that each linear operator $u : X \rightarrow \ell_2$ is 1-summing and satisfies $\pi_1(u) \leq K\|u\|$. We denote the least such constant K by $GT(X)$. According to Grothendieck's Theorem, \mathcal{L}_1 spaces are GT spaces, but there are several instances of GT

spaces which are not \mathcal{L}_1 spaces, for example L_1/H^1 or the quotient of L_1 by a reflexive subspace (see [22]).

It is shown in [22] that X is a GT space if and only if X^{**} is, and in that case, $\text{GT}(X) = \text{GT}(X^{**})$. Therefore, by [5, Proposition 20.18], every linear operator $u : X^* \rightarrow \ell_1$ is 2-dominated and $\Delta_2(u) \leq \text{GT}(X)\|u\|$. By the above remarks, we have

PROPOSITION 2.2. *If X is a GT space, then every linear operator $u : c_0 \rightarrow X$ is 2-dominated and $\Delta_2(u) \leq \text{GT}(X)\|u\|$.*

For $q \geq 2$, a Banach space is said to have *cotype q* if there exists a constant $K > 0$ such that, for each choice of $(x_i)_{i=1}^m \subset X$, we have

$$\left(\sum_{i=1}^m \|x_i\|^q\right)^{1/q} \leq K \left(\int_0^1 \left\| \sum_{i=1}^m r_i(t)x_i \right\|^2 dt\right)^{1/2},$$

where $(r_i)_{i=1}^\infty$ is the sequence of Rademacher functions. The least such constant K will be denoted by $C_q(X)$.

Finally, we recall Khinchin’s and Kahane’s inequalities.

THEOREM 2.3 (Khinchin’s Inequality). *For each $0 < p < \infty$, there are positive constants A_p, B_p such that, for each sequence $(a_i)_{i=1}^\infty \in \ell_2$, we have*

$$A_p \left(\sum_{i=1}^\infty |a_i|^2\right)^{1/2} \leq \left(\int_0^1 \left| \sum_{i=1}^\infty a_i r_i(t) \right|^p dt\right)^{1/p} \leq B_p \left(\sum_{i=1}^\infty |a_i|^2\right)^{1/2}.$$

THEOREM 2.4 (Kahane’s Inequality). *If $0 < p, q < \infty$, there exists a constant $K_{p,q} > 0$ such that*

$$\left(\int_0^1 \left\| \sum_{i=1}^m r_i(t)x_i \right\|^q dt\right)^{1/q} \leq K_{p,q} \left(\int_0^1 \left\| \sum_{i=1}^m r_i(t)x_i \right\|^p dt\right)^{1/p}$$

for each Banach space X and each choice of elements $x_1, \dots, x_m \in X$.

More notation and terminology will be introduced when needed.

3. Main results. To give the proof of the main result of this paper, we will need an integral formulation of the definition of multiple summing operators, in the spirit of [22, Proposition 1.2].

PROPOSITION 3.1. *Let $T \in \Pi_p^n(X_1, \dots, X_n; Y)$ and let $(\Omega_j, \Sigma_j, \mu_j)$ be measure spaces for each $1 \leq j \leq n$. Then*

$$(2) \quad \left(\int_{\Omega_1} \cdots \int_{\Omega_n} \|T(f_1(\omega_1), \dots, f_n(\omega_n))\|^p d\mu_n(\omega_n) \cdots d\mu_1(\omega_1)\right)^{1/p} \\ \leq \pi_p(T) \prod_{j=1}^n \sup_{x^* \in B_{X^*}} \left(\int_{\Omega_j} |\langle x^*, f_j(\omega_j) \rangle|^p d\mu_j(\omega_j)\right)^{1/p}$$

for every $f_j \in L_p(\mu_j, X_j)$.

Proof. It is enough to prove the result for simple functions of the form $f_j = \sum_{i_j=1}^{m_j} x_{i_j}^j \chi_{A_{i_j}^j}$, where, for each j , the sets $A_{i_j}^j \subset \Omega_j$ are measurable and disjoint and $x_{i_j}^j \in X_j$. Then (2) reduces to

$$\begin{aligned} & \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(\mu_1(A_{i_1}^1)^{1/p} x_{i_1}^1, \dots, \mu_n(A_{i_n}^n)^{1/p} x_{i_n}^n)\|^p \right)^{1/p} \\ & \leq \pi_p(T) \prod_{j=1}^n \sup_{x^* \in B_{X^*}} \left(\sum_{i_j=1}^{m_j} |\langle x^*, \mu_j(A_{i_j}^j)^{1/p} x_{i_j}^j \rangle|^p \right)^{1/p}, \end{aligned}$$

which is true by Definition 2.1. ■

Our main tool will be probability in Banach spaces. For this, all we will need is covered by [20, Chapter 21].

The first result we will need is the existence of q -stable random variables:

PROPOSITION 3.2. *For $1 \leq p < q \leq 2$, there exist a positive constant $c_{p,q}$, a measure space (Ω, Σ, μ) and a sequence $(\theta_i)_{i=1}^\infty$ of measurable scalar-valued functions such that*

$$(3) \quad \left\| \sum_{i=1}^m \alpha_i \theta_i \right\|_{L_p(\mu)} = c_{p,q} \left(\sum_{i=1}^m |\alpha_i|^q \right)^{1/q}$$

for each $(\alpha_i)_{i=1}^m \subset \mathbb{K}$.

Moreover, we will need the following fact, due to Maurey [12], which essentially says that every Banach space has cotype (q, p) for $1 \leq p < q < 2$:

PROPOSITION 3.3. *If $1 \leq p < q < 2$, there exists a constant $\tilde{K}_{p,q} > 0$, which only depends on p and q , such that if Z is a Banach space and $(z_i)_{i=1}^m \subset Z$, then*

$$(4) \quad \left(\sum_{i=1}^m \|z_i\|^q \right)^{1/q} \leq \tilde{K}_{p,q} \left(\int_{\Omega} \left\| \sum_{i=1}^m \theta_i(t) z_i \right\|^p d\mu(t) \right)^{1/p}.$$

We set $K_{p,q} = c_{p,q} \tilde{K}_{p,q}$.

THEOREM 3.4. *Let $1 \leq p < q < 2$. If X_1, \dots, X_n, Y are Banach spaces and $T \in \Pi_p^n(X_1, \dots, X_n; Y)$, then $T \in \Pi_q^n(X_1, \dots, X_n; Y)$ and $\pi_q(T) \leq K_{p,q}^n \pi_p(T)$.*

Proof. Take $T \in \Pi_p^n(X_1, \dots, X_n; Y)$ and $(x_{i_j}^j)_{i_j=1}^{m_j} \subset X_j$. We are going to show that

$$(5) \quad \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^q \right)^{1/q} \leq K_{p,q}^n \pi_p(T) \prod_{j=1}^n \|(x_{i_j}^j)_{i_j=1}^{m_j}\|_q.$$

By (4), it is not difficult to see by induction that

$$\begin{aligned} & \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^q \right)^{1/q} \\ & \leq \tilde{K}_{p,q}^n \left(\int_{\Omega} \cdots \int_{\Omega} \left\| T \left(\sum_{i_1=1}^{m_1} \theta_{i_1}(\omega_1) x_{i_1}^1, \dots, \sum_{i_n=1}^{m_n} \theta_{i_n}(\omega_n) x_{i_n}^n \right) \right\|^p \right. \\ & \qquad \qquad \qquad \left. d\mu(\omega_n) \cdots d\mu(\omega_1) \right)^{1/p} \end{aligned}$$

Then Proposition 3.1 tells us that

$$\begin{aligned} & \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^q \right)^{1/q} \\ & \leq \tilde{K}_{p,q}^n \pi_p(T) \prod_{j=1}^n \sup_{x^* \in B_{X^*}} \left(\int_{\Omega} \left| \sum_{i_j=1}^{m_j} \theta_{i_j}(\omega_j) \langle x^*, x_{i_j}^j \rangle \right|^p d\mu(\omega_j) \right)^{1/p} \end{aligned}$$

and using (3) we obtain (5). ■

REMARK 3.5. It is important to note that we cannot ensure that the constant $K_{p,q}$ in Theorem 3.4 is bounded as q increases to 2.

We can improve the constant in Theorem 3.4 using a tricky induction argument. This new result gives the constant 1 in the case $n = 1$.

THEOREM 3.6. *Let $1 \leq p < q < 2$. If X_1, \dots, X_n, Y are Banach spaces and $T \in \Pi_p^n(X_1, \dots, X_n; Y)$, then $T \in \Pi_q^n(X_1, \dots, X_n; Y)$ and $\pi_q(T) \leq K_{p,q}^{n-1} \pi_p(T)$.*

The induction step will use the following lemma, which is trivial from the definitions.

LEMMA 3.7. *Let $T : X_1 \times \cdots \times X_n \rightarrow Y$ be a multilinear operator and let $1 \leq p < \infty$. The following are equivalent:*

- (i) *T is multiple p -summing.*
- (ii) *There exists a constant $K > 0$ such that for every finite sequence $(x_i^1)_{i=1}^m \subset X_1$ with $\|(x_i^1)_{i=1}^m\|_p^\omega \leq 1$ the associated $(n - 1)$ -linear operator*

$$T_{(x_i^1)} : X_2 \times \cdots \times X_n \rightarrow \ell_q^m(Y)$$

given by

$$T_{(x_i^1)}(x^2, \dots, x^n) = (T(x_i^1, x^2, \dots, x^n))_{i=1}^m$$

is multiple p -summing and satisfies

$$(6) \qquad \qquad \qquad \pi_p(T_{(x_i^1)}) \leq K.$$

In that case, $\pi_p(T) = \min\{K : K \text{ satisfies (6)}\}$.

Proof of Theorem 3.6. We reason by induction. The case $n = 1$ is [6, Theorem 2.8]. To deduce the n -case from the $(n - 1)$ -case, we will use Lemma 3.7. So consider $(x_i^1)_{i=1}^m \subset X_1$ such that $\|(x_i^1)_{i=1}^m\|_q^w \leq 1$. We want to show that

$$(7) \quad \pi_q(T(x_i^1)) \leq K_{p,q}^{n-1} \pi_p(T).$$

Set $h_i = \theta_i/c_{p,q}$, $E = \text{span}\{h_1, \dots, h_m\} \subset L_p(\mu)$ and let $h : \ell_q^m \xrightarrow{1} E$ be the isometric inclusion given by $e_i \mapsto h_i$. Then

$$(8) \quad \|h\| = \|h^{-1}\| = 1.$$

Let $\varepsilon > 0$. Choose simple functions $S_i \in L_p(\mu)$ such that $\|S_i - h_i\|_{L_p(\mu)} \leq \varepsilon/m$. Clearly, there exist $A_1, \dots, A_N \subset \Omega$, measurable and disjoint, such that

$$S_i = \sum_{j=1}^N \frac{\alpha_i^j}{\mu(A_j)^{1/p}} \chi_{A_j} \quad \text{for } i = 1, \dots, m.$$

We define $u : E \rightarrow \ell_p^N$ by $u(h_i) = (\alpha_i^j)_{j=1}^N$ and $v : \ell_p^N \xrightarrow{1} L_p(\mu)$ by $v(e_j) = \chi_{A_j}/\mu(A_j)^{1/p}$. It is clear that v is an isometric inclusion. Moreover,

$$(9) \quad \|u\| \leq 1 + \varepsilon,$$

because

$$\begin{aligned} \left\| u \left(\sum_{i=1}^m \beta_i h_i \right) \right\|_{\ell_p^N} &= \left\| v \circ u \left(\sum_{i=1}^m \beta_i h_i \right) \right\|_{L_p(\mu)} = \left\| \sum_{i=1}^m \beta_i S_i \right\|_{L_p(\mu)} \\ &\leq \left\| \sum_{i=1}^m \beta_i h_i \right\|_{L_p(\mu)} + \left\| \sum_{i=1}^m \beta_i (S_i - h_i) \right\|_{L_p(\mu)} \\ &\leq \left\| \sum_{i=1}^m \beta_i h_i \right\|_{L_p(\mu)} + \left(\sum_{i=1}^m |\beta_i|^q \right)^{1/q} \sum_{i=1}^m \|S_i - h_i\|_{L_p(\mu)} \\ &\leq (1 + \varepsilon) \left\| \sum_{i=1}^m \beta_i h_i \right\|_{L_p(\mu)} \quad \text{by (8)}. \end{aligned}$$

Moreover, for $y = \sum_{i=1}^n h_i \otimes y_i \in L_p(\mu, Y)$, by (8) and [5] we have

$$\max_i \|y_i\|_Y \leq \left\| \sum_{i=1}^n e_i \otimes y_i \right\|_{\ell_q^m \otimes_\varepsilon Y} \leq \left\| \sum_{i=1}^n h_i \otimes y_i \right\|_{E \otimes_\varepsilon Y} \leq \|y\|_{L_p(\mu, Y)}.$$

Therefore,

$$(10) \quad \begin{aligned} (1 - \varepsilon) \|y\|_{L_p(\mu, Y)} &\leq \|y\|_{L_p(\mu, Y)} - \sum_{i=1}^m \|y_i\|_Y \|h_i - S_i\|_{L_p(\mu)} \\ &\leq \left\| \sum_{i=1}^m S_i \otimes y_i \right\|_{L_p(\mu, Y)} = \|(v \circ u) \otimes \text{id}(y)\|_{L_p(\mu, Y)}. \end{aligned}$$

Finally, using (4) we see that, for $y = \sum_{i=1}^m e_i \otimes y_i \in \ell_q^m(Y)$,

$$(11) \quad \|h \otimes \text{id}(y)\|_{L_p(\mu, Y)} \geq \frac{1}{c_{p,q} \widetilde{K}_{p,q}} \|y\|_{\ell_q^m(Y)}.$$

We can now prove that $\pi_q(T_{(x_i^1)}) \leq K_{p,q}^{n-1} \pi_p(T)$.

By (8) and (9) the linear operator $w = u \circ h : \ell_q^m \rightarrow \ell_p^N$ satisfies $\|w\| \leq 1 + \varepsilon$. Denote by $l : \ell_{q^*}^m \rightarrow X_1$ the linear operator defined by $l(e_i) = x_i^1$. By [6, Proposition 2.2], $\|l\| = \|(x_i^1)_{i=1}^m\|_q^w \leq 1$ and so $\|l \circ w^*\| \leq 1 + \varepsilon$.

Now, $l \circ w^* : \ell_{p^*}^N \rightarrow X_1$ and so if we set $z_j^1 = l \circ w^*(e_j)$, then $\|(z_j^1)_{j=1}^N\|_{p^*}^\omega \leq 1 + \varepsilon$, again by [6, Proposition 2.2]. But $z_j^1 = \sum_{i=1}^m \alpha_i^j x_i^1$ for each $1 \leq j \leq N$ and $(w \otimes \text{id}) \circ T_{(x_i^1)} : X_2 \times \dots \times X_n \rightarrow \ell_p^N(Y)$ is given by

$$(w \otimes \text{id}) \circ T_{(x_i^1)}(x^2, \dots, x^n) = \left(T \left(\sum_{i=1}^m \alpha_i^j x_i^1, x^2, \dots, x^n \right) \right)_{j=1}^N.$$

Consequently, $(w \otimes \text{id}) \circ T_{(x_i^1)} = T_{(z_j^1)} : X_2 \times \dots \times X_n \rightarrow \ell_p^N(Y)$, and we can deduce from Lemma 3.7 that

$$(12) \quad \pi_p((w \otimes \text{id}) \circ T_{(x_i^1)}) \leq (1 + \varepsilon) \pi_p(T).$$

We now consider $(x_{i_2}^2)_{i_2=1}^{m_2} \subset X_2, \dots, (x_{i_n}^n)_{i_n=1}^{m_n} \subset X_n$ such that

$$\|(x_{i_j}^j)_{i_j=1}^{m_j}\|_p^\omega \leq 1 \quad \text{for each } 2 \leq j \leq n.$$

We have

$$\begin{aligned} & \left(\sum_{i_2, \dots, i_n=1}^{m_2, \dots, m_n} \|T_{(x_i^1)}(x_{i_2}^2, \dots, x_{i_n}^n)\|_{\ell_q^m(Y)}^p \right)^{1/p} \\ & \leq K_{p,q} \left(\sum_{i_2, \dots, i_n=1}^{m_2, \dots, m_n} \|(h \otimes \text{id}) \circ T_{(x_i^1)}(x_{i_2}^2, \dots, x_{i_n}^n)\|_{L_p(\mu, Y)}^p \right)^{1/p} \quad \text{by (11)} \\ & \leq \frac{K_{p,q}}{1 - \varepsilon} \left(\sum_{i_2, \dots, i_n=1}^{m_2, \dots, m_n} \|(vu \otimes \text{id}) \circ (h \otimes \text{id}) \circ T_{(x_i^1)}(x_{i_2}^2, \dots, x_{i_n}^n)\|_{L_p(\mu, Y)}^p \right)^{1/p} \\ & \hspace{20em} \text{by (10)} \end{aligned}$$

$$= \frac{K_{p,q}}{1 - \varepsilon} \left(\sum_{i_2, \dots, i_n=1}^{m_2, \dots, m_n} \|(w \otimes \text{id}) \circ T_{(x_i^1)}(x_{i_2}^2, \dots, x_{i_n}^n)\|_{\ell_p^N(Y)}^p \right)^{1/p}$$

as $v \otimes \text{id}$ is an isometry

$$\leq \frac{K_{p,q}}{1 - \varepsilon} (1 + \varepsilon) \pi_p(T) \quad \text{by (12)}.$$

Therefore, as ε is arbitrary, we have $\pi_p(T(x_i^1)) \leq K_{p,q}\pi_p(T)$. Now the induction hypothesis tells us that $\pi_q(T(x_i^1)) \leq K_{p,q}^{n-2}\pi_p(T(x_i^1))$ and we obtain the estimate (7). ■

We can improve Theorem 3.4 if the target space has cotype 2. First of all, we can cover the case $q = 2$. What is even more important, we obtain uniform bounds for the inclusions (we have seen in Remark 3.5 that we do not have this in general).

We will need the following *multiple cotype inequality*.

PROPOSITION 3.8. *If Y has cotype 2, $p \leq 2$ and $(y_{i_1, \dots, i_n})_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \subset Y$, then*

$$\left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|y_{i_1, \dots, i_n}\|^2 \right)^{1/2} \leq (K_{p,2}C_2(Y))^n \left(\int_0^1 \cdots \int_0^1 \left\| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} r_{i_1}(t_1) \cdots r_{i_n}(t_n) y_{i_1, \dots, i_n} \right\|^p dt_1 \cdots dt_n \right)^{1/p},$$

where $K_{p,2}$ is the constant in Kahane’s inequality 2.4.

Proof. We argue by induction. The case $n = 1$ is just the definition of cotype and Kahane’s inequality. Let us prove case n assuming case $n - 1$. Again by the definition of cotype and Kahane’s inequality we get

$$\begin{aligned} & (K_{p,2}C_2(Y))^p \int_0^1 \cdots \int_0^1 \left\| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} r_{i_1}(t_1) \cdots r_{i_n}(t_n) y_{i_1, \dots, i_n} \right\|^p dt_1 \cdots dt_n \\ & \geq \int_0^1 \cdots \int_0^1 \left(\sum_{i_n=1}^{m_n} \left\| \sum_{i_1, \dots, i_{n-1}=1}^{m_1, \dots, m_{n-1}} r_{i_1}(t_1) \cdots r_{i_{n-1}}(t_{n-1}) y_{i_1, \dots, i_n} \right\|^2 \right)^{p/2} dt_1 \cdots dt_{n-1} \end{aligned}$$

and using Minkowski’s inequality for integrals, we obtain

$$\begin{aligned} & \int_0^1 \cdots \int_0^1 \left(\sum_{i_n=1}^{m_n} \left\| \sum_{i_1, \dots, i_{n-1}=1}^{m_1, \dots, m_{n-1}} r_{i_1}(t_1) \cdots r_{i_{n-1}}(t_{n-1}) y_{i_1, \dots, i_n} \right\|^2 \right)^{p/2} dt_1 \cdots dt_{n-1} \\ & \geq \left(\sum_{i_n=1}^{m_n} \left(\int_0^1 \cdots \int_0^1 \left\| \sum_{i_1, \dots, i_{n-1}=1}^{m_1, \dots, m_{n-1}} r_{i_1}(t_1) \cdots r_{i_{n-1}}(t_{n-1}) y_{i_1, \dots, i_n} \right\|^p dt_1 \cdots dt_{n-1} \right)^{2/p} \right)^{p/2}. \end{aligned}$$

The induction hypothesis yields the result. ■

Using Minkowski’s inequality exactly as in the proof of Proposition 3.8 we can obtain the following multiple version of Proposition 3.2.

PROPOSITION 3.9. *If $(\alpha_{i_1, \dots, i_n})_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \subset \mathbb{K}$, then*

$$(13) \quad \left(\int_{\Omega} \cdots \int_{\Omega} \left| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \alpha_{i_1, \dots, i_n} \theta_{i_1}(\omega_1) \cdots \theta_{i_n}(\omega_n) \right|^p d\mu(\omega_1) \cdots d\mu(\omega_n) \right)^{1/p} \\ \geq c_{p,q}^n \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} |\alpha_{i_1, \dots, i_n}|^q \right)^{1/q}.$$

Clearly, we also have (3) and (13) for $(\varepsilon_i \theta_i)_{i=1}^{\infty}$, independently of the choice of signs $(\varepsilon_i)_{i=1}^{\infty} \in \{-1, 1\}^{\mathbb{N}}$.

We can now prove our improvement of Theorem 3.4.

THEOREM 3.10. *If $1 \leq p < q \leq 2$, the space Y has cotype 2 and $T \in \Pi_p^n(X_1, \dots, X_n; Y)$, then $T \in \Pi_q^n(X_1, \dots, X_n; Y)$ and*

$$\pi_q(T) \leq (K_{p,2} C_2(Y))^n \pi_p(T),$$

where $K_{p,2}$ is the constant in Kahane’s inequality 2.4.

Proof. Using Proposition 3.2 and reasoning as in the proof of Theorem 3.4, we obtain

$$(14) \quad \left(\int_0^1 \cdots \int_0^1 \int_{\Omega} \cdots \int_{\Omega} \left\| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} r_{i_1}(t_1) \cdots r_{i_n}(t_n) \theta_{i_1}(\omega_1) \cdots \right. \right. \\ \left. \left. \cdots \theta_{i_n}(\omega_n) T(x_{i_1}^1, \dots, x_{i_n}^n) \right\|^p d\mu(\omega_1) \cdots d\mu(\omega_n) dt_1 \cdots dt_n \right)^{1/p} \\ \leq \pi_p(T) c_{p,q}^n \prod_{j=1}^n \|(x_{i_j}^j)_{i_j=1}^{m_j}\|_q^{\omega}.$$

Using now Proposition 3.8 we find that the left side of (14) is greater than or equal to

$$(15) \quad \frac{1}{((K_{p,2} C_2(Y))^n)^n} \left(\int_{\Omega} \cdots \int_{\Omega} \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} |\theta_{i_1}(\omega_1) \cdots \right. \right. \\ \left. \left. \cdots \theta_{i_n}(\omega_n) \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^2 \right)^{p/2} d\mu(\omega_1) \cdots d\mu(\omega_n) \right)^{1/p}$$

Now, using Khinchin’s inequality 2.3, together with the fact that $B_p = 1$, we see that (15) is again greater than or equal to

$$(16) \quad \frac{1}{(K_{p,2} C_2(Y))^n} \left(\int_{\Omega} \cdots \int_{\Omega} \int_0^1 \cdots \int_0^1 \left| \sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} r_{i_1}(t_1) \cdots r_{i_n}(t_n) \theta_{i_1}(\omega_1) \cdots \right. \right. \\ \left. \left. \cdots \theta_{i_n}(\omega_n) \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^2 \right|^p dt_1 \cdots dt_n d\mu(\omega_1) \cdots d\mu(\omega_n) \right)^{1/p}$$

and Proposition 3.9 shows that (16) is greater than or equal to

$$\frac{1}{(K_{p,2}C_2(Y))^n} C_{p,q}^n \left(\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\|^q \right)^{1/q},$$

which finishes the proof. ■

REMARK 3.11. By [10], we can take $K_{p,2} = \sqrt{2}$.

4. Applications. The first application of the previous results will be a multilinear version of the main theorem of [15]. We need to define first the class of Hilbert–Schmidt multilinear operators.

DEFINITION 4.1. A multilinear operator $T : H_1 \times \dots \times H_n \rightarrow H$ between Hilbert spaces is said to be *Hilbert–Schmidt* if there exists $K > 0$ such that

$$(17) \quad \left(\sum_{i_1 \in I_1, \dots, i_n \in I_n} \|T(e_{i_1}^1, \dots, e_{i_n}^n)\|^2 \right)^{1/2} < K,$$

where $(e_{i_j}^j)_{i_j \in I_j} \subset H_j$ is an orthonormal basis ($1 \leq j \leq n$). In that case, the least constant K satisfying (17) is the *Hilbert–Schmidt norm* of T , which will be denoted by $\|T\|_{S_2}$.

The first observation we make is that the definition does not depend on the orthonormal basis chosen [11, Proposition 5.1]. It is easy to see that the class of Hilbert–Schmidt multilinear operators $S_2^n(H_1, \dots, H_n; H)$, with its norm $\|\cdot\|_{S_2}$, is a Hilbert space. The scalar product is given by

$$(T | S) = \sum_{i_1 \in I_1, \dots, i_n \in I_n} (T(e_{i_1}^1, \dots, e_{i_n}^n) | S(e_{i_1}^1, \dots, e_{i_n}^n)).$$

THEOREM 4.2. Let H_1, \dots, H_n, H be Hilbert spaces and $T : H_1 \times \dots \times H_n \rightarrow H$ a multilinear operator. The following are equivalent:

- (i) $T \in S_2^n(H_1, \dots, H_n; H)$.
- (ii) $T \in \Pi_p^n(H_1, \dots, H_n; H)$ for every $p \in [1, \infty)$.
- (iii) $T \in \Pi_p^n(H_1, \dots, H_n; H)$ for some $p \in [1, \infty)$.

Proof. (i) \Rightarrow (ii) is [11, Proposition 5.6] and (ii) \Rightarrow (iii) is trivial. To see that (iii) \Rightarrow (i) we take $T \in \Pi_p^n(H_1, \dots, H_n; H)$. If $p \geq 2$, then [11, Proposition 5.7] tells us that $T \in S_2^n(H_1, \dots, H_n; H)$. If $1 \leq p \leq 2$, Theorem 3.10 yields $T \in \Pi_2^n(H_1, \dots, H_n; H)$. Then [11, Proposition 5.5] allows us to conclude that also in that case $T \in S_2^n(H_1, \dots, H_n; H)$. ■

Now, we are going to give the announced improvement of Theorem 1.1(ii). First, we need the following

LEMMA 4.3. Let H_1, \dots, H_n, H be Hilbert spaces, X_1, \dots, X_n Banach spaces, $T : X_1 \times \dots \times X_n \rightarrow H$ a multilinear operator and $u_j : H_1 \rightarrow X_j$

linear operators such that u_j^* is 2-summing for each $1 \leq j \leq n$. Then $S = T(u_1, \dots, u_n)$ is Hilbert–Schmidt and

$$\|S\|_{S_2} \leq \left(\prod_{j=1}^n \pi_2(u_j^*) \right) \|T\|.$$

Proof. The case $n = 1$ is well known (see [6]). We reason by induction. If we suppose the case $n - 1$ holds, we can consider the following diagram:

$$\begin{array}{ccc} H_1 & \xrightarrow{S_1} & S_2^{n-1}(H_2, \dots, H_n; H) \\ u_1 \downarrow & & \uparrow u \\ X_1 & \xrightarrow{T_1} & \mathcal{L}^{n-1}(X_2, \dots, X_n; H) \end{array}$$

where T_1, S_1 are the linear operators associated to T and S respectively, $u(R) = R(u_2, \dots, u_n)$ and

$$\|u\| \leq \prod_{j=2}^n \pi_2(u_j^*).$$

As $S_2^{n-1}(H_2, \dots, H_n; H)$ is again a Hilbert space, the case $n = 1$ shows that S_1 is Hilbert–Schmidt and

$$\|S_1\|_{S_2} \leq \left(\prod_{j=1}^n \pi_2(u_j^*) \right) \|T\|.$$

Now [11, Proposition 5.3] finishes the proof. ■

We also need the following result proved in [2].

THEOREM 4.4. *Let $u_j \in \Pi_q(X_j, Y_j)$ and $T \in \Pi_p^n(Y_1, \dots, Y_n; Z)$ and let $1 \leq r < \infty$ be such that $1/r = 1/p + 1/q$. Then $S = \hat{T}(u_1, \dots, u_n)$ is multiple r -summing and $\pi_r(S) \leq \pi_p(T) \prod_{j=1}^n \pi_q(u_j)$.*

THEOREM 4.5. *Let X_1, \dots, X_n be GT spaces. Then, for any Hilbert space H , every multilinear operator $T : X_1 \times \dots \times X_n \rightarrow H$ is multiple 1-summing and*

$$\pi_1(T) \leq \left(\prod_{j=1}^n \text{GT}(X_j) \right) \|T\|.$$

Moreover, the constant $\prod_{j=1}^n \text{GT}(X_j)$ is optimal.

Proof. Take finite sequences $(x_{i_j}^j)_{i_j=1}^{m_j} \subset X_j$ with $\|(x_{i_j}^j)_{i_j=1}^{m_j}\|_1^\omega \leq 1$ and define the operators $u_j : \ell_\infty^{m_j} \rightarrow X_j$ by $u_j(e_{i_j}) = x_{i_j}^j$. By [6, Proposition 2.2], $\|u_j\| \leq 1$.

By Proposition 2.2, u_j is 2-dominated and $\Delta_2(u_j) \leq \text{GT}(X_j)$. Therefore, for any $\varepsilon > 0$, there exist Hilbert spaces H_j and operators $v_j : \ell_\infty^{m_j} \rightarrow H_j$

and $w_j : H_j \rightarrow X_j$ such that $u_j = w_j v_j$ and v_j, w_j^* are 2-summing with

$$\pi_2(v_j)\pi_2(w_j^*) \leq \text{GT}(X_j) + \varepsilon.$$

By Lemma 4.3, $T(w_1, \dots, w_n)$ is Hilbert–Schmidt and

$$\|T(w_1, \dots, w_n)\|_{S_2} \leq \left(\prod_{j=1}^n \pi_2(w_j^*) \right) \|T\|.$$

Now, Theorems 1.2 and 4.4 tell us that $T(u_1, \dots, u_n)$ is multiple 1-summing and

$$\pi_1(T(u_1, \dots, u_n)) \leq \left(\prod_{j=1}^n (\text{GT}(X_j) + \varepsilon) \right) \|T\|.$$

As ε is arbitrary and $\|(e_{i_j})_{i_j=1}^{m_j}\|_1^\omega = 1$ in $\ell_\infty^{m_j}$, we conclude that

$$\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\| \leq \pi_1(T(u_1, \dots, u_n)) \leq \left(\prod_{j=1}^n \text{GT}(X_j) \right) \|T\|.$$

To see the optimality of the constant we consider, for $\varepsilon > 0$, operators $u_j : X_j \rightarrow \ell_2$ and finite sequences $(x_{i_j}^j)_{i_j=1}^{m_j} \subset X_j$ such that $\|(x_{i_j}^j)_{i_j=1}^{m_j}\|_1^\omega \leq 1$ and

$$\sum_{i_j=1}^{m_j} \|u_j(x_{i_j}^j)\| \geq \text{GT}(X_j) - \varepsilon.$$

The operator $T = u_1 \otimes \dots \otimes u_n : X_1 \times \dots \times X_n \rightarrow \ell_2(\mathbb{N} \times \dots \times \mathbb{N})$ has $\|T\| \leq 1$ and

$$\sum_{i_1, \dots, i_n=1}^{m_1, \dots, m_n} \|T(x_{i_1}^1, \dots, x_{i_n}^n)\| = \prod_{j=1}^n \sum_{i_j=1}^{m_j} \|u_j(x_{i_j}^j)\| \geq \prod_{j=1}^n (\text{GT}(X_j) - \varepsilon). \blacksquare$$

Grothendieck’s Theorem tells us that ℓ_1 is a GT space with $\text{GT}(\ell_1) = K_G$. Thus we have the following

COROLLARY 4.6. *Every multilinear operator $T : \ell_1 \times \dots \times \ell_1 \rightarrow \ell_2$ is multiple 1-summing and*

$$\pi_1(T) \leq K_G^n \|T\|.$$

Moreover, the constant K_G^n is optimal.

Finally, as another application of Theorem 3.10, we have the following general version of Theorem 1.1(ii).

COROLLARY 4.7. *Let X_1, \dots, X_n be GT spaces, H a Hilbert space, $T : X_1 \times \dots \times X_n \rightarrow H$ a multilinear operator and $1 \leq p \leq 2$. Then T is multiple p -summing and*

$$\pi_p(T) \leq \left(\prod_{j=1}^n \sqrt{2} \text{GT}(X_j) \right) \|T\|.$$

5. Open problems

5.1. Schatten classes of multilinear operators. We have seen in Theorem 4.2 a close relation between Hilbert–Schmidt and multiple summing multilinear operators. In the linear case we have a similar relation with the Schatten spaces S_q . Specifically, we have

THEOREM 5.1 ([6, Theorem 10.3]). *If $2 \leq q < \infty$ and H_1, H_2 are Hilbert spaces, then $S_q(H_1, H_2) \stackrel{1}{=} \Pi_{(q,2)}(H_1, H_2)$.*

It is natural to ask if we can obtain a multilinear analogue of this result. The first question here is to find the right definition for the multilinear Schatten classes. The problem of finding this definition was first stated in [21]. By the same time, and motivated by the application of the Schatten spaces to the study of Hankel operators, this problem appeared again in [9], [13] and [14]. [3] and [4] are the first steps towards a solution.

The idea in all these papers is to define the Schatten classes S_q of multilinear forms (they only consider the case of forms) by interpolation between the nuclear forms S_1 and the compact forms S_∞ . The first question, still open, is whether the spaces obtained by the real and complex interpolation methods coincide. What is known (see [3]) is that the definition of S_2 by interpolation coincides with Definition 4.1 for both the real and complex method.

Another possible way of defining the Schatten classes, which also appears in the above mentioned works, is to define them using approximation numbers. This alternative definition, as it appears in [14], is also closed under interpolation. However, the exact relation between these two definitions, and the relation of both of them to the class of multiple $(q, 2)$ -summing operators are not clear at all.

5.2. On the Inclusion Theorem. All what is known about inclusions between the classes of multiple p -summing operators is contained in Theorems 3.4, 3.10 and [17, Theorem 3.6]. There are still many open questions. We point out the following:

- Q1. Is it essential for Y to have cotype 2 in Theorem 3.10?
- Q2. Does there exist an Inclusion Theorem for $2 < p \leq q < \infty$? If not, we could ask for weak versions of the Inclusion Theorem. Specifically, we do not know if, or under which conditions, the following is true: *If $T \in \Pi_p^n(X_1, \dots, X_n; Y) \cap \Pi_q^n(X_1, \dots, X_n; Y)$ and $1 \leq p \leq r \leq q < \infty$, then $T \in \Pi_r^n(X_1, \dots, X_n; Y)$.*

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