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A note on the strong maximal operator on $\mathbb{R}^n$

by

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Abstract. We prove that for $f \in L\ln^+ L(\mathbb{R}^n)$ with compact support, there is a $g \in L\ln^+ L(\mathbb{R}^n)$ such that (a) g and f are equidistributed, (b) $M_S(g) \in L^1(E)$ for any measurable set E of finite measure.

1. Introduction. For a function $f \in L_{\mathrm{loc}}(\mathbb{R}^n)$, its *Hardy–Littlewood maximal function* is defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int\limits_Q |f(y)|\,dy,$$

where Q is a cube with sides parallel to the coordinate axes, and its *strong maximal function* is defined by

$$M_S(f)(x) = \sup_{P \ni x} \frac{1}{|P|} \int\limits_P |f(y)|\,dy,$$

where P is a rectangle with sides parallel to the coordinate axes. In addition, let $M^*(f)(x) = M_n \circ \cdots \circ M_1(f)(x)$, where M_j is the Hardy–Littlewood maximal operator on $\mathbb{R}^1$ acting on the jth coordinate x_j.

It is well known that for f with compact support,

- $M(f) \in L^1(E)$ for any measurable set E of finite measure $\Leftrightarrow f \in L\ln^+ L(\mathbb{R}^n)$ (see Stein [5]).
- $M^*(f) \in L^1(E)$ for any measurable set E of finite measure $\Leftrightarrow f \in L(\ln^+ L)^n(\mathbb{R}^n)$ (see Jessen–Marcinkiewicz–Zygmund [4] and Fava–Gatto–Gutiérrez [2]).
- $f \in L(\ln^+ L)^n(\mathbb{R}^n) \Rightarrow M_S(f) \in L^1(E)$ for any measurable set E of finite measure, because $M_S(f) \le M^*(f)$.

It was conjectured that for $f \in L(\ln^+ L)^{n-1}(\mathbb{R}^n)$, $M_S(f) \in L^1(E)$ for any measurable set E of finite measure $\Rightarrow f \in L(\ln^+ L)^n(\mathbb{R}^n)$ (see [2]). In [1]

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and [3], Bagby and Gomez independently proved that there are many functions $f \in L\ln^+ L(\mathbb{R}^2)$ such that $M_S(f) \in L^1(E)$ for any measurable set E of finite measure.

In this paper, in a different way which can be easily applied to higher dimensions, we shall prove that the conjecture is also not true for $n > 2$. An interesting thing is that we do not need $f \in L(\ln^+ L)^{n-1}(\mathbb{R}^n)$.

THEOREM 1. *For $f \in L\ln^+ L(\mathbb{R}^n)$ with compact support, there is a $g \in L\ln^+ L(\mathbb{R}^n)$ such that* (a) *g and f are equidistributed*, (b) $M_S(g) \in L^1(E)$ *for any measurable set E of finite measure.*

2. Proof of the Theorem. Before proving the above theorem, we first introduce some notations and give some lemmas. Let

$$A_t = \Big\{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i = t\Big\},$$

$$D = \Big\{(x_1, \ldots, x_n) : \sum_{i=1}^n x_i \ge n-1,\ x_i \le 1\ (i = 1, \ldots, n)\Big\},$$

$$t(x) = \sum_{i=1}^n x_i, \qquad v(x) = \mu_n(\{y \in D : t(y) < t(x)\}),$$

where μ_n denotes the Lebesgue measure on $\mathbb{R}^n$. Without loss of generality, we may assume that

$$\mu_n(\{x \in \mathbb{R}^n : |f(x)| > 0\}) \le \mu_n(D).$$

Take

$$g(x) = \begin{cases} f^*(v(x)) & \text{for } x \in D, \\ 0 & \text{for } x \notin D, \end{cases}$$

where f^* is the rearrangement function of f, i.e.

$$f^*(r) = \lambda_f^{-1}(r) := \inf\{s : \lambda_f(s) \le r\},$$
$$\lambda_f(s) = \mu_n(\{x \in \mathbb{R}^n : |f(x)| > s\}),$$

for $r, s > 0$. It is not difficult to show that f and g have the same distribution function, i.e.

$$\mu_n(\{x \in \mathbb{R}^n : |f(x)| > s\}) = \mu_n(\{x \in \mathbb{R}^n : |g(x)| > s\})$$

for all $s > 0$.

Let $\widetilde{g}(s) = \sup\{g(x) : t(x) = s\}$. It is easy to check that $\operatorname{supp}(\widetilde{g}) \subseteq [n-1, n]$, $g \in L\ln^+ L(\mathbb{R}^n) \Rightarrow \widetilde{g} \in L\ln^+ L(\mathbb{R}^1)$, and $\widetilde{g} \in L\ln^+ L(\mathbb{R}^1) \Rightarrow g \in L\ln^+ L(\mathbb{R}^n)$ if $\mu_n(\{x \in \mathbb{R}^n : |f(x)| > 0\}) > \mu_n(D)$.

LEMMA 2. *$M_S(g)(x) \le C_n M(\widetilde{g})(t(x))$, where M_S is the strong maximal function operator on $\mathbb{R}^n$ and M is the Hardy–Littlewood maximal function operator on $\mathbb{R}^1$.*

Proof. For $x \in \mathbb{R}^n$, $t \in \mathbb{R}^1$, and $P = \prod_{i=1}^n [a_i, b_i] \ni x$, let

$$d_t = \sup_{y \in P} d(y, A_t).$$

It is easy to see that if $P \cap A_t \neq \emptyset$, we have

$$d_t \ge \frac{1}{2\sqrt{n}}\Big(\sum_{i=1}^n b_i - \sum_{i=1}^n a_i\Big), \qquad d_t \cdot \mu_{n-1}(A_t \cap P) \le \mu_n(P).$$

So, we have

$$\mu_{n-1}(A_t \cap P) \le 2\sqrt{n} \cdot \mu_n(P) \Big/ \Big(\sum_{i=1}^n b_i - \sum_{i=1}^n a_i\Big).$$

Now, let $e_0 = (\sqrt{n^{-1}}, \ldots, \sqrt{n^{-1}})$, $L_0 = (\mathbb{R}^1 e_0)^{\perp}$, and $\mathbb{R}^n \ni x = re_0 \dot{+} z$, where $z \in L_0$. Noting that $P \ni x$ implies that $t(x) \in [\sum_{i=1}^n a_i, \sum_{i=1}^n b_i]$, we have

$$\begin{aligned}
\frac{1}{\mu_n(P)} \int_P g(y)\,dy &= \frac{1}{\mu_n(P)} \int_{\mathbb{R}^1 e_0 \times L_0} \chi_P(x) g(x)\,dx \\
&= \frac{1}{\mu_n(P)} \int_{\mathbb{R}^1 e_0} \int_{L_0} \chi_D(re_0 \dot{+} z) g(re_0 \dot{+} z)\,dr\,dz \\
&\le \frac{1}{\mu_n(P)} \int_{\sum_{i=1}^n a_i/\sqrt{n}}^{\sum_{i=1}^n b_i/\sqrt{n}} \mu_{n-1}(\{z : re_0 \dot{+} z \in P\}) \widetilde{g}(r\sqrt{n})\,dr \\
&\le \frac{1}{\sqrt{n}\,\mu_n(P)} \int_{\sum_{i=1}^n a_i}^{\sum_{i=1}^n b_i} \mu_{n-1}\Big(\Big\{z : \frac{r}{\sqrt{n}} e_0 \dot{+} z \in P\Big\}\Big) \widetilde{g}(r)\,dr \\
&= \frac{1}{\sqrt{n}\,\mu_n(P)} \int_{\sum_{i=1}^n a_i}^{\sum_{i=1}^n b_i} \mu_{n-1}(A_r \cap P) \widetilde{g}(r)\,dr \\
&\le \frac{2}{\sum_{i=1}^n b_i - \sum_{i=1}^n a_i} \int_{\sum_{i=1}^n a_i}^{\sum_{i=1}^n b_i} \widetilde{g}(t)\,dt \le 2M(\widetilde{g})(t(x)).
\end{aligned}$$

LEMMA 3. *For $|x| > 2n$,*

$$M_S(g)(x) \le C_n |x|^{-1} \|g\|_1.$$

Proof. Without loss of generality, we may assume that $x_1 > |x|/n$ for $|x| > 2n$, and furthermore, we may assume that $a_1 < 1$ and $a_1 + \sum_{i=2}^n b_i >$

$n-1$ for $P=\prod_{i=1}^n[a_i,b_i]$ containing x and such that $P\cap D\neq\emptyset$. Let $z=(1,b_1,\ldots,b_n)$. We have

$$\begin{aligned}\frac{1}{\mu_n(P)}\int_P g(y)\,dy &\le \frac{1-a_1}{b_1-a_1}\,\frac{1}{\mu_n([a_1,1]\times\prod_{i=2}^n[a_i,b_i])}\int\limits_{[a_1,1]\times\prod_{i=2}^n[a_i,b_i]} g(y)\,dy\\ &\le \frac{1-a_1}{b_1-a_1}\,M_S(g)(z)\le \frac{1-a_1}{b_1-a_1}\,C_nM(\widetilde{g})(t(z))\\ &\le \frac{1-a_1}{|x_1|-1}\,C_n\,\frac{\sqrt{n}}{t(z)-(n-1)}\,\|\widetilde{g}\|_1\\ &\le C_n'\,\frac{1}{|x|}\,\frac{1-(n-1-\sum_{i=2}^n b_i)}{1+\sum_{i=2}^n b_i-(n-1)}\,\|\widetilde{g}\|_1\le C_n\,\frac{1}{|x|}\,\|g\|_1.\end{aligned}$$

From Lemmas 2–3, we can easily get the Theorem.

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