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Notes on q-deformed operators

by

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Abstract. The paper concerns operators of deformed structure like q-normal and q-hyponormal operators with the deformation parameter q being a positive number different from 1. In particular, an example of a q-hyponormal operator with empty spectrum is given, and q-hyponormality is characterized in terms of some operator inequalities.

1. Introduction. The formal algebraic relation

$$xx^* = qx^*x, \tag{1}$$

with the parameter $q > 0$, $q \neq 1$, appears in several different situations (cf. [1], [4] and [7]). If x is an operator in a Hilbert space, this leads to the study of q-normal operators. Non-trivial q-normal operators must necessarily be unbounded and they have many basic properties that are different from those of usual ($q = 1$) normal ones (see [5]). For instance, q-normal operators have *large spectrum* (e.g., the spectrum of every q-normal weighted shift is the complex plane $\mathbb{C}$). On the other hand, there is a bounded q-hyponormal operator T such that $\sigma(T) = \{0\}$. It turns out that this can be pushed to the very extreme: in Section 3 we give an example of a q-hyponormal operator having *empty spectrum*.

In Section 4 direct sums of q-deformed operators are studied and it is shown that the direct sum of q-quasinormal operators is q-quasinormal. This provides a way to construct new deformed operators from old ones. In particular, in the case of $q > 1$, we get existence of an unbounded q-quasinormal operator which is *not* q-normal. This has to be related to the fact that a q-normal operator is always unbounded, which means that q-quasinormal

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operators exist and that, on the other hand, bounded q-quasinormal operators exist as well [5].

There are different possibilities to define order relations among unbounded operators. The aim of Section 5 is to characterize q-hyponormality in terms of these relations.

2. q-deformed operators. Throughout this paper, we suppose that q is a positive real number such that $q \neq 1$ and that all operators are linear. For an operator T in a Hilbert space $\mathcal{H}$, the domain and kernel of T are denoted by $\mathcal{D}(T)$ and $\ker T$, respectively. The usual inner product of $\mathcal{H}$ is denoted by $\langle \cdot, - \rangle$. For operators S and T in $\mathcal{H}$, the relation $S \subset T$ means that $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $S\eta = T\eta$ for all $\eta \in \mathcal{D}(S)$. We write $\mathbb{C}$ and $\mathbb{Z}$ for the set of complex numbers and the set of integers, respectively.

We give a brief review of q-deformed operators (this means that q is regarded as a deformation parameter). A densely defined operator T in a Hilbert space $\mathcal{H}$ is said to be *q-normal* if

$$\mathcal{D}(T) = \mathcal{D}(T^*), \quad \|T^*\eta\| = \sqrt{q}\,\|T\eta\| \quad \text{for } \eta \in \mathcal{D}(T). \tag{2}$$

Because this implies that a q-normal operator must be closed we can state the definition in an equivalent way (see [10] or [9] for an argument): A closed densely defined operator T in $\mathcal{H}$ is q-normal if and only if

$$TT^* = qT^*T. \tag{3}$$

Condition (3) reminds the formal relation (1). It, or alternatively (2), is equivalent to

$$|T^*| = \sqrt{q}\,|T|. \tag{4}$$

Let T be a closed densely defined operator in $\mathcal{H}$ with polar decomposition $T = U|T|$. If T satisfies the relation

$$U|T| = \sqrt{q}\,|T|U,$$

then T is called a *q-quasinormal operator*.

Weakening the requirement of (2) we say that a densely defined operator T is *q-hyponormal* if it satisfies

$$\mathcal{D}(T) \subset \mathcal{D}(T^*), \quad \|T^*\eta\| \le \sqrt{q}\,\|T\eta\| \tag{5}$$

for all $\eta \in \mathcal{D}(T)$. A q-hyponormal operator is closable and moreover a q-quasinormal operator is q-hyponormal.

Let T be a q-hyponormal operator in $\mathcal{H}$. Then there exists a unique contraction K_T such that

$$T^* \supset \sqrt{q}\,K_T T, \quad \ker K_T \supset \ker T^*.$$

K_T is called the *contraction attached to* T. For a closed q-hyponormal operator T, T is q-quasinormal if and only if $K_T = (U^*)^2$ (U is the partial isometry in the polar decomposition of T).

It follows immediately from (5) that a non-trivial q-hyponormal operator with $0 < q < 1$ is unbounded. Similarly (2) implies that a non-trivial q-normal operator is always unbounded (unless $q = 1$, which is not the case of our interest in this paper). Because the spectrum of a q-normal weighted shift is equal to $\mathbb{C}$ we can say that q-normal operators have sufficiently large spectrum. Moreover, every q-normal operator T is unitarily equivalent to qT. We refer to [5] and [6] for further details in this matter.

3. A q-deformed operator with empty spectrum. Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$. Recall that the resolvent set $\varrho(T)$ of T is defined as the set of all $\lambda \in \mathbb{C}$ for which $\ker(\lambda - T) = \{0\}$, $\mathcal{R}(\lambda - T) = \mathcal{H}$ and the inverse $(\lambda - T)^{-1}$ is bounded on $\mathcal{H}$. Consequently, $0 \in \varrho(T)$ if and only if there is a bounded operator S on $\mathcal{H}$ such that

$$ST \subset 1, \quad TS = 1. \tag{6}$$

It is clear that for any $q > 1$ every bounded hyponormal operator is a q-hyponormal operator. The converse is not true in general. Even more, we have

PROPOSITION 3.1. *Let T be a non-zero bounded operator on a Hilbert space $\mathcal{H}$. If T is q-quasinormal, then T cannot be hyponormal.*

Proof. Suppose T is hyponormal. It is well known that

$$\|T\| = \gamma(T),$$

where $\gamma(T)$ is the spectral radius of T. Since T is q-quasinormal, T is quasinilpotent by [5, Corollary 9.2]. Therefore, $T = 0$. This is a contradiction. ∎

The following result is in [9, Proposition 1.6].

LEMMA 3.2. *Let T be a closed densely defined operator in a Hilbert space $\mathcal{H}$ such that $0 \in \varrho(T)$. If $\sigma(T^{-1}) = \{0\}$, then $\sigma(T) = \emptyset$.*

Let S be a closed densely defined operator in a separable Hilbert space $\mathcal{H}$. If there are an orthonormal basis $\{e_n\}$ ($n \in \mathbb{Z}$) and a sequence $\{w_n\}$ ($w_n \neq 0$, $n \in \mathbb{Z}$) of complex numbers such that

$$\mathcal{D}(S) = \Big\{ \sum_{n=-\infty}^{\infty} \alpha_n e_n \in \mathcal{H} : \sum_{n=-\infty}^{\infty} |\alpha_n|^2 |w_n|^2 < \infty \Big\}$$

and

$$Se_n = w_n e_{n+1} \quad \text{for all } n \in \mathbb{Z},$$

then S is called a *bilateral* (*injective*) *weighted shift* with weight sequence $\{w_n\}$ (with respect to $\{e_n\}$). A unilateral weighted shift is defined analogously. A bilateral (resp. unilateral) weighted shift is q-hyponormal if and only if $\sqrt{q}\,|w_{n+1}| \geq |w_n|$ for all $n \in \mathbb{Z}$ (resp. for all $n \geq 0$) (see [5, Section 4] for further details).

Let $q > 1$. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. Take numbers r and ℓ such that

$$\ell > 1 > r \geq 1/\sqrt{q}. \tag{7}$$

Put

$$w_n = \begin{cases} r^n & \text{if } n \geq 0, \\ \ell^n & \text{if } n \leq -1, \end{cases}$$

and consider the weighted shift S_0 with weight sequence $\{w_n\}$. Then, clearly, S_0 is bounded with $\mathcal{D}(S_0) = \mathcal{H}$. Since the sequence $\{w_n\}$ tends to zero as $|n| \to \infty$, S_0 is compact and so $\sigma(S_0)$ is countable. On the other hand, by [8, Corollary 2],

$$\sigma(S_0) = c\sigma(S_0)$$

for all $c \in \mathbb{C}$ with $|c| = 1$. It follows that $\sigma(S_0) = \{0\}$.

Since $\ker(S_0) = \ker(S_0^*) = \{0\}$, S_0 is injective and has dense range. This means that the inverse S_0^{-1} is closed and densely defined. Hence, it follows from Lemma 3.2 that S_0^{-1} has empty spectrum. On the other hand, we have

$$\frac{w_{n+1}}{w_n} = \begin{cases} r \geq 1/\sqrt{q} & \text{for } n \geq 0, \\ \ell > 1 > 1/\sqrt{q} & \text{for } n \leq -1. \end{cases}$$

These inequalities imply that S_0 is q-hyponormal. Therefore, by [5, Proposition 3.10], S_0^{-1} is also q-hyponormal. Thus, we obtain

THEOREM 3.3. *For every $q > 1$ there exists a q-hyponormal operator with empty spectrum.*

REMARK 3.4. The argument given in the above proof shows that if a bounded q-hyponormal operator T with $\sigma(T) = \{0\}$ is injective and has dense range then T^{-1} is a closed densely defined q-hyponormal operator and it satisfies $\sigma(T^{-1}) = \emptyset$.

4. Direct sums of q-deformed operators. Let S and T be densely defined operators in a Hilbert space $\mathcal{H}$. Then $S \oplus T$ is a densely defined operator in the direct sum Hilbert space $\mathcal{H} \oplus \mathcal{H}$ defined by

$$(S \oplus T)(\xi \oplus \eta) = S\xi \oplus T\eta$$

for $\xi \in \mathcal{D}(S)$ and $\eta \in \mathcal{D}(T)$.

THEOREM 4.1. *Let T_1 and T_2 be q-hyponormal operators in a Hilbert space $\mathcal{H}$. Then $T_1 \oplus T_2$ is also q-hyponormal in $\mathcal{H} \oplus \mathcal{H}$ and*

$$K_{T_1 \oplus T_2} = K_{T_1} \oplus K_{T_2}.$$

Moreover, $T_1 \oplus T_2$ is q-normal (*resp. q-quasinormal*) *if and only if both T_1 and T_2 are q-normal* (*resp. q-quasinormal*).

Proof. Since $T_i{}^* \supset \sqrt{q} K_{T_i} T_i$ and $\ker K_{T_i} \supset \ker T_i{}^*$ $(i = 1,\ 2)$, we have

$$\begin{aligned}(T_1 \oplus T_2)^* &= T_1{}^* \oplus T_2{}^* \\ &\supset \sqrt{q}(K_{T_1} T_1 \oplus K_{T_2} T_2) = \sqrt{q}(K_{T_1} \oplus K_{T_2})(T_1 \oplus T_2)\end{aligned}$$

and

$$\ker(K_{T_1} \oplus K_{T_2}) \supset \ker T_1{}^* \oplus \ker T_2{}^* = \ker(T_1 \oplus T_2)^*.$$

Hence, $K_{T_1} \oplus K_{T_2}$ is the contraction attached to $T_1 \oplus T_2$. Therefore, $T_1 \oplus T_2$ is q-hyponormal and $K_{T_1 \oplus T_2} = K_{T_1} \oplus K_{T_2}$.

Let $T_i = U_i |T_i|$ be the polar decomposition of T_i $(i = 1, 2)$. Then $T_1 \oplus T_2$ has the polar decomposition

$$T_1 \oplus T_2 = (U_1 \oplus U_2)(|T_1| \oplus |T_2|).$$

If T_1 and T_2 are q-quasinormal, then

$$K_{T_1} \oplus K_{T_2} = (U_1^*)^2 \oplus (U_2^*)^2 = (U_1 \oplus U_2)^{*2}.$$

Since $K_{T_1} \oplus K_{T_2}$ is the contraction attached to $T_1 \oplus T_2$, it follows that $T_1 \oplus T_2$ is q-quasinormal. The converse is easily proved analogously.

Finally, from the definition of q-normality it is not difficult to see that $T_1 \oplus T_2$ is q-normal if and only if both T_1 and T_2 are q-normal. ∎

REMARK 4.2. For $0 < q < 1$ a non-trivial q-hyponormal operator is always unbounded and the 2-dimensional Lebesgue measure of its spectrum is positive ([6]).

If $q > 1$ a q-quasinormal unilateral weighted shift is always bounded ([5]). On the other hand q-normal operators which are always q-quasinormal must necessarily be unbounded. Using Theorem 4.1 one can construct an unbounded q-quasinormal operator which is not q-normal. For this take T_1 to be any q-normal operator (which is unbounded) and T_2 to be any bounded q-quasinormal operator.

In view of Theorem 3.3 there exists a q-hyponormal operator, again with $q > 1$, which has empty spectrum; this is in contrast to the fact that every closed densely defined hyponormal operator $(q = 1)$ has non-empty spectrum ([9]).

5. Order relations for q-deformed operators. For unbounded operators there are several ways to define order relations. Besides the relations considered by Kato and Rellich ([2] and [11]), where for operators S and T in $\mathcal{H}$,

$$S \ll T \quad \text{means} \quad \mathcal{D}(T) \subset \mathcal{D}(S) \text{ and } \|S\eta\| \leq \|T\eta\| \text{ for } \eta \in \mathcal{D}(T)$$

and

$$S \preceq T \quad \text{means} \quad \mathcal{D}(T^{1/2}) \subset \mathcal{D}(S^{1/2}) \text{ and } \|S^{1/2}\eta\| \leq \|T^{1/2}\eta\| \text{ for } \eta \in \mathcal{D}(T^{1/2})$$
$$\text{provided } S \text{ and } T \text{ are selfadjoint and non-negative,}$$

we consider the relation

$$S \leq T \quad \text{means} \quad \mathcal{D}(T) \subset \mathcal{D}(S) \text{ and } \langle S\eta, \eta\rangle \leq \langle T\eta, \eta\rangle \text{ for } \eta \in \mathcal{D}(T) \tag{8}$$
$$\text{provided } S \text{ and } T \text{ are symmetric.}$$

Let H be a symmetric operator in $\mathcal{H}$ such that $\langle H\eta, \eta\rangle = 0$ for all $\eta \in \mathcal{D}(H)$. Then it follows that $\langle H\eta, \xi\rangle = 0$ for all $\eta, \xi \in \mathcal{D}(H)$. Since $\mathcal{D}(H)$ is dense in $\mathcal{H}$, $H = 0$. This shows that, if symmetric operators S and T satisfy $S \leq T$ and $T \leq S$, then $S = T$. Therefore, $\leq$ is an order relation.

Because q-normality means $TT^* = q\,T^*T$ a question is under which meaning of "$\leq$" the condition

$$TT^* \leq qT^*T \tag{9}$$

characterizes q-hyponormality.

PROPOSITION 5.1. *For a closed densely defined operator T in $\mathcal{H}$ consider the following statements*:

(a) *T is q-hyponormal*,
(b) $|T^*| \ll \sqrt{q}\,|T|$,
(c) $|T^*| \leq \sqrt{q}\,|T|$,
(d) $|T^*| \preceq \sqrt{q}\,|T|$,

Then (a)⇔(b)⇒(c)⇒(d). *If T is a weighted shift*, *unilateral or bilateral*, *then all these statements are equivalent.*

Proof. The equivalence of statements (a) and (b) is elementary. The implication (b)⇒(c) follows from [3, Chapter 5, Theorem 4.12].

We show the implication (c)⇒(d). Suppose $|T^*| \leq \sqrt{q}\,|T|$. Take η in $\mathcal{D}(|T|^{1/2})$. Since $\mathcal{D}(|T|)$ is a core for $|T|^{1/2}$, there is a sequence $\{\eta_n\}$ in $\mathcal{D}(|T|)$ such that $\eta_n \to \eta$ and $|T|^{1/2}\eta_n \to |T|^{1/2}\eta$. By our assumption, we have

$$\eta_n \in \mathcal{D}(|T^*|) \subset \mathcal{D}(|T^*|^{1/2})$$

and

$$\|\,|T^*|^{1/2}\eta_n\|^2 = \langle |T^*|\eta_n, \eta_n\rangle \leq \sqrt{q}\,\langle |T|\eta_n, \eta_n\rangle = \sqrt{q}\,\|\,|T|^{1/2}\eta_n\|^2.$$

Hence, the sequence $\{|T^*|^{1/2}\eta_n\}$ is Cauchy. It follows that

$$\eta \in \mathcal{D}(|T^*|^{1/2}), \qquad |T^*|^{1/2}\eta_n \to |T^*|^{1/2}\eta.$$

Therefore, $\|\,|T^*|^{1/2}\eta\| \leq \sqrt[4]{q}\,\|\,|T|^{1/2}\eta\|$. Thus $|T^*| \preceq \sqrt{q}\,|T|$.

Next, suppose that T is a bilateral weighted shift with weight sequence $\{w_n\}$ and statement (d) holds true. Clearly, $e_n \in \mathcal{D}(|T|) \cap \mathcal{D}(|T^*|)$. Hence,

$$\langle |T^*| e_n, e_n \rangle = \| \, |T^*|^{1/2} e_n \|^2 \le \sqrt{q} \, \| \, |T|^{1/2} e_n \|^2 = \sqrt{q} \, \langle |T| e_n, e_n \rangle.$$

Since $|T| e_n = |w_n| e_n$ and $|T^*| e_n = |w_{n-1}| e_n$, we have $\sqrt{q} \, |w_{n+1}| \ge |w_n|$ for all integers n. Thus, T is q-hyponormal. In the case of a unilateral weighted shift, an analogous argument shows that all the statements are equivalent. This completes the proof. ∎

By the same arguments as in the proof of implication (c)⇒(d) above, we have

PROPOSITION 5.2. *If a closed densely defined operator T in $\mathcal{H}$ satisfies condition* (9) *with "$\le$" defined as in* (8), *then T is q-hyponormal.*

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