

Some aspects of nuclear vector groups

by

LYDIA AUSSENHOFER (Tübingen)

Abstract. In [2] W. Banaszczyk introduced nuclear groups, a Hausdorff variety of abelian topological groups which is generated by all nuclear vector groups (cf. 2.3) and which contains all nuclear vector spaces and all locally compact abelian groups.

We prove in 5.6 that the Hausdorff variety generated by all nuclear vector spaces and all locally compact abelian groups (denoted by \mathcal{V}_1) is strictly smaller than the Hausdorff variety of all nuclear groups (denoted by \mathcal{V}_2). More precisely, we characterize those nuclear vector groups belonging to \mathcal{V}_1 (5.5). (These are called special nuclear vector groups.) It is proved that special nuclear vector groups can be embedded into a product of nuclear and of discrete vector spaces (2.5).

The sequence space Σ_0 is introduced (2.6) and it is proved that it is a nuclear but not a special nuclear vector group (2.12). Moreover, together with all discrete vector spaces it generates the Hausdorff variety of all nuclear groups (3.3).

We show that the Hausdorff variety \mathcal{V}_0 generated by all nuclear vector spaces is strictly contained in \mathcal{V}_1 (4.5).

1. Notation.

Let us first fix some notation.

We consider real vector spaces only. For a (real) vector space E and a subset $X \subseteq E$ we denote by $\langle X \rangle_{\mathbb{R}}$ the linear span of X and by $\langle X \rangle_{\mathbb{Z}}$ the subgroup generated by X . If $L \subseteq E$ is a linear subspace, we write $L \leq E$. For a normed space E , the closed unit ball is denoted by B_E . Accordingly, if p is a (semi)norm on E then $B_p := \{x \in E : p(x) \leq 1\}$.

E' stands for the topological dual of the topological vector space E . The polar X^0 of $X \subseteq E$ is defined by $\{\varphi \in E' : \varphi(X) \subseteq [-1, 1]\}$.

For an abelian topological group G , the set of neighborhoods of the zero element 0 is denoted by $\mathcal{U}_G(0)$.

For a symmetric convex subset U of a vector space E , we denote by p_U the Minkowski functional of U .

$\dim E$ is the cardinality of a basis of E and $|X|$ stands for the cardinality of the set X . $\mathbb{R}^{(X)}$ denotes the subspace of \mathbb{R}^X which consists of those families where at most finitely many entries are $\neq 0$.

Finally, $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2. (Special) nuclear vector groups. In this section we introduce nuclear vector groups and the subclass of special nuclear vector groups. We will see that the sequence space Σ_0 (defined in 2.6) is a nuclear vector group but not a special nuclear vector group (2.12).

DEFINITION 2.1. For symmetric convex subsets X, Y of a vector space E and $k \in \mathbb{N}$, we put $d_k(X, Y) := \inf\{c > 0 : \text{there exists a subspace } L_c \leq E \text{ such that } \dim L_c < k \text{ and } X \subseteq cY + L_c\} \in [0, \infty]$.

$d_k(X, Y)$ is called the k -th *Kolmogorov diameter* of X with respect to Y in E .

If $T : E_1 \rightarrow E_2$ is a linear operator between the normed spaces E_1, E_2 , we set $d_k(T) := d_k(T(B_{E_1}), B_{E_2})$.

Let us list some properties of the Kolmogorov diameter. One easily verifies that $d_k(T(X), T(Y)) \leq d_k(X, Y)$ for every linear operator T . Hence, if T is an isomorphism, then $d_k(T(X), T(Y)) = d_k(X, Y)$. For every $c > 0$ and all symmetric convex subsets X, Y we have

$$d_k(cX, Y) = cd_k(X, Y) \quad \text{and} \quad d_k(X, cY) = \frac{1}{c}d_k(X, Y).$$

For symmetric convex subsets X, Y, Z, W such that $Z \subseteq X$ and $Y \subseteq W$ we have $d_k(Z, Y) \leq d_k(X, Y)$ and $d_k(X, W) \leq d_k(X, Y)$ for all $k \in \mathbb{N}$.

Recall that a (semi)norm p on a vector space V is named a *pre-Hilbert (semi)norm* if it satisfies the parallelogram law:

$$p(x + y)^2 + p(x - y)^2 = 2p(x)^2 + 2p(y)^2.$$

LEMMA 2.2. (i) *Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of symmetric convex sets in a vector space E satisfying $d_k(U_{n+1}, U_n) \leq 1/k$ for all $k, n \in \mathbb{N}$. For every $m \in \mathbb{N}$ there exists a subsequence $(U_{n_j})_{j \in \mathbb{N}}$ such that $n_1 = 1$ and $d_k(U_{n_{j+1}}, U_{n_j}) \leq k^{-m}$ for all $j, k \in \mathbb{N}$.*

(ii) *For every natural number $m \geq 3$, there exists a constant $\gamma_m > 0$ such that for all symmetric convex subsets X, Y of a vector space which satisfy $d_k(X, Y) \leq ck^{-m}$ for all $k \in \mathbb{N}$ and a positive constant $c > 0$, there are pre-Hilbert seminorms p and q defined on $\langle X \rangle_{\mathbb{R}}$ such that $X \subseteq B_p, B_q \subseteq Y$, and $d_k(B_p, B_q) \leq c\gamma_m k^{-m+3}$ for all $k \in \mathbb{N}$.*

(iii) *Let p be an arbitrary seminorm and let q be a pre-Hilbert seminorm on a vector space E such that $d_1(B_p, B_q) < \infty$. For every subspace $E_0 \leq E$, we have $d_k(B_p \cap E_0, B_q \cap E_0) \leq d_k(B_p, B_q)$.*

Proof. (i) Cf. (7.1.1) in [9] or 20.2 in [1].

A proof of (ii) can be found in [1] (Lemma 18.32). In [2] (Lemma 2.14), a slightly stronger version is formulated.

(iii) This is Lemma 2.13 of [2] or Lemma 18.18 of [1]. ■

DEFINITIONS 2.3. A locally convex Hausdorff vector space E is called a *nuclear vector space* if for every symmetric convex neighborhood $U \in$

$\mathcal{U}_E(0)$, there exists a symmetric convex neighborhood $W \in \mathcal{U}_E(0)$ such that $d_k(W, U) \leq k^{-1}$ for all $k \in \mathbb{N}$.

A vector space E endowed with a Hausdorff group topology having a 0-neighborhood basis consisting of symmetric convex sets is named a *nuclear vector group* if it has the following additional property: for every symmetric convex neighborhood $U \in \mathcal{U}_E(0)$, there exists a symmetric convex neighborhood $W \in \mathcal{U}_E(0)$ such that $d_k(W, U) \leq k^{-1}$ for all $k \in \mathbb{N}$.

We call a nuclear vector group E a *special nuclear vector group* if there exists a 0-neighborhood basis \mathcal{U}_0 consisting of symmetric convex sets such that for every $U_0 \in \mathcal{U}_0$ there exists a sequence $(U_n)_{n \in \mathbb{N}}$ of symmetric convex neighborhoods which satisfy $\langle U_0 \rangle_{\mathbb{R}} = \langle U_n \rangle_{\mathbb{R}}$ for all $n \in \mathbb{N}$ and $d_k(U_n, U_{n-1}) \leq 1/k$ for all $k, n \in \mathbb{N}$.

REMARKS 2.4. (i) The main difference between nuclear vector spaces and nuclear vector groups is that in a nuclear vector group, the linear span $\langle W \rangle_{\mathbb{R}}$ of a smaller neighborhood is in general strictly smaller than $\langle U \rangle_{\mathbb{R}}$.

(ii) In general, a nuclear vector group is not connected. But if it is connected, then every neighborhood of 0 is absorbing. Conversely, if every neighborhood is absorbing, it is easily verified that the scalar multiplication is continuous and hence the nuclear vector group is a topological (and therefore nuclear) vector space.

(iii) For every neighborhood W of 0 in a nuclear vector group and for every $c > 0$, the set cU is a neighborhood of 0.

(iv) It is clear that every nuclear vector space and every discrete vector space (i.e. a vector space endowed with the discrete topology) is a special nuclear vector group.

It follows easily from Lemma 2.7 of [2] that every product of (special) nuclear vector groups is a (special) nuclear vector group. Trivially, every Hausdorff quotient space of a (special) nuclear vector group is a (special) nuclear vector group.

According to 2.2 and (iii), every subspace of a (special) nuclear vector group is a (special) nuclear vector group.

PROPOSITION 2.5. *Every special nuclear vector group E can be embedded into a product of nuclear and of discrete vector spaces.*

Proof. Fix a neighborhood basis \mathcal{U}_0 as in the definition. For $U_0 \in \mathcal{U}_0$, we denote by E_{U_0} the largest subspace contained in U_0 . As a consequence of $E_{U_0} \subseteq \{x \in U_0 : \lambda x \in U_0 \ \forall \lambda \in \mathbb{R}\} \subseteq p_{U_0}^{-1}(0) \subseteq E_{U_0}$ and the continuity of p_{U_0} , we see that E_{U_0} is a closed linear subspace. Hence E/E_{U_0} is a special nuclear vector group again (2.4(iv)). Let $\psi_{U_0} : E \rightarrow E/E_{U_0}$ be the canonical projection.

We fix a sequence $(U_n)_{n \in \mathbb{N}}$ of neighborhoods which satisfies

$$d_k(U_n, U_{n-1}) \leq 1/k \quad \forall k, n \in \mathbb{N} \quad \text{and} \quad \langle U_n \rangle_{\mathbb{R}} = \langle U_0 \rangle_{\mathbb{R}} \quad \forall n \in \mathbb{N}.$$

Obviously, the sequence $(\psi_{U_0}(U_n))_{n \in \mathbb{N}}$ has the corresponding properties in E/E_{U_0} . So the family $(\varepsilon\psi_{U_0}(U_n) : \varepsilon > 0, n \in \mathbb{N})$ forms a 0-neighborhood basis of a nuclear vector space topology on $F_0 := \langle \psi_{U_0}(U_0) \rangle_{\mathbb{R}}$ (2.4(ii)). (Observe further that $\psi_{U_0}(U_0)$ contains only the trivial subspace, which implies that this topological space is necessarily Hausdorff.) Let \tilde{F}_{U_0} be an algebraic complement such that $E/E_{U_0} = F_{U_0} \oplus \tilde{F}_{U_0}$ and let π_{F_0} , respectively $\pi_{\tilde{F}_0}$, denote the corresponding projections onto F_{U_0} , respectively \tilde{F}_{U_0} . Due to the definition, F_{U_0} is open in E/E_{U_0} . We endow \tilde{F}_{U_0} with the discrete topology. Then

$$\varphi_{U_0} : E \rightarrow F_{U_0} \times \tilde{F}_{U_0}, \quad x \mapsto (\pi_{F_0}(\psi_{U_0}(x)), \pi_{\tilde{F}_0}(\psi_{U_0}(x))),$$

is a well defined, continuous linear mapping with kernel E_{U_0} ; moreover $\varphi_{U_0}(U_0)$ is a neighborhood of 0. So

$$\varphi : E \rightarrow \prod_{U_0 \in \mathcal{U}_0} F_{U_0} \times \tilde{F}_{U_0}, \quad x \mapsto (\varphi_{U_0}(x))_{U_0 \in \mathcal{U}_0},$$

is a continuous monomorphism. Since

$$\varphi(\tilde{U}_0) = \varphi(E) \cap \left(\left(\prod_{U_0 \in \mathcal{U}_0, U_0 \neq \tilde{U}_0} F_{U_0} \times \tilde{F}_{U_0} \right) \times \varphi_{\tilde{U}_0}(\tilde{U}_0) \right)$$

for every $\tilde{U}_0 \in \mathcal{U}_0$, φ is an embedding. ■

Next we present a concrete example of a nuclear vector group.

EXAMPLE 2.6. For $r \in \mathbb{N}_0$, let

$$p_r : \Sigma_r \rightarrow \mathbb{R}, \quad (x_n)_{n \in \mathbb{N}} \mapsto \sup\{n^r|x_n| : n \in \mathbb{N}\},$$

where

$$\Sigma_r := \{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : (n^r x_n)_{n \in \mathbb{N}} \text{ is bounded}\}.$$

For $\Sigma := \bigcap_{r \in \mathbb{N}} \Sigma_r$ we have

$$d_k(B_{p_{r+1}}, B_{p_r}) \leq k^{-1} \quad \text{and} \quad d_k(B_{p_{r+1}} \cap \Sigma, B_{p_r} \cap \Sigma) \leq k^{-1}.$$

[Observe that $B_{p_{r+1}} \subseteq k^{-1}B_{p_r} + \langle e_1, \dots, e_{k-1} \rangle_{\mathbb{R}}$ (where $(e_j)_{j \in \mathbb{N}}$ denotes the sequence of the standard unit vectors); the analogous inclusions hold for the intersection with Σ .]

We endow the vector space $\Sigma_0 := \mathbb{R}^{\mathbb{N}}$ with the group topology induced by the neighborhoods $(\varepsilon B_{p_r} : r \in \mathbb{N}, \varepsilon > 0)$ and observe that it is a nuclear vector group. Σ endowed with the topology induced by the norms $p_r|_{\Sigma}$ is a nuclear vector space. Moreover, it is easily verified that Σ is the connected component of 0 in Σ_0 .

Furthermore, for $r \in \mathbb{N}_0$, we define $V_r := \{(x_n)_{n \in \mathbb{N}} : \sum_{n \in \mathbb{N}} (n^r x_n)^2 < \infty\}$ and

$$q_r : V_r \rightarrow \mathbb{R}, \quad ((x_n)_{n \in \mathbb{N}}) \mapsto \sqrt{\sum_{n \in \mathbb{N}} (n^r x_n)^2}.$$

It follows easily that

$$V_r \subseteq \Sigma_r \subseteq V_{r-1}, \quad q_r \geq p_r|_{V_r}, \quad p_r \geq \frac{\sqrt{6}}{\pi} q_{r-1}|_{\Sigma_r}$$

for all $r \in \mathbb{N}$. Hence the family $(\varepsilon B_{q_r} : \varepsilon > 0, r \in \mathbb{N}_0)$ generates the given topology on Σ_0 and for every $r \in \mathbb{N}_0$, (V_r, q_r) is a Hilbert space.

LEMMA 2.7. *Let $T : H_1 \rightarrow H_2$ be a continuous linear operator between inner product spaces. If T' denotes the adjoint operator, then $d_k(T') = d_k(T)$ for all $k \in \mathbb{N}$.*

Proof. The proof is straightforward. Alternatively, it is a consequence of (11.7.6) and (11.3.4) in [8] and of (18.16) in [1]. ■

LEMMA 2.8. *For every normed space E and every finite-dimensional subspace $L \leq E$, the sum of the closed unit ball B with L is closed.*

Proof. This is an easy consequence of the Bolzano–Weierstraß theorem. ■

LEMMA 2.9. *Let $T : H_1 \rightarrow H_2$, $x \mapsto \sum_{j \in \mathbb{N}} \alpha_j(x, x_j)_1 \cdot y_j$, be a continuous linear operator between the Hilbert spaces $(H_1, (\cdot, \cdot)_1)$ and $(H_2, (\cdot, \cdot)_2)$, where $(x_j)_{j \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ are orthonormal systems in H_1 and H_2 respectively and $(\alpha_j)_{j \in \mathbb{N}}$ is a decreasing sequence of non-negative real numbers. Then $d_k(T) = \alpha_k$ for all $k \in \mathbb{N}$.*

Proof. This is Proposition (11.3.3) in [8]. ■

We intend to apply these results to the sequence space Σ_0 ; we use the notation introduced in 2.6.

LEMMA 2.10. *For every $r \in \mathbb{N}_0$ and all $k \in \mathbb{N}$ we have $d_k(B_{q_r}, B_{q_0}) = k^{-r}$.*

Proof. Since $(k^{-r} e_k : k \in \mathbb{N})$ is an orthonormal basis of (V_r, q_r) , the inclusion $V_r \rightarrow V_0$ can be written in the form

$$(x_n) = \sum_{k \in \mathbb{N}} ((x_n), k^{-r} e_k)_r \cdot k^{-r} e_k = \sum_{k \in \mathbb{N}} k^{-r} ((x_n), k^{-r} e_k)_r \cdot e_k.$$

(The scalar product associated with q_r is denoted by $(\cdot, \cdot)_r$.) It follows from 2.9 that $d_k(B_{q_r}, B_{q_0}) = k^{-r}$ for all $k \in \mathbb{N}$. ■

LEMMA 2.11. *For every $r \in \mathbb{N}_0$ and every symmetric, convex, absorbing subset $X \subseteq V_r$, there exists $\varepsilon > 0$ such that $d_k(X, B_{q_0}) \geq \varepsilon k^{-r}$ for all $k \in \mathbb{N}$.*

Proof. Fix $r \in \mathbb{N}_0$. Since $V_r = \bigcup_{n \in \mathbb{N}} n \bar{X}^{q_r}$ and (V_r, q_r) is a Hilbert space, in particular a Baire space, there exists $n \in \mathbb{N}$ such that the interior of $n \bar{X}^{q_r}$

is not empty. It follows easily that $\overline{X}^{qr} \supseteq \varepsilon B_{q_r}$ for suitable $\varepsilon > 0$. (The norm as upper index denotes with respect to which topology the closure is taken.)

Next we prove that $d_k(X, B_{q_0}) = d_k(\overline{X}^{qr}, B_{q_0})$. The inequality “ \leq ” is trivial. Conversely, let $c > 0$, and $L \leq \Sigma_0$ be of finite dimension such that $X \subseteq cB_{q_0} + L$. Without loss of generality we may assume that $L \leq V_0$. Then $\overline{X}^{qr} \subseteq \overline{(cB_{q_0} + L) \cap V_r}^{qr} \subseteq \overline{cB_{q_0} + L}^{q_0} \stackrel{2.8}{=} cB_{q_0} + L$, which implies “ \geq ”.

Combining these results yields

$$d_k(X, B_{q_0}) = d_k(\overline{X}^{qr}, B_{q_0}) \geq d_k(\varepsilon B_{q_r}, B_{q_0}) = \varepsilon d_k(B_{q_r}, B_{q_0}) \stackrel{2.10}{=} \varepsilon k^{-r}$$

for all $k \in \mathbb{N}$, which was to be shown. ■

THEOREM 2.12. *The nuclear vector group Σ_0 is not a special nuclear vector group.*

Proof. Suppose the converse. This means: there exists a sequence $(U_n)_{n \in \mathbb{N}_0}$ of symmetric convex neighborhoods which satisfies:

- (i) $U_0 \subseteq B_{q_0}$,
- (ii) $d_k(U_{n+1}, U_n) \leq 1/k$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, and
- (iii) $\langle U_n \rangle_{\mathbb{R}} = \langle U_0 \rangle_{\mathbb{R}}$ for all $n \in \mathbb{N}$.

Applying 2.2(i) inductively enables us to find a subsequence of $(U_n)_{n \in \mathbb{N}}$ (again denoted by $(U_n)_{n \in \mathbb{N}}$) which satisfies

$$(ii)' \quad d_k(U_n, U_0) \leq k^{-n} \text{ for all } k, n \in \mathbb{N}.$$

There exists $r \in \mathbb{N}$ such that $V_r \subseteq \langle U_0 \rangle_{\mathbb{R}}$. Hence we get

$$d_k(U_n \cap V_r, B_{q_0}) \stackrel{(i)}{\leq} d_k(U_n, U_0) \stackrel{(ii)'}{\leq} k^{-n} \quad \text{for all } k, n \in \mathbb{N}.$$

Since the $U_n \cap V_r$ satisfy the assumptions of 2.11, there is a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive numbers such that

$$\varepsilon_n k^{-r} \leq d_k(U_n \cap V_r, B_{q_0}) \leq k^{-n} \quad \forall k, n \in \mathbb{N}.$$

This contradiction completes the proof. ■

3. Embedding of nuclear vector groups. We want to show in this section that every nuclear vector group can be embedded into a product of copies of Σ_0 and of discrete vector spaces.

LEMMA 3.1. *Every nuclear vector group E has a 0-neighborhood basis \mathcal{U}_0 consisting of symmetric convex sets such that for every $W \in \mathcal{U}_0$ there exists a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of linear forms defined on $\langle W \rangle_{\mathbb{R}}$ which satisfies:*

- (i) $(n^r \varphi_n)_{n \in \mathbb{N}}$ is equicontinuous for every $r \in \mathbb{N}$ and
- (ii) $\{x \in \langle W \rangle_{\mathbb{R}} : |\varphi_n(x)| \leq (2n)^{-1} \forall n \in \mathbb{N}\} \subseteq W$.

Proof. Let U be a symmetric convex neighborhood of 0. We have to show that there exists a symmetric convex neighborhood W of 0, contained in U , which has the properties stated above.

We may assume that U contains only the trivial subspace.

$[p_U^{-1}(0) = \{x \in E : \lambda x \in U \ \forall \lambda \in \mathbb{R}\}]$ is a closed linear subspace, and hence $E/p_U^{-1}(0)$ is a nuclear vector group again. Let $\pi : E \rightarrow E/p_U^{-1}(0)$ be the canonical projection. Then $\pi(U)$ contains only the trivial subspace. If we find $\widetilde{W} \subseteq \pi(U)$ and $(\widetilde{\varphi}_n)$ which satisfy the assertions in the quotient space, then $\pi^{-1}(\widetilde{W})$ and $\varphi_n := \widetilde{\varphi}_n \circ \pi$ have the desired properties in E .]

According to 2.2, there is a sequence $s_n : E_n \rightarrow \mathbb{R}$ ($n \in \mathbb{N}_0$) of continuous pre-Hilbert norms defined on a decreasing sequence $(E_n)_{n \in \mathbb{N}_0}$ of subspaces such that

$$(1) \quad B_{s_0} \subseteq U \quad \text{and} \quad d_k(B_{s_n}, B_{s_0}) \leq k^{-2n} \quad \text{for all } k \in \mathbb{N}, n \in \mathbb{N}_0.$$

Moreover, we may assume that $E_0 = E_1$. [Replacing E_0 by E_1 and s_0 by $s_0|_{E_1}$ implies that $B_{s_0|_{E_1}} = B_{s_0} \cap E_1$ is a neighborhood of 0 and

$$d_k(B_{s_0|_{E_1}}, B_{s_1}) = d_k(B_{s_0} \cap E_1, B_{s_1}) \leq d_k(B_{s_0}, B_{s_1}).]$$

For $n \in \mathbb{N}$, let $T_n : (E_n, s_n) \rightarrow (E_0, s_0)$ be the inclusion map. For the adjoint operator $T'_n : (E_0, s_0)' \rightarrow (E_n, s_n)'$ we have

$$(2) \quad d_k(T'_n) \stackrel{2.7}{=} d_k(T_n) = d_k(B_{s_n}, B_{s_0}) \stackrel{(1)}{\leq} k^{-2n} \quad \text{for all } k, n \in \mathbb{N}.$$

According to (9.1.4) of [7], T'_n is a compact operator for all $n \in \mathbb{N}$.

If one of the spaces E_n has finite dimension, then the assertion is clear, since the E_n are open in E and the weak topology on a finite-dimensional vector space coincides with the given one. So let us assume the converse.

Due to the spectral decomposition theorem (cf. (8.3.1) in [7]), for all $n \in \mathbb{N}$ there are orthonormal systems $(\varphi_{n,k})_{k \in \mathbb{N}}$ in $(E_0, s_0)'$ and $(\psi_{n,k})_{k \in \mathbb{N}}$ in $(E_n, s_n)'$ and a decreasing sequence $(\alpha_{n,k})_{k \in \mathbb{N}}$ of non-negative numbers such that

$$(3) \quad T'_n(\varphi) = \sum_{k \in \mathbb{N}} \alpha_{n,k}(\varphi, \varphi_{n,k})_0 \cdot \psi_{n,k} \quad \forall n \in \mathbb{N}, \forall \varphi \in (E_0, s_0)'$$

where $(\cdot, \cdot)_n$ denotes the inner product of the Hilbert space $(E_n, s_n)'$ (for $n \in \mathbb{N}_0$). Furthermore, according to 2.9 and (2), we have

$$(4) \quad \alpha_{n,k} = d_k(T'_n) \leq k^{-2n}.$$

Since T_1 is surjective, the adjoint operator T'_1 is injective and hence $(\varphi_{1,k})_{k \in \mathbb{N}}$ is an orthonormal basis of $(E_0, s_0)'$. Hence the closed linear span of the set $\{\varphi_{n,k} : n, k \in \mathbb{N}\}$ equals $(E_0, s_0)'$. We form the Gram-Schmidt

“orthonormalization” of the $\varphi_{n,k}$ with respect to $(\cdot, \cdot)_0$ in the following order:

$$\begin{array}{ccccccc} \varphi_{1,1} & & & & & & \\ \varphi_{2,1} & \varphi_{2,2} & \varphi_{1,2} & & & & \\ \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3} & \varphi_{2,3} & \varphi_{1,3} & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \end{array}$$

Observe that the family $(\varphi_{n,k} : n, k \in \mathbb{N})$ need not be linearly independent. (In this case it happens that some φ_l are 0 and hence not of norm 1.) Anyhow, the set $\{\varphi_l : l \in \mathbb{N}\} \setminus \{0\}$ forms an orthonormal basis of $(E_0, s_0)'$.

Since the k -th row has $2k - 1$ entries and $\varphi_{n,k}$ appears in the $\max(n, k)$ -th row, we get

$$(\varphi_l, \varphi_{n,k})_0 = 0 \quad \text{for } l > \sum_{j=1}^{\max(n,k)} (2j - 1) = \max(n, k)^2.$$

Moreover, putting φ_l in (3) yields

$$(5) \quad T'_n(\varphi_l) = \varphi_l|_{E_n} = \sum_{k \geq \sqrt{l}} \alpha_{n,k} (\varphi_l, \varphi_{n,k})_0 \cdot \psi_{n,k} \quad \text{for all } l > n^2$$

and hence for all $l > n^2$ we have

$$\begin{aligned} (\varphi_l|_{E_n}, \varphi_l|_{E_n})_n &\stackrel{(5)}{=} \sum_{k \geq \sqrt{l}} (\alpha_{n,k} (\varphi_l, \varphi_{n,k})_0)^2 \stackrel{(4)}{\leq} \sum_{k \geq \sqrt{l}} k^{-4n} ((\varphi_l, \varphi_{n,k})_0)^2 \\ &\leq l^{-2n} \sum_{k \geq \sqrt{l}} ((\varphi_l, \varphi_{n,k})_0)^2 \leq l^{-2n} \end{aligned}$$

by the Bessel inequality. This implies $l^n \varphi_l \in B_{s_n}^0$ for all $l > n^2$ and hence the set $\{l^r \varphi_l : l \in \mathbb{N}\}$ is equicontinuous for all $r \in \mathbb{N}$.

Now, let $x \in E_0$ satisfy $|\varphi_n(x)| \leq 1/(2n)$ for all $n \in \mathbb{N}$ and let $\varphi \in B_{s_0}^0$. Since φ can be represented in the form $\varphi = \sum_{l \in \mathbb{N}} (\varphi, \varphi_l)_0 \cdot \varphi_l$, we get

$$|\varphi(x)| \leq \sqrt{\sum_{l \in \mathbb{N}} ((\varphi, \varphi_l)_0)^2} \sqrt{\sum_{l \in \mathbb{N}} \varphi_l(x)^2} \leq (\varphi, \varphi)_0 \sqrt{\frac{\pi^2}{4 \cdot 6}} < 1.$$

By the Hahn–Banach theorem, x is an element of B_{s_0} and hence $W := B_{s_0}$ and $(\varphi_l)_{l \in \mathbb{N}}$ have the desired properties. ■

COROLLARY 3.2. *With the notation introduced above, for every $W \in \mathcal{U}_0$ there exists a continuous linear mapping $\Phi_W : \langle W \rangle_{\mathbb{R}} \rightarrow \Sigma_0$ which satisfies $W \supseteq \Phi_W^{-1}(\frac{1}{2}B_{p_1})$ (B_{p_1} as in 2.6); in particular, $\Phi_W(x) \neq 0$ for all $x \in \langle W \rangle_{\mathbb{R}} \setminus W$.*

Proof. Let $(\varphi_n)_{n \in \mathbb{N}}$ have the properties stated in 3.1 and put

$$\Phi_W : \langle W \rangle_{\mathbb{R}} \rightarrow \Sigma_0, \quad x \mapsto (\varphi_n(x))_{n \in \mathbb{N}}.$$

It is clear that Φ_W is a linear operator.

Fix $r \in \mathbb{N}$. By the equicontinuity of the sequence $(n^r \varphi_n)_{n \in \mathbb{N}}$, there exists a neighborhood U_r of 0 in $\langle W \rangle_{\mathbb{R}}$ such that $\varphi_n(x) \leq n^{-r}$ for all $x \in U_r$. Hence $\Phi_W(U_r) \subseteq B_{p_r}$ and the continuity of Φ_W follows.

According to 3.1(ii), $W \supseteq \Phi_W^{-1}(\frac{1}{2}B_{p_1})$, which implies $\Phi_W(x) \neq 0$ for all $x \in \langle W \rangle_{\mathbb{R}} \setminus W$. ■

THEOREM 3.3. *Every nuclear vector group E can be embedded into a product of copies of Σ_0 and of discrete vector spaces.*

Proof. Let \mathcal{U}_0 be a neighborhood basis having the properties stated in 3.1. For $W \in \mathcal{U}_0$, we fix a complementary subspace E_W such that $E = \langle W \rangle_{\mathbb{R}} \oplus E_W$ (algebraically). Endowing E_W with the discrete topology makes the above decomposition topological. Let $\pi_W : E \rightarrow \langle W \rangle_{\mathbb{R}}$ denote the projection with kernel E_W and for Φ_W as in 3.2, put

$$\Phi : E \rightarrow \prod_{W \in \mathcal{U}_0} (\Sigma_0 \times E_W), \quad x \mapsto (\Phi_W(\pi_W(x)), x - \pi_W(x))_{W \in \mathcal{U}_0}.$$

According to 3.2, Φ is a continuous monomorphism and

$$\Phi(W) \supseteq \text{im } \Phi \cap \left(\left(\frac{1}{2}B_{p_1} \times \{0\} \right) \times \prod_{W \neq W' \in \mathcal{U}_0} (\Sigma_0 \times E_{W'}) \right)$$

for all $W \in \mathcal{U}_0$. [Observe that an element of the right hand side is of the form $\Phi(x)$ where $x \in \langle W \rangle_{\mathbb{R}}$; moreover, $\Phi_W(x) \in \frac{1}{2}B_{p_1}$ implies $x \in W$.] Hence Φ is an embedding. ■

COROLLARY 3.4 (T. and Y. Kōmura). *Every nuclear vector space E can be embedded into a product of copies of Σ .*

Proof. We use the notation introduced in the proof of 3.3. One has to observe that every neighborhood of 0 in E is absorbing, which implies that $E_W = \{0\}$ for every $W \in \mathcal{U}_0$. Moreover, $\Phi_W(E)$ is a topological vector space and a subspace of Σ_0 , hence contained in the component Σ (see 2.4). ■

This corollary was first proved by T. and Y. Kōmura in [5].

4. On the variety generated by all nuclear vector spaces

NOTATION 4.1. Let \mathcal{C} be a class of topological groups. By $\mathcal{V}(\mathcal{C})$ we denote the smallest class of topological groups which contains \mathcal{C} and which is closed under forming arbitrary products, Hausdorff quotient groups, arbitrary subgroups, and which is closed under topological isomorphisms. Then $\mathcal{V}(\mathcal{C})$ is called the *Hausdorff (group-) variety generated by \mathcal{C}* .

It is well known that every member of $\mathcal{V}(\mathcal{C})$ is topologically isomorphic to a Hausdorff quotient group of a subgroup of a product of elements of \mathcal{C} (cf. [4], p. 65). In particular, $\mathcal{V}(\mathcal{C})$ exists. If \mathcal{C} consists of abelian Hausdorff groups, so does $\mathcal{V}(\mathcal{C})$.

In what follows we use the following abbreviations:

- \mathcal{V}_0 for the Hausdorff variety generated by all nuclear vector spaces,
- \mathcal{V}_1 for the Hausdorff variety generated by all nuclear vector spaces and all discrete abelian groups, and
- \mathcal{V}_2 for the Hausdorff variety generated by all nuclear vector groups.

In this section we are going to show that \mathcal{V}_0 is a proper subclass of \mathcal{V}_1 .

LEMMA 4.2. *Let E be a nuclear vector space which has a neighborhood $U \in \mathcal{U}_E(0)$ containing only the trivial subspace. Then $\dim E \leq |\mathbb{R}|$.*

Proof. Choose linear functionals $\varphi_n : E \rightarrow \mathbb{R}$ as in 3.1 associated with a neighborhood W contained in U . Since W contains only the trivial subspace, we conclude from 3.1(ii) that $\Phi := (\varphi_n) : E \rightarrow \mathbb{R}^{\mathbb{N}}$ is injective. Hence $\dim E \leq \dim \mathbb{R}^{\mathbb{N}} \leq |\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$. ■

LEMMA 4.3. *For a symmetric convex subset C of a vector space E and a subgroup H of E , put $U := H \cap C$, $H_U := \{x \in H : \langle x \rangle_{\mathbb{Z}} \subseteq U\}$, and $E_C := \{x \in E : \langle x \rangle_{\mathbb{R}} \subseteq C\}$. Then H_U is the largest subgroup contained in U , E_C is the largest subgroup (subspace) contained in C , and $E_C \cap H = H_U$.*

Proof. Since $E_C = p_C^{-1}(0)$, E_C is the largest linear subspace contained in C .

Obviously, $E_C \cap H \subseteq H_U$. Conversely, for $x \in H_U$, we have $\langle x \rangle_{\mathbb{Z}} \subseteq U \subseteq C$ and hence $p_C(kx) \leq 1$ for all $k \in \mathbb{Z}$. This implies $p_C(x) = 0$ and hence $x \in E_C \cap H$. Combining these inclusions yields $E_C \cap H = H_U$.

This shows that H_U is a subgroup of H contained in U ; by the definition of H_U , it is the largest one. ■

Using the above notation, we get the following commutative square where all mappings are the canonical ones and where τ is injective.

$$\begin{array}{ccc} E & \xrightarrow{\pi_0} & E/E_C \\ \uparrow \iota & & \uparrow \tau \\ H & \xrightarrow{\pi} & H/H_U \end{array}$$

THEOREM 4.4. *A discrete abelian group D does not belong to \mathcal{V}_0 if $|D| > |\mathbb{R}|$.*

Proof. For $D \in \mathcal{V}_0$, there exists a nuclear vector space E , a subgroup $H \leq E$ and a projection $\varphi : H \rightarrow D$. Since D is discrete, there is a continuous seminorm p on E such that $\varphi(H \cap B_p) = \{0\}$. Put $U := H \cap B_p$ and $C := B_p$.

Since φ induces an epimorphism $\tilde{\varphi} : H/H_U \rightarrow D$, we get $|D| \leq |H/H_U| \leq |E/E_C|$ (by the injectivity of τ). According to 4.2, the nuclear vector space E/E_C is of dimension $\leq |\mathbb{R}|$ and hence $|E/E_C| \leq |\mathbb{R}^{(\mathbb{R})}| = |\mathbb{R}|$. This implies the assertion. ■

COROLLARY 4.5. *The variety \mathcal{V}_0 is properly contained in \mathcal{V}_1 .*

5. On the variety generated by all nuclear groups

REMARK 5.1. In [2], nuclear groups have been introduced and it is shown there that the class of nuclear groups forms a Hausdorff variety (Chapter 7) which coincides with \mathcal{V}_2 (Theorem 9.6 of [2]).

As a consequence of the structure theorem for locally compact abelian groups, these groups are nuclear (7.10 in [2]); even more, the same argument shows that they belong to the (possibly) smaller variety \mathcal{V}_1 . Of course, nuclear vector spaces are also nuclear groups. Since these examples all belong to \mathcal{V}_1 , the question arose whether every nuclear group belonged to \mathcal{V}_1 .

In this section we show that \mathcal{V}_1 is a proper subclass of \mathcal{V}_2 .

LEMMA 5.2. *For every subgroup H of a nuclear vector group (E, \mathcal{O}) , the sets $(\text{conv } U)_{U \in \mathcal{U}_H(0)}$ form a 0-neighborhood basis of a nuclear vector group topology $\mathcal{O}_{\text{conv}}$; the inclusion $H \rightarrow (E, \mathcal{O}_{\text{conv}})$ is an embedding and the identity mapping $(E, \mathcal{O}_{\text{conv}}) \rightarrow (E, \mathcal{O})$ is continuous.*

Proof. For $U', U \in \mathcal{U}_H(0)$ such that $U' + U' \subseteq U$ we get $U' \subseteq \frac{1}{2} \text{conv } U$ and hence $\text{conv } U' + \text{conv } U' \subseteq \text{conv } U$. Now it is easy to verify that the sets $(\text{conv } U : U \in \mathcal{U}_H(0))$ form a neighborhood basis of a group topology $\mathcal{O}_{\text{conv}}$ on E . Since for every symmetric convex neighborhood $U_E \in \mathcal{U}_E(0)$, we have $\text{conv}(U_E \cap H) \subseteq U_E$, this topology is finer than the given topology on E . In particular, $(E, \mathcal{O}_{\text{conv}})$ is a Hausdorff space. Obviously, the inclusion mapping $H \rightarrow (E, \mathcal{O}_{\text{conv}})$ is continuous. Moreover, since $U_E \cap H = \text{conv}(U_E \cap H) \cap H$, it is an embedding.

It remains to show that $(E, \mathcal{O}_{\text{conv}})$ is a nuclear vector group. Therefore, let p and q be continuous pre-Hilbert seminorms on an open subspace $E_0 \leq E$ such that $d_k(B_p, B_q) \leq \frac{1}{2}k^{-1}$ (see 2.2). Then

$$d := \sum_{k \in \mathbb{N}} d_k(B_p, B_q)^2 \leq \frac{1}{4} \cdot \frac{\pi^2}{6} < \frac{1}{2}.$$

According to 3.20 of [2], we get

$$d_k(\text{conv}(B_p \cap H), \text{conv}(B_q \cap H)) \leq \frac{1}{1-d} d_k(B_p, B_q) \leq k^{-1},$$

which completes the proof. ■

LEMMA 5.3. *Let p and q be pre-Hilbert seminorms on a vector space E such that*

$$\frac{21}{2\pi} \sum_{k \in \mathbb{N}} d_k(B_p, B_q) < \frac{1}{4}$$

and let H be a subgroup of E . For every homomorphism $\chi : H \rightarrow \mathbb{T}$ satisfying $\chi(H \cap B_q) \subseteq \{e^{2\pi it} : |t| \leq 1/4\}$, there exists a linear homomorphism $f : E \rightarrow \mathbb{R}$ such that $e^{2\pi if}|_H = \chi$ and $f(B_p) \subseteq [-1/4, 1/4]$.

Proof. This is Lemma 19.14(ii) in [1]; cf. also 8.1 in [2].

REMARKS 5.4. (i) The variety \mathcal{V}_1 is generated by all nuclear and all discrete vector spaces.

[Every discrete abelian group is a quotient group of a free abelian group and free abelian groups can be considered as subgroups of vector spaces.]

(ii) The variety \mathcal{V}_1 is generated by all special nuclear vector groups.

[According to 2.5, every special nuclear vector group belongs to \mathcal{V}_1 . Conversely, according to 2.4(iv), every nuclear and every discrete vector space is a special nuclear vector group and hence the assertion follows from (i).]

LEMMA 5.5. *A nuclear vector group V belongs to \mathcal{V}_1 if and only if it is a special nuclear vector group.*

Proof. One direction was proved in 2.5.

So let us prove the other implication. According to 5.4(i), there exist a nuclear vector space E , a family of discrete vector spaces D_j ($j \in J$), a subgroup $H \leq E \times \prod_{j \in J} D_j$, and a quotient mapping $\varphi : H \rightarrow V$.

We denote the free abelian group over H by $A(H)$; let $\eta : H \rightarrow A(H)$ be an injective mapping onto the basis $\eta(H)$ of $A(H)$. Further, we denote by $L(H)$ a vector space with basis $\eta(H)$ and we assume $A(H)$ to be a subgroup of $L(H)$. So we have

$$H \xrightarrow{\eta} A(H) \xrightarrow{\tau} L(H)$$

where τ denotes the canonical inclusion.

Now we endow $A(H)$ and $L(H)$ with the initial topology \mathcal{O} induced by the inclusion τ and the linear operator ψ :

$$A(H) \xrightarrow{\tau} L(H) \xrightarrow{\psi} \left(E \times \prod_{j \in J} D_j \right) \times \mathbb{R}^{\mathcal{C}(H, \mathbb{R})}$$

where

$$\psi \left(\sum_{x \in H} \mu_x \eta(x) \right) = \left(\sum_{x \in H} \mu_x x, \left(\sum_{x \in H} \mu_x f(x) \right)_{f \in \mathcal{C}(H, \mathbb{R})} \right) \quad \text{for } (\mu_x) \in \mathbb{R}^{(H)}$$

and $\mathcal{C}(H, \mathbb{R})$ denotes the space of continuous functions $H \rightarrow \mathbb{R}$. The space $\mathbb{R}^{\mathcal{C}(H, \mathbb{R})}$ is endowed with the product topology.

Taking into consideration that H is a completely regular space, there exists for every $0 \neq \sum \mu_x \eta(x) \in L(H)$ a continuous mapping $f : H \rightarrow \mathbb{R}$ such that $f(x) = \text{sign}(\mu_x)$ for all x such that $\mu_x \neq 0$. Hence

$$\sum \mu_x f(x) = \sum |\mu_x| > 0.$$

This shows that ψ is injective.

According to 2.4(iv), $(E \times \prod_{j \in J} D_j) \times \mathbb{R}^{\mathcal{C}(H, \mathbb{R})}$ is a special nuclear vector group and so is $L(H)$.

Since $\psi(\tau(A(H)))$ is a subgroup of $H \times \mathbb{R}^{\mathcal{C}(H, \mathbb{R})}$ (and the topology on $A(H)$ is induced by the embedding $A(H) \xrightarrow{\psi \circ \tau} H \times \mathbb{R}^{\mathcal{C}(H, \mathbb{R})}$), the projection

$$\pi_H : A(H) \rightarrow H, \quad \sum k_x \eta(x) \mapsto \sum k_x x,$$

is well defined and continuous. We now show that it is also open. Fix $U \in \mathcal{U}_H(0)$ and a finite subset $C \subseteq \mathcal{C}(H, \mathbb{R})$, and let $\varepsilon > 0$. Since

$$\begin{aligned} \pi_H \left(\left\{ \sum k_x \eta(x) : \sum k_x x \in U \text{ and } \left| \sum k_x f(x) \right| < \varepsilon \text{ for all } f \in C \right\} \right) \\ \supseteq \{x = x - 0 : x \in U \text{ and } |f(x) - f(0)| < \varepsilon \text{ for all } f \in C\} \end{aligned}$$

is a neighborhood of 0 in H , the projection π_H is open.

This yields that the composition $\varphi \circ \pi_H$ is a continuous and open epimorphism. Let $\Phi : L(H) \rightarrow V$ be the (unique) linear extension of $\varphi \circ \pi_H$ and consider the following diagram:

$$\begin{array}{ccc} & (L(H), \mathcal{O}) & \\ \iota_{\text{conv}} \uparrow & & \\ (L(H), \mathcal{O}_{\text{conv}}) & & \\ \uparrow \iota & \searrow \Phi & \\ A(H) & \xrightarrow{\varphi \circ \pi_H} & V \end{array}$$

For $\mathcal{O}_{\text{conv}}$ as in 5.2, the inclusion ι is an embedding, the identity ι_{conv} is continuous, and $\tau = \iota_{\text{conv}} \circ \iota$. In particular, by the definition of the 0-neighborhood basis in $\mathcal{O}_{\text{conv}}$ and since V has a neighborhood basis of symmetric convex sets, the linear mapping Φ is continuous and, of course, open.

We fix a symmetric convex neighborhood $U \in \mathcal{U}_V(0)$. By the continuity of Φ , there exists a continuous (with respect to $\mathcal{O}_{\text{conv}}$) seminorm q on a subspace L_q of $L(H)$ such that $\Phi(B_q) \subseteq U$. According to 5.2 and by assumption, $\mathcal{O}_{\text{conv}}$ and \mathcal{O} induce the same topology on $A(H)$.

Since $(L(H), \mathcal{O})$ is a special nuclear vector group, there exists, due to 2.2(i) and (ii) and 2.4(iii), a sequence $(p_n)_{n \in \mathbb{N}_0}$ of pre-Hilbert seminorms defined on (the same) subspace $L_p \leq L(H)$ which are continuous with respect to \mathcal{O} and which satisfy:

- (i) $A(H) \cap B_q \supseteq A(H) \cap B_{p_0}$,
- (ii) $(21/(2\pi)) \sum_{k \in \mathbb{N}} d_k(B_{p_1}, B_{p_0}) < 1/4$, and
- (iii) $d_k(B_{p_{n+1}}, B_{p_n}) \leq k^{-1}$ for all $k, n \in \mathbb{N}$.

The subspace $L_0 := \langle A(H) \cap B_q \cap B_{p_1} \rangle_{\mathbb{R}} \leq L_q \cap L_p$ is open with respect to $\mathcal{O}_{\text{conv}}$ by the continuity of ι_{conv} .

We wish to show

$$(*) \quad L_0 \cap B_q \supseteq L_0 \cap B_{p_1}.$$

Therefore, let $f : L_0 \rightarrow \mathbb{R}$ be a linear functional such that $f(B_q \cap L_0) \subseteq I := [-1/4, 1/4]$. Due to (i), we have $f(A(H) \cap B_{p_0} \cap L_0) \subseteq f(A(H) \cap B_q \cap L_0) \subseteq I$.

Since

$$\frac{21}{2\pi} \sum_{k \in \mathbb{N}} d_k(B_{p_1} \cap L_0, B_{p_0} \cap L_0) \stackrel{2.2(\text{iii})}{\leq} \frac{21}{2\pi} \sum_{k \in \mathbb{N}} d_k(B_{p_1}, B_{p_0}) < \frac{1}{4},$$

we can apply 5.3 to the character $e^{2\pi i f}|_{A(H) \cap L_0}$ and get a linear functional $\tilde{f} : L_0 \rightarrow \mathbb{R}$ which satisfies $\tilde{f}(B_{p_1} \cap L_0) \subseteq I$ and $e^{2\pi i \tilde{f}(x)} = e^{2\pi i f(x)}$ for all $x \in A(H) \cap L_0$. This implies that $(f - \tilde{f})(A(H) \cap L_0)$ is a subgroup of \mathbb{Z} . By assumption,

$$(f - \tilde{f})(A(H) \cap B_q \cap B_{p_1}) \subseteq (I - I) \cap \mathbb{Z} = \{0\},$$

which shows that $f|_{L_0} = \tilde{f}|_{L_0}$. In particular, we have $f(B_{p_1} \cap L_0) \subseteq I$.

This inclusion implies $(B_q \cap L_0)^0 \subseteq (B_{p_1} \cap L_0)^0$ and an application of the Hahn–Banach theorem to the seminormed space $(L_0, q|_{L_0})$ (cf. Chap. II, §3, Corollary 4 of [3]) yields $B_{p_1} \cap L_0 \subseteq B_q \cap L_0$ (since $B_q \cap L_0$ is closed).

By the continuity of ι_{conv} , the sets $B_{p_n} \cap L_0$ are neighborhoods of 0 with respect to $\mathcal{O}_{\text{conv}}$. They satisfy

$$d_k(B_{p_{n+1}} \cap L_0, B_{p_n} \cap L_0) \stackrel{2.2(\text{iii})}{\leq} d_k(B_{p_{n+1}}, B_{p_n}) \stackrel{(\text{iii})}{\leq} 1/k$$

for all $k, n \in \mathbb{N}$ and $\langle B_{p_n} \cap L_0 \rangle_{\mathbb{R}} = L_p \cap L_0 = L_0$ for all $n \in \mathbb{N}$.

As a consequence of (*) we get $\Phi(L_0 \cap B_{p_1}) \subseteq \Phi(L_0 \cap B_q) \subseteq U$. Since Φ is open, it is clear that the sequence $(\Phi(B_{p_n} \cap L_0))_{n \in \mathbb{N}}$ of neighborhoods has the desired properties. ■

COROLLARY 5.6. *The nuclear vector group Σ_0 does not belong to \mathcal{V}_1 and hence \mathcal{V}_1 is properly contained in \mathcal{V}_2 .*

Proof. According to 2.12, Σ_0 is not a special nuclear group. As a consequence of the above lemma, $\Sigma_0 \notin \mathcal{V}_1$. But Σ_0 is an element of \mathcal{V}_2 , so the assertion follows. ■

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Mathematisches Institut
Auf der Morgenstelle 10
D-72076 Tübingen, Germany
E-mail: lydia.aussenhofer@uni-tuebingen.de

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