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The range of a derivation on a Jordan–Banach algebra

by

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Abstract. The questions when a derivation on a Jordan–Banach algebra has quasinilpotent values, and when it has the range in the radical, are discussed.

1. Introduction. In 1955 I. M. Singer and J. Wermer [35] proved that a continuous derivation on a commutative Banach algebra has the range in the (Jacobson) radical of the algebra. Another related result was obtained just a little later independently by D. C. Kleinecke [21] and F. V. Shirokov [32]. One possible way to state this result is the following (cf. [20]): If D is a continuous derivation of a Banach algebra A and $a \in A$ is such that $D(a)a = aD(a)$, then $D(a)$ is quasinilpotent (in the literature this result is more often stated only for inner derivations so that the condition reads $(ab - ba)a = a(ab - ba)$ for some $a, b \in A$). Both these classical results, the Singer–Wermer theorem and the Kleinecke–Shirokov theorem, were conjectured by I. Kaplansky who was inspired by results in [17, 31, 41, 42]. Let us remark that the second result, although it deals only with some local property of a derivation, clearly implies the first one which describes a global property of derivations. Another way of generalizing the Singer–Wermer theorem to noncommutative algebras was that by A. M. Sinclair [33]: Every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant.

The question whether the continuity assumption is redundant in these results has been of permanent interest for almost 50 years now, and it has been one of the leading motives for the development of the theory of automatic continuity. Already in [35] Singer and Wermer wrote that it seems probable that the hypothesis of continuity in their theorem is superfluous. B. E. Johnson [18] proved this for semisimple algebras, but it took more than 30 years before this classical Singer–Wermer conjecture was finally settled for any commutative Banach algebra by M. P. Thomas [36]. For the

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other two results, the Kleinecke–Shirokov theorem and Sinclair's theorem, the question concerning continuity is still open. The conjecture that every (not necessarily continuous) derivation of a Banach algebra leaves primitive ideals invariant is known as the *noncommutative Singer–Wermer conjecture*. It is known that for each derivation there can only be finitely many noninvariant primitive ideals each of which is of finite codimension [37], but whether such derivations and ideals actually exist is still an open question.

A number of authors have extended the results mentioned above in various directions (see [24] for a full account). Let us mention here only two more results, connected with the present paper. In [8] the first author and J. Vukman proved a kind of a global Kleinecke–Shirokov theorem: If a continuous derivation D of a Banach algebra A is such that $D(a)a - aD(a)$ lies in the radical for every $a \in A$, then D maps A into its radical. The conjecture that the continuity is superfluous in this result is equivalent to the noncommutative Singer–Wermer conjecture (see [24]). On the other hand, if one assumes that $D(a)a - aD(a)$ is 0 for each a, or even slightly more generally, that $D(a)a - aD(a)$ is always a central element, then one can prove that D maps into the radical without assuming the continuity. This was done by M. Mathieu and V. Runde [26].

It is our aim in the present paper to treat analogous problems in the context of Jordan–Banach algebras. A very rough summary of the results mentioned could be that derivations of Banach algebras are rather rare on commutative algebras, and that only in some special cases can they satisfy certain commuting relations. Note that the concepts of a commutative Banach algebra and an associative Jordan–Banach algebra coincide. Therefore, by analogy one might expect that derivations of Jordan–Banach algebras can only exceptionally satisfy some "associating" relations. This is the main idea behind the present paper. In the study of derivations on noncommutative Banach algebras the concept of the *commutator* of elements, i.e. $[a, b] = ab - ba$, plays an important role. A similar role in the present paper will be played by the *associator* of elements in the Jordan–Banach algebra, i.e. $[a, b, c] = (a \cdot b) \cdot c - a \cdot (b \cdot c)$ (here, $\cdot$ denotes the product in the Jordan–Banach algebra). Let us remark that in the case of a special Jordan algebra the commutator and the associator are closely related, namely, $[a, b, c] = \frac{1}{4}[[c, a], b]$.

In Section 2 we review some facts concerning Jordan–Banach algebras and also fix the notation and terminology. Section 3 treats local properties of derivations on Jordan–Banach algebras. Some of the conditions treated can be viewed as Jordan analogues of the condition appearing in the Kleinecke–Shirokov theorem. Let us mention that the second author has recently obtained an extension of this celebrated theorem to Jordan–Banach algebras [40]. In Section 4 we prove two theorems on global properties of derivations. We remark that the second one (Theorem 4.14) generalizes the result of Mathieu and Runde [26] mentioned above. Finally, in Section 5 we treat what we call the Singer–Wermer conjecture for Jordan–Banach algebras (see Section 2). In particular, a number of assertions equivalent to the truthfulness of this conjecture are found.

In most of the paper we deal with derivations without assuming that they are continuous. A standard approach when treating possibly discontinuous derivations is to consider their separating spaces (which are closed ideals), and often it is also necessary to treat separating spaces of their powers (which do not have such nice algebraic structure). In the present paper we certainly also deal with them, but what seems to be new is the introduction of the closed ideal generated by the separating spaces of all powers of a derivation, which turns out to be quite useful.

Since every Banach algebra can be transformed into a Jordan–Banach algebra (see below), all results obtained in this paper make sense also in the associative context; moreover, many of them seem to be new. As a matter of fact, one of our main reasons for treating derivations on nonassociative algebras is that we believe that this may prove to be useful for understanding derivations on associative algebras. Let us try to justify this admittedly somewhat speculative idea by an analogy. As already mentioned at the very beginning, both the Kleinecke–Shirokov theorem and Sinclair's theorem imply the Singer–Wermer theorem. On the other hand, Thomas' theorem on derivations on commutative Banach algebras [36] can be deduced from another theorem of Thomas [37] which treats a local property of derivations on any (possibly noncommutative) Banach algebra. Therefore, the results and especially the methods of the theory of noncommutative algebras have proved to be useful in the study of commutative algebras. Perhaps, similarly it may turn out that the study of derivations on nonassociative algebras will give a better understanding of derivations on noncommutative associative algebras, especially in the cases where, as in attempts to prove the noncommutative Singer–Wermer conjecture, standard approaches have failed to produce the final conclusion.

2. Jordan algebra preliminaries. A *Jordan algebra* is a nonassociative algebra J whose product satisfies

$$a \cdot b = b \cdot a \quad \text{and} \quad (a \cdot b) \cdot a^2 = a \cdot (b \cdot a^2)$$

for all $a, b \in J$. Such algebras were introduced in 1934 by P. Jordan, J. von Neumann, and E. Wigner motivated by quantum mechanics [19]. The knowledge of the structure of Jordan algebras became fairly complete when E. I. Zel'manov [44] provided his characterization of prime nondegenerate Jordan algebras. A *Jordan–Banach* algebra is a real or complex Jordan algebra J whose underlying linear space is a Banach space with respect to a norm $\|\cdot\|$ satisfying $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ for all $a, b \in J$. Every associative algebra A becomes a Jordan algebra, denoted by A^+, with respect to the product $a \cdot b = \frac{1}{2}(ab + ba)$. Moreover A^+ is a Jordan–Banach algebra in the case where A is a Banach algebra. For an account of how Jordan structures arise in analysis we refer the reader to [29].

Let J be a Jordan algebra. By $L(J)$ we denote the associative algebra of all linear operators on J. We write $R \circ S$ for the compositon of $R, S \in L(J)$, and $[R, S]$ for $R \circ S - S \circ R$. For each $a \in J$, we define the operator $R_a \in L(J)$ by $R_a(b) = b \cdot a$ for all $b \in J$. The *unital multiplication algebra* of J is the subalgebra $M_1(J)$ of $L(J)$ generated by the identity operator and all the multiplication operators R_a $(a \in J)$. It should be noted that $M_1(J)$ is a subalgebra of the Banach algebra $BL(J)$ of all bounded linear operators on J in the case where J is a Jordan–Banach algebra.

Every nonunital Jordan algebra J can be embedded into a unital Jordan algebra J^1 by externally adjoining an identity. The standard concept of invertibility in associative algebras was extended to the context of Jordan algebras by N. Jacobson. An element a in a unital Jordan algebra J is said to be *invertible* if there exists $b \in J$ such that $a \cdot b = 1$ and $a^2 \cdot b = a$. This is equivalent to the invertibility of the operator U_a from J to itself given by $U_a x = 2a \cdot (a \cdot x) - a^2 \cdot x$ for all $x \in J$. An element a in a nonunital Jordan algebra J is said to be *quasi-invertible* if $1 - a$ is invertible in its unitization J^1.

The standard spectral theory and analytic functional calculus can be extended to the context of complex Jordan–Banach algebras. This follows from the fact that if J is a complex Jordan–Banach algebra and $a \in J$, then there exists a closed associative subalgebra A of J^1 containing 1 and a. Therefore the spectral theory and analytic functional calculus run as in the associative case. The *spectrum* $\mathrm{Sp}(a)$ of an element a in a complex Jordan–Banach algebra J is defined as in the associative case and it is a nonempty compact subset of the complex plane. The *spectral radius* of a is given by $r(a) = \max\{|\lambda| : \lambda \in \mathrm{Sp}(a)\} = \lim \|a^n\|^{1/n}$. The element a is said to be *quasinilpotent* if $r(a) = 0$. By $\mathcal{Q}(J)$ we denote the set of all quasinilpotent elements in J. For each function f which is analytic in a neighbourhood Ω of $\mathrm{Sp}(a)$ we can define the element $f(a)$ of J^1 by $(2\pi i)^{-1} \int_\gamma f(\lambda)(\lambda - a)^{-1}\, d\lambda$, where γ is any positively oriented curve contained in Ω and surrounding $\mathrm{Sp}(a)$. For a discussion of this theory we refer the reader to [1].

K. McCrimmon [27] proved that in each Jordan algebra J there exists the largest ideal consisting of quasi-invertible elements. This ideal is called the *Jacobson–McCrimmon radical* of J and will be denoted by $\mathrm{Rad}(J)$. Of course, $\mathrm{Rad}(J) \subset \mathcal{Q}(J)$. We say that J is *semisimple* if $\mathrm{Rad}(J) = 0$. If A is an associative algebra, then $\mathrm{Rad}(A^+)$ coincides with the classical Jacobson radical of A [27].

E. I. Zel'manov [43] introduced the notion of primitiveness for unital Jordan algebras to derive his characterization of prime Jordan algebras. This concept was extended to nonunital Jordan algebras by L. Hogben and K. McCrimmon [15]. A linear subspace I of J is said to be an *inner ideal* of J if $U_I(J^1) \subset I$. We call an ideal P of J *primitive* if it is the largest ideal of J contained in a maximal-modular inner ideal of J (see [15] for the definition of modularity). It turns out that $\mathrm{Rad}(J)$ is the intersection of all primitive ideals of J and that the classical primitive ideals of an associative algebra A are primitive ideals of the Jordan algebra A^+ [15]. Primitive ideals of Jordan–Banach algebras are closed [11]. We also make frequent use of the fact that

$$\mathrm{Sp}(a) = \mathrm{Sp}(\pi_{\mathrm{Rad}(J)}(a)) = \bigcup_{P \text{ primitive}} \mathrm{Sp}(\pi_P(a))$$

for any element a in a Jordan–Banach algebra J [40, Lemma 1], where $\pi_{\mathrm{Rad}(J)}$ and π_P denote the corresponding quotient maps. In general, whenever we arrive at a closed subspace M of a Banach space X, we write π_M for the quotient map from X onto the quotient Banach space X/M.

A linear map D from a Jordan algebra J to itself is said to be a *derivation* on J if it satisfies

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

for all $a, b \in J$. Derivations of the Jordan algebra A^+, where A is an associative algebra, are called *Jordan derivations* of A. I. N. Herstein [13] showed that any Jordan derivation on a prime ring of characteristic different from 2 is a derivation. J. M. Cusack [10] extended this result to 2-torsion free semiprime rings (see also [3]). From the Jordan algebra axioms it can be deduced that if $a, c \in J$, then the map $[R_c, R_a]$ is a derivation on J. We write

$$[a, b, c] = [R_c, R_a](b) = (a \cdot b) \cdot c - a \cdot (b \cdot c)$$

for the *associator* of elements $a, b, c \in J$. Note that for any fixed $a, c \in J$, the set $\{b \in J : [a, b, c] = 0\}$, being a kernel of a derivation $[R_c, R_a]$, is a subalgebra of J. Further, note that every derivation D on J satisfies

$$D([a, b, c]) = [D(a), b, c] + [a, D(b), c] + [a, b, D(c)]$$

for all $a, b, c \in J$. Hence we see at once that if M and N are subsets of J which are both invariant under D, then so are the sets $\{a \in J : [a, M, N] = 0\}$ and $\{a \in J : [M, a, N] = 0\}$. This observation will be frequently used (without explicit reference) in the next section. Let us also mention another useful formula

$$R_{D(a)} = [D, R_a],$$

which will play a very important role. In particular, it implies that for any $R \in M_1(J)$, $\Delta(R) = [D, R]$ lies in $M_1(J)$. Indeed, the formula shows this for the generators R_a of the algebra $M_1(J)$, and so from $\Delta(R_1 \circ R_2) = \Delta(R_1) \circ R_2 + R_1 \circ \Delta(R_2)$ it can be easily deduced that it holds for any $R \in M_1(J)$. Thus, Δ is a derivation on the algebra $M_1(J)$.

We can measure the continuity of a linear map T from a Banach space X to a Banach space Y by considering its *separating subspace* which is defined as the subspace $\mathcal{S}(T)$ of those $y \in Y$ for which there exists a sequence (x_n) in X such that $\lim x_n = 0$ and $\lim T(x_n) = y$. The closed graph theorem shows that T is continuous if and only if $\mathcal{S}(T) = 0$.

Let D be a derivation on a complex Jordan–Banach algebra J. Suppose that I is a closed ideal of J which is invariant under D. Then we can define a derivation D_I on the quotient Jordan–Banach algebra J/I by $D_I(\pi_I(a)) = D(\pi_I(a))$. Studying derivations D_I for appropriate invariant ideals I is often very useful. Namely, D_I usually inherits some properties of the original derivation D, but the quotient algebras J/I may be more tractable than the algebra J. We shall be primarily concerned with the case when I is a primitive ideal. It turns out that the invariance of primitive ideals under a derivation D is closely related to the closed ideal of J generated by $\{\mathcal{S}(D^n) : n \in \mathbb{N}\}$. We denote this ideal by $\mathcal{I}(D)$. It is easy to check that $\mathcal{I}(D)$ is the closure in J of the linear subspace of J generated by $\{R(a) : R \in M_1(J), a \in \mathcal{S}(D^n), n \in \mathbb{N}\}$.

In the following result we summarize some basic properties of derivations on Jordan–Banach algebras, proved by the second author [39].

THEOREM 2.1. *Let J be a complex Jordan–Banach algebra and let D be a derivation on J. Then*:

(i) *A primitive ideal P of J is invariant under D if and only if $\mathcal{I}(D) \subset P$.*

(ii) *$D(P) \subset P$ for each primitive ideal P of J except possibly finitely many exceptional primitive ideals. Moreover*, *if P is an exceptional primitive ideal then J/P is simple and either it is finite-dimensional or it is the Jordan–Banach algebra of a continuous nondegenerate symmetric bilinear form f on a complex Banach space X of dimension greater than one.*

(iii) *If D is continuous*, *then $D(P) \subset P$ for each primitive ideal P of J.*

(iv) *If J is semisimple*, *then D is automatically continuous.*

It is well known that for linear operators R and T between Banach spaces we have $\overline{R\mathcal{S}(T)} = \mathcal{S}(RT)$ provided that R is continuous [34, Lemma 1.3]. Using this, we see that the first assertion of Theorem 2.1 is just another way to formulate [39, Theorem 6]. The other assertions are stated more explicitly in [39].

Can the exceptional primitive ideals described in Theorem 2.1 really exist? For us, this is the principal open question concerning derivations of Jordan–Banach algebras. It seems appropriate to call the conjecture that there are no exceptional primitive ideals, that is, that *any* derivation of *any* Jordan–Banach algebra J leaves each primitive ideal of J invariant, the *Singer–Wermer conjecture for Jordan–Banach algebras.* We remark that since every primitive ideal of a Banach algebra A is also a primitive ideal of the Jordan–Banach algebra A^+, the truthfulness of this conjecture would imply the truthfulness of the noncommutative Singer–Wermer conjecture.

3. Local properties. Let D be a derivation on a Jordan–Banach algebra J. Suppose that T is an associative subalgebra of J which is invariant under D. Then there exists a closed associative subalgebra $A(T)$ of J which contains T and is also invariant under D. This follows immediately from [40, Lemmas 3 and 4]. The proof is algebraic and elementary. More concretely, $A(T)$ is constructed as follows: one first defines the sets $C_1(T) = \{x \in J : [x, T, T] = 0\}$ and $C_2(T) = \{u \in J : [u, T, C_1(T)] = [u, C_1(T), T] = 0\}$, and then introduces $A(T)$ as $\{a \in J : [a, C_1(T), C_2(T)] = [a, C_2(T), C_1(T)] = 0\}$. Then one can check that $A(T)$ has all the properties described. Now, being a closed associative subalgebra of a Jordan–Banach algebra, $A(T)$ is actually a commutative Banach algebra. Therefore, by Thomas' theorem [36], D maps the algebra $A(T)$ into its (Jacobson) radical. Therefore, $D(A(T))$ consists of quasinilpotent elements. In particular, $D(T) \subset \mathcal{Q}(J)$.

Using this we can now easily derive our first theorem.

Theorem 3.1. *Let J be a complex Jordan–Banach algebra and let D be a derivation on J. If $a \in J$ is such that $[D(a), J, J] = 0$, then $D(a) \in \mathcal{Q}(J)$.*

Proof. Let T be the subalgebra of J generated by a and all $b \in J$ such that $[b, J, J] = 0$. It is easy to check that T is an associative subalgebra of J. Moreover, it is invariant under D since $[D(a), J, J] = 0$. But then, in view of the discussion above, $D(T) \subset \mathcal{Q}(J)$. In particular, $D(a) \in \mathcal{Q}(J)$. ■

Let us point out that we did not assume in Theorem 3.1 that D is continuous. If we do assume the continuity of D, then the conclusion $D(a) \in \mathcal{Q}(J)$ follows even from a milder assumption $[D(a), J, J] \subset \mathrm{Rad}(J)$. Indeed, from Theorem 2.1 we deduce, since D is continuous, that $D(\mathrm{Rad}(J)) \subset \mathrm{Rad}(J)$ and so we can consider the derivation $D_{\mathrm{Rad}(J)}$ on the Jordan–Banach algebra $J/\mathrm{Rad}(J)$ which clearly satisfies $[D_{\mathrm{Rad}(J)}(\pi_{\mathrm{Rad}(J)}(a)), J/\mathrm{Rad}(J), J/\mathrm{Rad}(J)] = 0$. Since $r(D(a)) = r(\pi_{\mathrm{Rad}(J)}(D(a))) = r(D_{\mathrm{Rad}(J)}(\pi_{\mathrm{Rad}(J)}(a)))$, it follows that there is no loss of generality in assuming that $\mathrm{Rad}(J) = 0$. Therefore, Theorem 2.1 can be applied. Let us record this observation:

Corollary 3.2. *Let J be a complex Jordan–Banach algebra and let D be a continuous derivation on J. If $a \in J$ is such that $[D(a), J, J] \subset \mathrm{Rad}(J)$, then $D(a) \in \mathcal{Q}(J)$.*

Using a different approach, this result can be sharpened as follows.

THEOREM 3.3. *Let J be a complex Jordan–Banach algebra and let D be a continuous derivation on J. If $a \in J$ is such that $[D(a), J, a] \subset \mathrm{Rad}(J)$, then $D(a) \in \mathcal{Q}(J)$.*

Proof. As in the proof of the preceding corollary, we can assume without loss of generality that $\mathrm{Rad}(J) = 0$.

According to our assumption, we have $[R_{D(a)}, R_a] = 0$. Since $R_{D(a)} = [D, R_a]$, we obtain $[[D, R_a], R_a] = 0$. The Kleinecke–Shirokov theorem (for inner derivations) implies that $R_{D(a)} = [D, R_a]$ is a quasinilpotent operator on J. For each $n \in \mathbb{N}$ we have

$$D(a)^{n+1} = R_{D(a)}^n(D(a))$$

and therefore

$$\|D(a)^{n+1}\| \le \|R_{D(a)}^n\| \cdot \|D(a)\|.$$

Consequently,

$$\begin{aligned} r(D(a)) &= \lim \|D(a)^{n+1}\|^{1/(n+1)} \\ &\le \lim(\|R_{D(a)}^n\|^{1/(n+1)}\|D(a)\|^{1/(n+1)}) \\ &= r(R_{D(a)}) = 0. \ \blacksquare \end{aligned}$$

THEOREM 3.4. *Let J be a complex Jordan–Banach algebra and let D be a continuous derivation on J. Suppose that M and N are subsets of J which are both invariant under D. If $a \in M$ is such that $D(a) \in N$ and $[M, a, N] \subset \mathrm{Rad}(J)$, then $D(a) \in \mathcal{Q}(J)$.*

Proof. Again, there is no loss of generality in assuming that $\mathrm{Rad}(J) = 0$. Let

$$H = \{x \in J : [M, x, N] = 0\}.$$

Note that H is a closed subalgebra of J. Further we set

$$A = \{T \in BL(J) : T(H) \subset H\}, \qquad I = \{T \in A : T(H) = 0\}.$$

Then A is a closed subalgebra of the Banach algebra $BL(J)$, $D \in A$ since M and N are invariant under D, and I is a closed two-sided ideal of A. From our initial assumption we see that $a \in H$. Since H is a subalgebra of J, it follows that $R_a \in A$. Since $D(a) \in N$, we have $[a, H, D(a)] = 0$, which can be written as $[R_a, R_{D(a)}] \in I$. We thus get $[\pi_I(R_a), \pi_I(R_{D(a)})] = 0$. Since $R_{D(a)} = [D, R_a]$, the preceding equality becomes

$$[\pi_I(R_a), [\pi_I(D), \pi_I(R_a)]] = 0.$$

Since A/I is a Banach algebra, the Kleinecke–Shirokov theorem implies that $\pi_I(R_{D(a)}) = [\pi_I(D), \pi_I(R_a)]$ is a quasinilpotent element of A/I. Let us show that this yields $D(a)$ is quasinilpotent. Since $D(a) \in H$, for all $T \in I$ and

$n \in \mathbb{N}$ we have

$$D(a)^{n+1} = R^n_{D(a)}(D(a)) = (R^n_{D(a)} + T)(D(a)),$$

whence

$$\|D(a)^{n+1}\| \le \|R^n_{D(a)} + T\| \cdot \|D(a)\|,$$

and so

$$\|D(a)^{n+1}\| \le \inf_{T\in I} \|R^n_{D(a)} + T\| \cdot \|D(a)\|,$$

that is,

$$\|D(a)^{n+1}\| \le \|\pi_I(R^n_{D(a)})\| \cdot \|D(a)\| = \|(\pi_I(R_{D(a)}))^n\| \cdot \|D(a)\|.$$

Consequently,

$$\begin{aligned} r(D(a)) &= \lim \|D(a)^{n+1}\|^{1/(n+1)} \\ &\le \lim(\|(\pi_I(R_{D(a)}))^n\|^{1/(n+1)}\|D(a)\|^{1/(n+1)}) \\ &= r(\pi_I(R_{D(a)})) = 0. \ \blacksquare \end{aligned}$$

Incidentally, by taking $M = J$ and $N = \{a \in J : [J, J, a] \in \mathrm{Rad}(J)\}$ we see that Theorem 3.4 also covers Corollary 3.2. However, another special case when $M = N = \{D^i(a) : i \ge 0\}$ seems to be of greater interest:

Corollary 3.5. *Let J be a complex Jordan–Banach algebra and let D be a continuous derivation on J. Suppose that $a \in J$ is such that $[D^i(a), a, D^j(a)] \in \mathrm{Rad}(J)$ for all $i, j \ge 0$. Then $D(a) \in \mathcal{Q}(J)$.*

Let us point out that Theorem 3.1 is the only result in this section which was proved without assuming the continuity of a derivation. In Section 5 we shall see that the problem whether the assumption of continuity can be removed in the other results is intimately connected with the Singer–Wermer conjecture for Jordan–Banach algebras.

4. Global properties. This section has two aims. The first one is to consider derivations of Jordan–Banach algebras whose range is an associative set, and the second one is to characterize derivations satisfying a (simplified version of the) condition of Theorem 3.3, but for every element a in the algebra. Throughout this section we assume that *D is derivation of a complex Jordan–Banach algebra J*. It should be pointed out that we do not assume that D is continuous.

Derivations whose range is associative. Our goal in this subsection is to prove Theorem 4.6. We shall do this in a series of lemmas; some of them may be of independent interest.

Lemma 4.1. *Suppose that there exists a continuous linear map F from J to a Banach space X such that $F \circ D = 0$. Then $F(\mathcal{I}(D)) = 0$.*

Proof. It suffices to prove that $F(R(\mathcal{S}(D^n))) = 0$ for all $R \in M_1(J)$ and $n \geq 1$. Since for $n = 0$ this is trivially true, we may assume that, for some $n \geq 0$, $F(R(\mathcal{S}(D^n))) = 0$ holds true for every $R \in M_1(J)$, and we have to show that this yields $F(R(\mathcal{S}(D^{n+1}))) = 0$ for every $R \in M_1(J)$.

Let $R \in M_1(J)$. Recall that $\Delta(R) = D \circ R - R \circ D$ also belongs to $M_1(J)$. Since $F \circ D = 0$ by our assumption, we have $F \circ \Delta(R) \circ D^n = -F \circ R \circ D^{n+1}$. Since $F \circ \Delta(R)$ and $F \circ R$ are continuous, [34, Lemma 1.3] gives

$$\begin{aligned}\overline{F(R(\mathcal{S}(D^{n+1})))} &= \mathcal{S}(F \circ R \circ D^{n+1}) = \mathcal{S}(-F \circ \Delta(R) \circ D^n) \\ &= \overline{-F(\Delta(R)(\mathcal{S}(D^n))} = 0.\end{aligned}$$

Hence $F(R(\mathcal{S}(D^{n+1}))) = 0$. ∎

LEMMA 4.2. *Let M be a closed subspace of J such that*

$$[D(J), D(J), D(J)] \subset M.$$

Then

$$[D(J) + \mathcal{I}(D), D(J) + \mathcal{I}(D), D(J) + \mathcal{I}(D)] \subset M.$$

Proof. Let $b, c \in D(J)$ and let F be a continuous linear map from J to the quotient Banach space J/M given by $F(a) = \pi_M([a, b, c])$ for each $a \in J$. Since $F \circ D = 0$, the preceding lemma gives $[\mathcal{I}(D), b, c] \subset M$.

Now let $b \in D(J) + \mathcal{I}(D)$, $c \in D(J)$, and let $F : J \to J/M$ be given by $F(a) = \pi_M([b, a, c])$. By what was proved above it follows that $F \circ D = 0$, and so the preceding lemma now gives $[b, \mathcal{I}(D), c] \subset M$.

Finally, let $b, c \in D(J) + \mathcal{I}(D)$ and define $F : J \to J/M$ by $F(a) = \pi_M([b, c, a])$. Again we have $F \circ D = 0$, and so $[b, c, \mathcal{I}(D)] \subset M$ by the preceding lemma. ∎

LEMMA 4.3. *Suppose that $[D(J), D(J), D(J)] \subset \mathrm{Rad}(J)$. Then $D(P) \subset P$ for each primitive ideal P of J except possibly finitely many primitive ideals each of which is* 1*-codimensional.*

Proof. By Theorem 2.1, there may only be finitely many primitive ideals of J which are not invariant under D. Suppose that P is such a primitive ideal. By Theorem 2.1, $\mathcal{I}(D) \not\subset P$ and $\pi_P(\mathcal{I}(D)) = J/P$. Further, from Lemma 4.2 it follows that $[\mathcal{I}(D), \mathcal{I}(D), \mathcal{I}(D)] \subset \mathrm{Rad}(J)$. Hence

$$[J/P, J/P, J/P] = [\pi_P(\mathcal{I}(D)), \pi_P(\mathcal{I}(D)), \pi_P(\mathcal{I}(D))] = 0$$

and so J/P is a primitive commutative complex Banach algebra. Consequently, J/P is isomorphic to $\mathbb{C}$. ∎

LEMMA 4.4. *Suppose that $[D(J), D(J), D(J)] \subset \mathrm{Rad}(J)$. If P is a primitive ideal of J such that $D(P) \subset P$, then $\pi_P(D^2(J))$ consists of quasi-nilpotent elements.*

Proof. Note that $[D_P(J/P), D_P(J/P), D_P(J/P)] = 0$. Since D_P is continuous according to Theorem 2.1, by taking $M = N = D_P(J/P)$ we see that Theorem 3.4 shows that $D^2_P(J/P) = \pi_P(D^2(J))$ consists of quasinilpotent elements. ∎

LEMMA 4.5. *Suppose that* $[D(J), D(J), D(J)] = 0$. *Then*

(i) $D(\mathcal{I}(D)) \cdot \mathcal{I}(D) \subset \mathrm{Rad}(\mathcal{I}(D))$.

(ii) $a^4 \in \mathrm{Rad}(\mathcal{I}(D))$ *for all* $a \in \mathcal{S}(D^n)$ *and* $n \in \mathbb{N}$; *accordingly* $\mathcal{S}(D^n)$ *consists of quasinilpotent elements for each* $n \in \mathbb{N}$.

(iii) $D(P) \subset P$ *for each primitive ideal* P *of* J.

Proof. Let $y \in \mathcal{I}(D)$ and let d be the map from $\mathcal{I}(D)$ to itself given by $d(a) = D(a) \cdot y$ for each $a \in \mathcal{I}(D)$. From Lemma 4.2 it follows immediately that $\mathcal{I}(D)$ is a commutative Banach algebra and that d is a derivation on $\mathcal{I}(D)$. Thomas' theorem [36] gives $d(\mathcal{I}(D)) \subset \mathrm{Rad}(\mathcal{I}(D))$, which proves the first statement.

In order to prove the second statement, we pick $a \in \mathcal{S}(D^n)$ and a sequence (a_k) in J such that $\lim a_k = 0$ and $\lim D^n(a_k) = a$. Then for each $k \in \mathbb{N}$ we have

$$a \cdot (a \cdot (a \cdot D^n(a_k))) = a \cdot (a \cdot D(a \cdot D^{n-1}(a_k))) - a \cdot (a \cdot (D(a) \cdot D^{n-1}(a_k))).$$

It is clear that $\lim a \cdot (a \cdot (a \cdot D^n(a_k))) = a^4$. From the first statement we deduce that $a \cdot (a \cdot D(a \cdot D^{n-1}(a_k))) \in \mathrm{Rad}(\mathcal{I}(D))$ for each $k \in \mathbb{N}$. If $n = 1$, then $\lim a \cdot (a \cdot (D(a) \cdot D^{n-1}(a_k))) = 0$ and therefore $a^4 \in \mathrm{Rad}(\mathcal{I}(D))$. If $n > 1$, then from Lemmas 4.2 and the first statement it may be concluded that $a \cdot (a \cdot (D(a) \cdot D^{n-1}(a_k))) = (a \cdot D(a)) \cdot (a \cdot D^{n-1}(a_k)) \in \mathrm{Rad}(\mathcal{I}(D))$, and hence $a^4 \in \mathrm{Rad}(\mathcal{I}(D))$ in this case as well.

It remains to prove the last statement. To obtain a contradiction, suppose there exists a primitive ideal P of J such that $D(P) \not\subset P$. Lemma 4.3 shows that J/P is isomorphic to $\mathbb{C}$. On the other hand, there exists $n \in \mathbb{N}$ such that $\mathcal{S}(D^n) \not\subset P$ and therefore $\pi_P(\mathcal{S}(D^n)) = J/P$, which contradicts the second statement. ∎

THEOREM 4.6. *Let* J *be a complex Jordan–Banach algebra and let* D *be a derivation on* J. *Suppose that* $[D(J), D(J), D(J)] = 0$. *Then* $D^2(J) \subset \mathcal{Q}(J)$.

Proof. From Lemmas 4.4 and 4.5 it follows that $\pi_P(D^2(J))$ consists of quasinilpotent elements for each primitive ideal P of J. Accordingly, $D^2(J)$ consists of quasinilpotent elements. ∎

Let us give two examples illustrating Theorem 4.6.

EXAMPLE 4.7. Suppose J is a Jordan–Banach algebra of a continuous symmetric bilinear form f on a nonzero complex Banach space X. That is, $J = \mathbb{C} \oplus X$ with the product given by $(\alpha, x) \cdot (\beta, y) = (\alpha\beta + f(x, y), \beta x + \alpha y)$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$. We remark that J is simple in the case when $\dim X > 1$ and f is nondegenerate. It is easy to see that a linear map $D : J \to J$ is a derivation if and only if there exists a linear map

$T : X \to X$ such that $D(\alpha, x) = (0, T(x))$ and $f(x, T(x)) = 0$ for all $\alpha \in \mathbb{C}$ and $x \in X$. Suppose, in addition, that $f(T(x), T(y)) = 0$ for all $x, y \in X$ (concretely, we can take $X = \mathbb{C}^4$, $f(x, y) = x_1y_3 - x_2y_4 + x_3y_1 - x_4y_2$, and $T(x) = (x_4, x_3, 0, 0)$). Note that in this case $D(\alpha, x) \cdot D(\beta, y) = 0$ for all $(\alpha, x), (\beta, y) \in J$. In particular, $[D(J), D(J), D(J)] = 0$. Further, linearizing $f(x, T(x)) = 0$ we get $f(x, T(y)) + f(y, T(x)) = 0$ for all $x, y \in X$, which together with $f(T(x), T(y)) = 0$ gives $f(x, T^2(y)) = 0$ for all $x, y \in X$. Using this it is easy to see that $D^2(J)$ lies in the radical of J.

The next example is taken from [12].

Example 4.8. Let J be the Jordan algebra of all symmetric elements of $M_{2n}(\mathbb{C})$, $n \geq 1$, with respect to symplectic involution. If $e_{1,n+1}$ denotes the matrix with 1 in position $(1, n+1)$ and 0's elsewhere, then the derivation D on J defined by $D(a) = [a, e_{1,n+1}]$ satisfies $D(a) \cdot D(b) = 0$ for all $a, b \in J$, $[D(J), D(J), D(J)] = 0$ and $D^2 = 0$.

Let us mention, incidentally, that these two examples show that there are nonzero derivations D on simple Jordan–Banach algebras J such that $D^2(a) = 0 = D(a)^2$ for all $a \in J$. This shows that one cannot extend a result of Posner [28] on the composition of derivations into the Jordan context, as well as that, unlike the case of Banach algebras [38, 25], there exist derivations on (even simple) Jordan–Banach algebras J whose range consists of quasinilpotent elements but it does not lie in the radical.

We have been unable, however, to find an example of a derivation D on a Jordan–Banach algebra J such that $[D(J), D(J), D(J)] = 0$ and $D^2(J) \not\subseteq \mathrm{Rad}(J)$. Therefore, we leave as an open question whether or not it is possible to improve Theorem 4.6 by showing that $D^2(J)$ is actually contained in the radical.

Derivations satisfying $[D(a), J, a] = 0$ *for each* a. Our next goal is to prove Theorem 4.14. The proof consists of two rather different parts. One part is the reduction to the case when J is primitive. Here, many arguments are similar to those used in the proof of Theorem 4.6. Assuming that J is a primitive algebra, it follows at once from Theorems 3.3 and 2.1 that $D(a)$ is quasinilpotent for each $a \in J$. However, as noted in the examples above, this condition is not sufficient to conclude that $D = 0$. Therefore, some other methods are also needed, and we shall rely heavily on the structure theory for primitive Jordan–Banach algebras in the spirit of E. Zel'manov obtained recently by M. Cabrera, A. Moreno and A. Rodríguez [9].

Lemma 4.9. *Suppose there exists a continuous anti-symmetric bilinear map* G *from* $J \times J$ *to a Banach space* X *such that* $G(D(a), a) = 0$ *for each* $a \in J$. *Then* $G(\mathcal{I}(D), J) = 0$.

Proof. Linearizing $G(D(a),a)=0$ for each $a\in J$ we get $G(D(a),b)+G(D(b),a)=0$ and therefore $G(D(a),b)=G(a,D(b))$ for all $a,b\in J$. Letting G_b, for $b\in J$, denote the continuous linear map from J to X given by $G_b(a)=G(a,b)$, we see that the identity above can be written as $G_b\circ D=G_{D(b)}$ for every $b\in J$.

It suffices to prove that for each $n\in\mathbb{N}$, $G(R(\mathcal{S}(D^n)),J)=0$ for all $R\in M_1(J)$. Since this is trivially true when $n=0$, we may assume that it is true for some $n\geq 0$ and we have to show that then it is also true for $n+1$. Again involving the derivation Δ on $M_1(J)$ we see that for any $b\in J$ we have

$$G_b\circ R\circ D^{n+1}=G_b\circ D\circ R\circ D^n-G_b\circ\Delta(R)\circ D^n=(G_{D(b)}\circ R-G_b\circ\Delta(R))\circ D^n.$$

Since the maps $G_b\circ R$ and $G_{D(b)}\circ R-G_b\circ\Delta(R)$ are continuous, [34, Lemma 1.3] implies that

$$\begin{aligned}\overline{G_b(R\mathcal{S}(D^{n+1}))}&=\mathcal{S}(G_b\circ R\circ D^{n+1})=\mathcal{S}((G_{D(b)}\circ R-G_b\circ\Delta(R))\circ D^n)\\&=\overline{(G_{D(b)}\circ R-G_b\circ\Delta(R))(\mathcal{S}(D^n))}=0,\end{aligned}$$

because $G(R(\mathcal{S}(D^n)),D(b))=G(\Delta(R)(\mathcal{S}(D^n)),b)=0$ by our assumption. Thus, $G_b(R\mathcal{S}(D^{n+1}))=0$ for each $b\in J$ and $R\in M_1(J)$, meaning that $G(R(\mathcal{S}(D^{n+1})),J)=0$. ■

LEMMA 4.10. *Suppose that there exists a closed subspace M of J such that $[D(a),J,a]\subset M$ for each $a\in J$. Then $[\mathcal{I}(D),J,J]\subset M$.*

Proof. Let $c\in J$ and let G be the continuous anti-symmetric bilinear map from $J\times J$ to the quotient Banach space J/M given by $G(a,b)=\pi_M([a,c,b])$ for all $a,b\in J$. Since $G(D(a),a)=0$ for each $a\in J$, the preceding lemma gives $G(\mathcal{I}(D),J)=0$ and therefore $[\mathcal{I}(D),c,J]\subset M$. ■

LEMMA 4.11. *Suppose that $[D(a),J,a]\subset \mathrm{Rad}(J)$ for each $a\in J$. Then $D(P)\subset P$ for each primitive ideal P of J except possibly finitely many primitive ideals each of which is* 1*-codimensional.*

Proof. Suppose that $D(P)\not\subset P$ for some primitive ideal P of J. Then $\mathcal{I}(D)\not\subset P$ and therefore $\pi_P(\mathcal{I}(D))=J/P$. On the other hand, from Lemma 4.10 it follows that $[\mathcal{I}(D),J,J]\subset\mathrm{Rad}(J)$. Hence

$$[J/P,J/P,J/P]=[\pi_P(\mathcal{I}(D)),\pi_P(J),\pi_P(J)]=0$$

and so J/P is a primitive commutative complex Banach algebra. Consequently, J/P is isomorphic to $\mathbb{C}$. The fact that there can only be finitely many such exceptional ideals follows from Theorem 2.1. ■

LEMMA 4.12. *Suppose that $[D(a),J,a]\subset\mathrm{Rad}(J)$ for each $a\in J$ and that $D(P)\subset P$ for some primitive ideal P of J. Then $D(J)\subset P$.*

Proof. Replacing D by the derivation D_P on J/P we see that there is no loss of generality in assuming that J is primitive. Our goal is to show that $D=0$.

According to the structure theorem for primitive Jordan–Banach algebra [9], there are four cases to consider.

CASE 1. First we consider the case when J is the simple exceptional 27-dimensional complex Jordan–Banach algebra of all matrices of the form

$$\begin{pmatrix} \lambda & x & y \\ \overline{x} & \mu & z \\ \overline{y} & \overline{z} & \nu \end{pmatrix}$$

where $\lambda,\mu,\nu\in\mathbb{C}$ and $x,y,z\in\mathbb{O}$, with product given by $a\cdot b=\frac{1}{2}(ab+ab)$ for all $a,b\in J$. Here, $\mathbb{O}$ denotes the complex octonions. Let e_{ij}, $i,j=1,2,3$, denote the matrix with 1 in position (i,j) and 0's elsewhere. From $D(e_{11})=D(e_{11}^2)=2e_{11}\cdot D(e_{11})$ one can conclude that $D(e_{11})$ must be of the form

$$\begin{pmatrix} 0 & x & y \\ \overline{x} & 0 & 0 \\ \overline{y} & 0 & 0 \end{pmatrix}$$

with $x,y\in\mathbb{O}$. From the identity $[D(e_{11}),e_{22}+e_{33},e_{11}]=0$ it follows at once that $D(e_{11})=0$. Similarly we see that $D(e_{22})=D(e_{33})=0$. Now pick any $x\in\mathbb{O}$ and set

$$a=\begin{pmatrix} 0 & x & 0 \\ \overline{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Our goal is to show that $D(a)=0$. Since $a=2a\cdot e_{11}=2a\cdot e_{22}$, and $D(e_{11})=D(e_{22})=0$, it follows that $D(a)=2D(a)\cdot e_{11}=2D(a)\cdot e_{22}$, which implies that $D(a)$ is of the form

$$D(a)=\begin{pmatrix} 0 & w & 0 \\ \overline{w} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $w\in\mathbb{O}$. Linearizing $[D(b),c,b]=0$ we get $[D(b_1),c,b_2]+[D(b_2),c,b_1]=0$ for all $b_1,b_2,c\in J$. Letting $b_1=e_{11}$, $c=e_{13}+e_{31}$, and $b_2=a$ we thus get $[D(a),e_{13}+e_{31},e_{11}]=0$, which immediately gives $D(a)=0$. Similarly we see that D vanishes on matrices of the form

$$\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ \overline{y} & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & \overline{z} & 0 \end{pmatrix}$$

with $y, z \in \mathbb{O}$, and so $D = 0$ as claimed.

Case 2. Now we assume that J is a Jordan–Banach algebra of a continuous nondegenerate symmetric bilinear form f on a nonzero complex Banach space X. Using the same notation as in Example 4.7, we thus have $[(0, T(x)), (\beta, y), (\alpha, x)] = 0$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y \in X$, where T is a linear operator on X such that $f(x, T(x)) = 0$ for all $x \in X$. A direct computation shows that this yields $f(x, y)T(x) = f(y, T(x))x$ for all $x, y \in X$. Given $x \neq 0$ in X and choosing $y \in X$ so that $f(x, y) \neq 0$, we see that $T(x) = \lambda_x x$ for some $\lambda_x \in \mathbb{C}$. A standard argument shows that λ_x does not depend on x and therefore $T = \lambda I$ for some $\lambda \in \mathbb{C}$. If λ were nonzero, $f(x, T(x)) = 0$ would yield $f(x, x) = 0$ for all $x \in X$, a contradiction. Thus $\lambda = 0$ and so $D = 0$.

It should be pointed out that if J is a Jordan quadratic algebra then J is the Jordan algebra of a symmetric bilinear form on some linear space X. Indeed, there exist a linear functional $\tau : J \to \mathbb{C}$ and a functional $\mu : J \to \mathbb{C}$ such that $a^2 - \tau(a)a + \mu(a)1 = 0$ for all $a \in J$. Let $X = \{a \in J : \tau(a) = 0\}$. It is easily checked that J is the Jordan algebra of the symmetric bilinear form f on the linear space X given by $f(x, y) = \frac{1}{2}(-\mu(x + y) + \mu(x) + \mu(y))$ for all $x, y \in X$.

Case 3. We consider the case when there is a complex Banach space X and an associative subalgebra A of $BL(X)$ acting irreducibly on X such that J can be viewed as a Jordan subalgebra of $BL(X)$ containing A as an ideal, and the inclusion $J \hookrightarrow BL(X)$ is continuous. The identity $[D(a), J, a] = 0$ can now be written as $[[D(a), a], J] = 0$, where $[a, b]$ stands for $ab - ba$. We claim that this implies that $[D(a), a] = 0$. On the one hand, this follows from [4, Proposition 3.1], but a simple direct argument can also be given. Namely, since J contains a subalgebra of $BL(X)$ which acts irreducibly on X, $[[D(a), a], J] = 0$ implies that $[D(a), a]$ must be a scalar multiple of the identity. However, it is well known (and in fact it is a corollary to the Kleinecke–Shirokov theorem) that the commutator of two bounded linear operators cannot be a nonzero scalar multiple of the identity, so $[D(a), a]$ must be 0 for every $a \in J$.

Let $\mathcal{D}$ be the restriction of D to A. Then $\mathcal{D}$ is a Jordan derivation from A to J. Now, Herstein's theorem [13] is not directly applicable since it only tells us that every Jordan derivation of A into itself is a derivation (of associative algebras). However, just glancing through the proof given in [7] shows that this conclusion also holds for Jordan derivations from A into $BL(X)$. Thus,

$\mathcal{D}$ is a derivation. Therefore, we have arrived at a similar situation to the one in the well known result of Posner [28].

Linearizing $[D(a),a]=0$ we get $[D(a),b]+[D(b),a]=0$ for all $a,b\in J$. Now replace b by ba and assume that $a,b\in A$. Then we get

$$\begin{aligned}0&=[\mathcal{D}(a),ba]+[\mathcal{D}(b)a+b\mathcal{D}(a),a]\\&=[\mathcal{D}(a),b]a+[\mathcal{D}(b),a]a+[b,a]\mathcal{D}(a).\end{aligned}$$

However, since $[\mathcal{D}(a),b]+[\mathcal{D}(b),a]=0$, this reduces to $[b,a]\mathcal{D}(a)=0$. Replacing b by cb with $b,c\in A$ and using $[cb,a]=[c,a]b+c[b,a]$ we arrive at $[A,a]A\mathcal{D}(a)=0$ for every $a\in A$. The irreducibility of A implies that for each $a\in A$, either $\mathcal{D}(a)=0$ or a is a scalar multiple of the identity. However, in the latter case we also have $\mathcal{D}(a)=0$. Thus, $\mathcal{D}=0$. Therefore, $[D(a),b]+[D(b),a]=0$ for all $a,b\in J$ now yields $[D(b),a]=0$ for all $b\in J$, so that $D(J)$ consists of scalar multiples of the identity. However, D is a derivation and it is easy to see that this is possible only when $D=0$.

Let us mention that case 3 could also be handled in a similar way to case 4 below, and that perhaps the shortest proof could be given by applying the results on functional identities (see e.g. [5]); however, invoking this theory would make the paper less self-contained.

Case 4. Finally, we may assume that there exist a complex Banach space X and an associative subalgebra A of $BL(X)$ acting irreducibly on X such that J can be viewed as a Jordan subalgebra of $BL(X)$, the inclusion $J\hookrightarrow BL(X)$ is continuous, the identity map on J extends to a linear algebra involution $*$ on the subalgebra B of $BL(X)$ generated by J, A is a $*$-invariant subset of B, $H(A,*)$ is an ideal of J, and A is generated by $H(A,*)$. There is no loss of generality in assuming that J is not a quadratic algebra since otherwise it would belong to the class of Jordan algebras already treated in case 2. Therefore we have $\dim X>2$.

Set $H=H(A,*)$ and let H^2 be the linear span of all h^2, $h\in H$ (equivalently, H^2 is the linear span of all $h_1h_2+h_2h_1$, $h_1,h_2\in H$). Note that $D(H^2)\subset H$ since H is an ideal of J (this is the main reason why we deal with H^2). We claim that the (associative) subalgebra generated by H^2 contains a nonzero ideal of A. We shall prove this by using [6] (this could probably also be extracted from the proofs of Herstein's classical results [14] on rings with involution but we have been unable to find an appropriate direct reference). First note that, since $\dim X>2$, there exist elements in A which are not algebraic over $\mathbb{C}$ of degree ≤ 2 (for example, pick three linearly independent vectors $x_1,x_2,x_3\in X$ and use the Jacobson density theorem to obtain an element $a\in A$ such that $ax_1=x_2$ and $ax_2=x_3$). Consequently, it is clear from [6] that the following is true: If L is any nonzero subspace of A such that $ax+xa^*\in L$ for all $a\in A$, $x\in L$, then the subalgebra generated by L contains a nonzero ideal of A. We claim that $L = H^2$ has this property. Since any element in A can be written as a sum $h+k$ with $h^* = h$ and $k^* = -k$, it suffices to show that for any $h' \in H$ we have $hh'^2 + h'^2 h \in H^2$ and $kh'^2 - h'^2 k \in H^2$. The first relation is clear, while the second one is obvious from $kh'^2 - h'^2 k = (kh' - h'k)h' + h'(kh' - h'k)$, since $kh' - h'k \in H$.

Therefore, the subalgebra of A generated by H^2 also acts irreducibly on X. This implies, in particular, that scalar multiples of the identity are the only operators in $BL(X)$ that commute with every element from H^2. The same conclusion of course holds for H and J.

In the course of the proof below we shall arrive at the point where the concept of the extended centroid will be used. This concept was introduced by W. S. Martindale [23] and we refer the reader to [2] for a full account. In general, the extended centroid of a prime ring is a certain field containing the centre of the ring. In our situation, considering the algebra A, the extended centroid is just (isomorphic to) the complex field $\mathbb{C}$. Namely, if we identify $\mathbb{C}$ with the scalar multiples of the identity, it is obvious that $\mathbb{C}$ is contained in the extended centroid of A. On the other hand, [2, Corollary 4.1.2] implies that $\mathbb{C}$ contains the extended centroid of A. Thus, $\mathbb{C}$ is the extended centroid of A.

We now have enough information to start to treat a derivation D on J satisfying our condition. First, as in the proof of the preceding case one shows that then $[D(a), a] = 0$ for every $a \in J$, which yields

$$[D(a), b] + [D(b), a] = 0$$

for all $a, b \in J$. Replacing b by b^2 in this identity and using

$$[D(a), b^2] = [D(a), b]b + b[D(a), b],$$

$$[D(b)b + bD(b), a] = [D(b), a]b + D(b)[b, a] + [b, a]D(b) + b[D(b), a],$$

together with $[D(a), b] + [D(b), a] = 0$, we arrive at

$$D(b)[b, a] + [b, a]D(b) = 0$$

for all $a, b \in J$. Let us rewrite this identity as

$$\{-abD(b) + baD(b)\} + \{D(b)ba - D(b)ab\} = 0.$$

Now consider this relation for a fixed $b \in H^2$ and an arbitrary $a \in H$. Since $D(b) \in H$, [22, Lemma 3] can be applied. Therefore, we find that if b does not lie in $\mathbb{C}$, which is the extended centroid of A (again we used the identification of $\mathbb{C}$ with the scalar multiples of the identity), then $D(b) \in \mathbb{C}\, b + \mathbb{C}$. Of course, if $b \in \mathbb{C}$, then $D(b) = 0$, so that $D(b) \in \mathbb{C}\, b + \mathbb{C}$ for any $b \in H^2$. That is to say, for any $b \in H^2$ there is $\lambda_b \in \mathbb{C}$ such that $D(b) - \lambda_b b \in \mathbb{C}$.

Our next goal is to show that λ_b can be chosen independently of b. The argument is rather standard (see e.g. [22, pp. 2890–2891]), but we give it for the sake of completeness. Pick some $b \in H^2$ such that $b \notin \mathbb{C}$ (such a b certainly exists in view of the discussion above) and set $\lambda = \lambda_b$. Now, if $c \in H^2$ is such that $[b,c] \neq 0$, then we have

$$\lambda[b,c] = [D(b),c] = -[D(c),b] = \lambda_c[b,c]$$

and so $\lambda_c = \lambda$. On the other hand, if b and c commute, then we can pick $d \in H^2$ such that $[b,d] \neq 0$ since $b \notin \mathbb{C}$, hence also $[c+d,b] \neq 0$, so that $\lambda_{d+c} = \lambda_d = \lambda$. Thus, $D(c+d) \in \lambda(c+d) + \mathbb{C}$. On the other hand, $D(c+d) = D(c) + D(d) \in \lambda_c c + \lambda d + \mathbb{C}$. Comparing we see that $\lambda_c = \lambda$ unless $c \in \mathbb{C}$. But then $D(c) - \lambda c$ lies in $\mathbb{C}$ for every $c \in H^2$. Therefore, the relation $[D(c),a] + [D(a),c] = 0$ with $a \in J$, $c \in H^2$ can be written as $[D(a) - \lambda a, c] = 0$ for all $a \in J$, $c \in H^2$. Consequently, $\mu(a) = D(a) - \lambda a$ lies in $\mathbb{C}$ for every $a \in J$. Now consider $D(a^2)$ with $a \in J$. On the one hand this element is equal to $\lambda a^2 + \mu(a^2)$, and on the other hand to $2aD(a) = 2\lambda a^2 + 2\mu(a)a$. Hence

$$\lambda a^2 + 2\mu(a)a \in \mathbb{C}$$

for every $a \in J$. However, since it is assumed that the algebra J is not quadratic, it follows that $\lambda = 0$ and also $\mu(a) = 0$ for every $a \in J$ such that $a \notin \mathbb{C}$. Consequently, $D(a) = 0$ for every such a. Of course, if $a \in \mathbb{C}$ then $D(a)$ is certainly 0. Thus, $D(J) = 0$. ∎

LEMMA 4.13. *Suppose that* $[D(a), J, a] = 0$ *for each* $a \in J$. *Then*

(i) $D(\mathcal{I}(D)) \cdot \mathcal{I}(D) \subset \mathrm{Rad}(\mathcal{I}(D))$.

(ii) $a^4 \in \mathrm{Rad}(\mathcal{I}(D))$ *for all* $a \in \mathcal{S}(D^n)$ *and* $n \in \mathbb{N}$; *accordingly*, $\mathcal{S}(D^n)$ *consists of quasinilpotent elements for each* $n \in \mathbb{N}$.

(iii) $D(P) \subset P$ *for each primitive ideal* P *of* J.

Proof. Just follow the proof of Lemma 4.5 and apply Lemmas 4.10 and 4.11 instead of Lemmas 4.2 and 4.3 at appropriate places. ∎

THEOREM 4.14. *Let* J *be a complex Jordan–Banach algebra and let* D *be a derivation on* J. *Suppose that* $[D(a), J, a] = 0$ *for each* $a \in J$. *Then* $D(J) \subset \mathrm{Rad}(J)$.

Proof. An immediate consequence of Lemmas 4.13 and 4.12. ∎

5. On the Singer–Wermer conjecture for Jordan–Banach algebras

LEMMA 5.1. *Let* J *be a Jordan algebra and let* $I, P_1, \ldots, P_n$ *be pairwise different ideals of* J *such that* $I \not\subset P_i$, J/P_i *is simple and it has an identity for each* $i \in \{1, \ldots, n\}$. *Then the homomorphism* $a \mapsto (a + P_1, \ldots, a + P_n)$ *from* I *to* $J/P_1 \oplus \ldots \oplus J/P_n$ *is onto.*

Proof. Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. We claim that $I \cap P_i + I \cap P_j = I$. From the simplicity assumption, P_i and P_j are maximal ideals of J. Hence

$P_i + P_j = J$ and $P_i + I = J$. Since J/P_i has an identity, [39, Lemma 4] now shows that $P_i + I \cap P_j = J$, and so $I \cap P_i + I \cap P_j = I$.

On the other hand, for each $i \in \{1, \ldots, n\}$, the map $a + I \cap P_i \mapsto a + P_i$ is an isomorphism from $I/I \cap P_i$ onto J/P_i. Consequently, the ideals $I \cap P_1, \ldots, I \cap P_n$ of the Jordan algebra I satisfy the requirements of [39, Lemma 5] and therefore the homomorphism $a \mapsto (a + I \cap P_1, \ldots, a + I \cap P_n)$ from I to $I/I \cap P_1 \oplus \ldots \oplus I/I \cap P_n$ is onto, which completes the proof. ∎

THEOREM 5.2. *Let J be a complex Jordan–Banach algebra and let D be a derivation on J such that $[D(J), D(J), D(J)] \subset \mathrm{Rad}(J)$. Then there are pairwise orthogonal idempotents $e_1, \ldots, e_n$ in J such that*

$$D^2(J) \subset \mathbb{C}\, e_1 + \ldots + \mathbb{C}\, e_n + \mathcal{Q}(J).$$

Proof. On account of Lemma 4.4, we may assume that there exist primitive ideals of J which are not invariant under D. In view of Lemma 4.3 there are only finitely many such ideals $P_1, \ldots, P_n$ each of which is 1-codimensional. For each $i \in \{1, \ldots, n\}$ let ϕ_i be the character on J whose kernel is P_i. By Lemma 5.1 with $I = \mathcal{I}(D)$ (taking into account Theorem 2.1) there exists $u \in \mathcal{I}(D)$ such that $\phi_i(u) = i$ for each $i \in \{1, \ldots, n\}$. From [40, Lemma 1] (cf. Section 2) we have $\mathrm{Sp}(u) \subset \{0, 1, \ldots, n\}$. For each $i \in \{1, \ldots, n\}$ there is a function f_i analytic on a neighbourhood of $\mathrm{Sp}(u)$ which is identically 1 on a neighbourhood of $\{i\}$ but identically 0 on a neighbourhood of $\{0\} \cup (\mathrm{Sp}(u) \setminus \{i\})$. Hence the elements $e_1, \ldots, e_n \in J^1$ given by $e_i = f_i(u)$ for each $i \in \{1, \ldots, n\}$ are pairwise orthogonal idempotents of J^1. We could explicitly take

$$e_j = \frac{1}{2\pi i} \int\limits_{|\lambda - j| = 1/2} (\lambda - u)^{-1}\, d\lambda$$

for each $j \in \{1, \ldots, n\}$. Since $f_i(0) = 0$ it may be concluded that $e_i \in \mathcal{I}(D)$ for each $i \in \{1, \ldots, n\}$. Furthermore, $\phi_j(e_i) = f_i(\phi_j(u)) = f_i(j) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ and therefore $e_i \in P_j$ if $i \neq j$.

Let $a \in J$. We claim that $D^2(a) - \sum_{j=1}^{n} \phi_j(D^2(a)) e_j$ is quasinilpotent. Indeed, for each primitive ideal P such that $D(P) \subset P$ we have

$$\pi_P\Big(D^2(a) - \sum_{j=1}^{n} \phi_j(D^2(a)) e_j\Big) = \pi_P(D^2(a)),$$

which is quasinilpotent by Lemma 4.4. Further, for each $i \in \{1, \ldots, n\}$ we have

$$\phi_i\Big(D^2(a) - \sum_{j=1}^{n} \phi_j(D^2(a)) e_j\Big) = 0.$$

Accordingly, we conclude from [40, Lemma 1] that $D^2(a) - \sum_{j=1}^{n} \phi_j(D^2(a)) e_j$ is quasinilpotent, as claimed. ∎

THEOREM 5.3. *Let J be a complex Jordan–Banach algebra and let D be a derivation on J such that $[D(a), J, a] \subset \mathrm{Rad}(J)$ for each $a \in J$. Then there are pairwise orthogonal idempotents $e_1, \ldots, e_n$ in J such that*

$$D(J) \subset \mathbb{C}e_1 + \ldots + \mathbb{C}e_n + \mathrm{Rad}(J).$$

Proof. A simple modification of the proof of Theorem 5.2. ∎

We leave as an open question whether the appearance of idempotents e_i can be avoided in Theorems 5.2 and 5.3. Our next theorem shows, in particular, that this question and the Singer–Wermer conjecture for Jordan–Banach algebras are in fact equivalent problems.

THEOREM 5.4. *The following assertions are equivalent.*

(i) *Every derivation on a complex Jordan–Banach algebra leaves each primitive ideal invariant.*

(ii) *Every derivation on the unitization of a radical complex Jordan–Banach algebra leaves the radical invariant.*

(iii) *Every derivation D on a complex Jordan–Banach algebra J such that $[D(J), D(J), D(J)] \subset \mathrm{Rad}(J)$ satisfies $D^2(J) \subset \mathcal{Q}(J)$.*

(iv) *Every derivation D on a complex Jordan–Banach algebra such that $[D(a), J, a] \in \mathrm{Rad}(J)$ for each $a \in J$ satisfies $D(J) \subset \mathrm{Rad}(J)$.*

(v) *For every derivation D on a complex Jordan–Banach algebra J, $D(a) \in \mathcal{Q}(J)$ whenever $a \in J$ is such that $[D(a), J, J] \subset \mathrm{Rad}(J)$.*

(vi) *For every derivation D on a complex Jordan–Banach algebra J, $D(a) \in \mathcal{Q}(J)$ whenever $a \in J$ is such that $[D(a), J, a] \subset \mathrm{Rad}(J)$.*

(vii) *For every derivation D on a complex Jordan–Banach algebra J, $D(a) \in \mathcal{Q}(J)$ whenever $a \in J$ is such that $[D^i(a), a, D^j(a)] \in \mathrm{Rad}(J)$ for all $i, j \geq 0$.*

Proof. Assume that (i) holds. Let D be a derivation on a Jordan–Banach algebra J such that $[D(J), D(J), D(J)] \subset \mathrm{Rad}(J)$. For each primitive ideal P of J we clearly have $[D_P(J/P), D_P(J/P), D_P(J/P)] = 0$, and so Theorem 4.6 shows that $D_P^2(J/P) \subset \mathcal{Q}(J/P)$. Thus $\pi_P(D^2(J)) \subset \mathcal{Q}(J/P)$ for each primitive ideal P of J and hence [40, Lemma 1] yields $D^2(J) \subset \mathcal{Q}(J)$. Thus, (i) implies (iii). In a similar fashion, by applying results of previous sections, we see that (i) also implies each of the assertions (iv)–(vii) (for the proof of the last two implications one also has to use the fact (see Theorem 2.1) that derivations on primitive Jordan–Banach algebras are necessarily continuous). It is trivial that (i) also implies (ii).

Next we claim that each of the assertions (iii)–(vii) implies (ii). Let D be a derivation on the unitization J^1 of a radical Jordan–Banach algebra J. Note that $[J^1, J^1, J^1] \subset J = \mathrm{Rad}(J^1) = \mathcal{Q}(J^1)$. Using this we see that the validity of any of the assertions (iv)–(vii) implies immediately that $D(J) \subset J$, as desired. If we assume that (iii) holds, it first follows only that $D^2(J^1) \subset J$. However, from

$$(D(a))^2 = \tfrac{1}{2}D^2(a^2) - D^2(a)\cdot a \in J$$

we can conclude that $D(a) \in J$ for any $a \in J$ in this case as well.

It remains to prove that (ii) implies (i). Suppose that there exists a derivation D on some Jordan–Banach algebra J such that $D(P) \not\subset P$ for some primitive ideal P of J. Let $P = P_1, P_2, \dots, P_n$ be the only primitive ideals of J which are not invariant under D. For each $i \in \{1, \dots, n\}$, J/P_i is simple and either it is finite-dimensional or it is the Jordan–Banach algebra of a continuous nondegenerate symmetric bilinear form f on a complex Banach space X of dimension greater than one (cf. Theorem 2.1). We claim that there exists an idempotent element u of J/P such that $U_u(J/P) = \mathbb{C}u$. Indeed, if J/P is isomorphic to $\mathbb{C}$ then this is clear. If J/P is the Jordan algebra of a nondegenerate symmetric bilinear form f on a nonzero linear space X, then we can take $u = (1/2, x)$ with $x \in X$ and $f(x, x) = 1/4$. Otherwise, on account of Albert's theorem [16, Corollary V.6.2], J/P is isomorphic to the matrix algebra $\{(a_{ij}) \in M_n(\mathbb{D}) : (\overline{a}_{ji}) = (a_{ij})\}$, where $\mathbb{D}$ is a composition algebra over $\mathbb{C}$ of dimension 1, 2, or 4 if $n \geq 4$ and of dimension 1, 2, 4 or 8 if $n = 3$. In this case we can take u to be the matrix e_{11}. By Lemma 5.1 there is $a \in \mathcal{I}(D)$ such that $a + P = u$ and $a \in P_i$ if $i \neq 1$. Then $\mathrm{Sp}(a) \subset \{0, 1\}$ by [40, Lemma 1]. Let f be a function analytic on a neighbourhood of $\{0, 1\}$ which is identically 1 on a neighbourhood of 1 but identically 0 on a neighbourhood of 0. Hence $e = f(a)$ is an idempotent element in J^1. We could explicitly take

$$e = \frac{1}{2\pi i} \int\limits_{|\lambda-1|=1/2} (\lambda - a)^{-1}\, d\lambda.$$

Since $f(0) = 0$ it follows that $e \in \mathcal{I}(D)$ and $\pi_P(e) = f(u)$. Further,

$$f(u) = \frac{1}{2\pi i} \int\limits_{|\lambda-1|=1/2} (\lambda - u)^{-1}\, d\lambda$$

and since

$$(\lambda - u)^{-1} = (\lambda - 1)^{-1}u + \lambda^{-1}(1 - u)$$

for each $\lambda \in \mathbb{C}\backslash\{0, 1\}$, it follows that $f(u) = u$. Moreover $e \in \mathcal{I}(D) \cap P_2 \cap \ldots \cap P_n$. Indeed, $\pi_{\mathcal{I}(D)}(e) = f(\pi_{\mathcal{I}(D)}(a)) = f(0) = 0$ and $\pi_{P_i}(e) = f(\pi_{P_i}(a)) = f(0) = 0$ for each $i \neq 1$. Hence $U_e(P) \subset \mathrm{Rad}(J)$. From [16, Lemma III.1(i) and equality III.1.(5)] we deduce that $U_e(J)$ is a unital Jordan–Banach algebra with identity element e. Furthermore, $U_e(P)$ is an ideal of $U_e(J)$. We claim that the quotient map π_P induces an isomorphism from $U_e(J)/U_e(P)$ onto $U_u(J/P) \cong \mathbb{C}$. Indeed, it is clear that $\pi_P(U_e(J)) = U_u(J/P)$ and if

$a \in U_e(J)$ is such that $\pi_P(a) = 0$, then $a = U_e(a) \in U_e(P)$. Accordingly, $U_e(J)$ is the unitization of $U_e(P)$ which is a radical Jordan–Banach algebra, because $U_e(P) \subset \mathrm{Rad}(J)$. Consider the linear map D_e from $U_e(J)$ to itself defined by $D_e(x) = U_e(D(x))$ for each $x \in U_e(J)$. By using the Peirce decomposition of J relative to the idempotent e (see [16, Section III.1]) it follows immediately that D_e is a derivation. On the other hand, for each $x \in P$ we have

$$\begin{aligned} U_e(D(U_e(x))) &= U_e(2D(e)\cdot(e\cdot x) + 2e\cdot(D(e)\cdot x) - D(e)\cdot x + U_e(D(x))) \\ &= U_e(2D(e)\cdot(e\cdot x) + 2e\cdot(D(e)\cdot x) - D(e)\cdot x) + U_e(U_e(D(x))). \end{aligned}$$

As $2D(e)\cdot(e\cdot x)+2e\cdot(D(e)\cdot x)-D(e)\cdot x \in P$ and $U_e(U_e(D(x))) = U_e(D(x))$ we see that $\pi_P(D_e(U_e(x))) = \pi_P(U_e(D(x))) = U_u(\pi_P(D(x)))$. Therefore

$$\pi_P(D_e(U_e(P))) = U_u(\pi_P(D(P))) = U_u(J/P) = \mathbb{C}u.$$

Consequently, D_e does not map into the radical of $U_e(J)$. ■

Analysis similar to that in the preceding proof together with some results of the previous section when restricted to Banach algebras shows the following version of Theorem 5.4 for Banach algebras. Unfortunately, as far as we know the following result does not follow directly from the preceding one.

Theorem 5.5. *The following assertions are equivalent.*

(i) *Every derivation on a complex Banach algebra leaves each primitive ideal invariant.*

(ii) *Every derivation on the unitization of a radical complex Banach algebra leaves the radical invariant.*

(iii) *Every derivation D on a complex Banach algebra A such that $[D(A), D(A)] \subset \mathrm{Rad}(A)$ satisfies $D(A) \subset \mathrm{Rad}(A)$.*

(iv) *Every derivation D on a complex Banach algebra A such that $[D(a), a] \in \mathrm{Rad}(A)$ for each $a \in A$ satisfies $D(A) \subset \mathrm{Rad}(A)$.*

(v) *For every derivation D on a complex Banach algebra A, $D(a) \in \mathrm{Rad}(A)$ whenever $a \in A$ is such that $[D(a), A] \subset \mathrm{Rad}(A)$.*

(vi) *For every derivation D on a complex Banach algebra A, $D(a) \in \mathcal{Q}(A)$ whenever $a \in A$ is such that $[D(a), a] \subset \mathrm{Rad}(A)$.*

(vii) *For every derivation D on a complex Banach algebra A, $D(a) \in \mathcal{Q}(A)$ whenever $a \in A$ is such that $[D^i(a), D^j(a)] \in \mathrm{Rad}(A)$ for all $i, j \geq 0$.*

It should be mentioned that the equivalence between (i) and (ii) in Theorem 5.5 is already known. Some authors [24, 30] refer to it as to an unpublished result of M. Thomas.

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