On the statistical and $\sigma$-cores

by

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Abstract. In [11] and [7], the concepts of $\sigma$-core and statistical core of a bounded number sequence $x$ have been introduced and also some inequalities which are analogues of Knopp’s core theorem have been proved. In this paper, we characterize the matrices of the class $(S \cap m, V_\sigma)_{\text{reg}}$ and determine necessary and sufficient conditions for a matrix $A$ to satisfy $\sigma$-core$(Ax) \subseteq \text{st-core}(x)$ for all $x \in m$.

1. Introduction. Let $K$ be a subset of $\mathbb{N}$, the set of positive integers. The natural density $\delta$ of $K$ is defined by

$$\delta(K) = \lim_{n \to \infty} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where the vertical bars indicate the number of elements in the enclosed set. The number sequence $x = (x_k)$ is said to be statistically convergent to the number $l$ if for every $\varepsilon$, $\delta\{k : |x_k - l| \geq \varepsilon\} = 0$ (see [7]). In this case, we write $\text{st-lim} x = l$. We shall also write $S$ and $S_0$ to denote the sets of all statistically convergent sequences and of all sequences statistically convergent to zero. The statistically convergent sequences were studied by several authors (see [2], [7] and others).

Let $m$ and $c$ be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual supremum norm. Let $\sigma$ be a one-to-one mapping from $\mathbb{N}$ into itself. An element $\Phi \in m'$, the conjugate space of $m$, is called an invariant mean or a $\sigma$-mean if (i) $\Phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all $k$, (ii) $\Phi(e) = 1$, where $e = (1, 1, 1, \ldots)$, (iii) $\Phi((x_{\sigma(k)})) = \Phi(x)$ for all $x \in m$.

Throughout this paper we consider the mapping $\sigma$ such that $\sigma^p(k) \neq k$ for all positive integers $k \geq 0$ and $p \geq 1$, where $\sigma^p(k)$ is the $p$th iterate of $\sigma$ at $k$. Thus, a $\sigma$-mean extends the limit functional on $c$ in the sense that $\Phi(x) = \lim x$ for all $x \in c$ (see [12]). Consequently, $c \subset V_\sigma$ where $V_\sigma$ is the set of bounded sequences all of whose $\sigma$-means are equal.

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In case $\sigma(k) = k+1$, a $\sigma$-mean is often called a Banach limit and $V_\sigma$ is the set of almost convergent sequences, introduced by Lorentz [9]. If $x = (x_n)$, write $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown [15] that

$$V_\sigma = \{ x \in m : \lim_{p} t_{pn}(x) = s \text{ uniformly in } n, \ s = \sigma\text{-lim} \ x \}$$

where

$$t_{pn}(x) = (x_n + Tx_n + \ldots + T^px_n)/(p + 1), \quad t_{-1,n}(x) = 0.$$

We say that a bounded sequence $x = (x_k)$ is $\sigma$-convergent if $x \in V_\sigma$. By $Z$, we denote the set of $\sigma$-convergent sequences with $\sigma$-limit zero. It is well known [14] that $x \in m$ if and only if $Tx - x \in Z$.

Let $A$ be an infinite matrix of real entries $a_{nk}$ and $x = (x_k)$ be a real number sequence. Then $Ax = ((Ax)_n) = (\sum_{k} a_{nk} x_k)$ denotes the transformed sequence of $x$. If $X$ and $Y$ are two non-empty sequence spaces, then we use $(X,Y)$ to denote the set of all matrices $A$ such that $Ax$ exists and $Ax \in Y$ for all $x \in X$. Throughout, $\sum_k$ will denote summation from $k = 1$ to $\infty$.

A matrix $A$ is called (i) regular if $A \in (c,c)$ and $\lim Ax = \lim x$ for all $x \in c$, (ii) $\sigma$-regular if $A \in (c,V_\sigma)$ and $\sigma\text{-lim} Ax = \lim x$ for all $x \in c$, and (iii) $\sigma$-coercive if $A \in (m,V_\sigma)$. The regularity conditions for $A$ are well known [10].

The following two lemmas which were established in [15] will enable us to prove our results:

**Lemma 1.1 ([15, Th. 3]).** The matrix $A$ is $\sigma$-coercive if and only if

\[ (1.1) \quad \|A\| = \sup_{n} \sum_{k} |a_{nk}| < \infty, \]

\[ (1.2) \quad \sigma\text{-lim} a_{nk} = \alpha_k \quad \text{for each } k, \]

\[ (1.3) \quad \lim_{p} \sum_{k} \frac{1}{p + 1} \left| \sum_{i=0}^{p} (a_{\sigma^i(n),k} - \alpha_k) \right| = 0 \quad \text{uniformly in } n. \]

**Lemma 1.2 ([15, Th. 2]).** The matrix $A$ is $\sigma$-regular if and only if the conditions (1.1) and (1.2) hold with $\alpha_k = 0$ for each $k$ and

\[ (1.4) \quad \sigma\text{-lim} \sum_{k} a_{nk} = 1. \]

A matrix $A$ is called normal if $a_{nk} = 0 \ (k > n)$ and $a_{nn} \neq 0$ for all $n$. If $A$ is normal, then it has its reciprocal.

For any real number $\lambda$ we write $\lambda^- = \max\{-\lambda,0\}$, $\lambda^+ = \max\{0,\lambda\}$. Then $\lambda = \lambda^+ - \lambda^-$. We recall (see [11]) that a matrix $A$ is said to be $\sigma$-uniformly positive if

$$\lim_{p} \sum_{k} a^-_{(p,n,k)} = 0 \quad \text{uniformly in } n$$
where

\[ a(p, n, k) = \frac{1}{p + 1} \sum_{i=0}^{p} a_{\sigma^i(n)}. \]

It is known [11] that a \( \sigma \)-regular matrix \( A \) is \( \sigma \)-uniformly positive if and only if

\[
\lim_p \sum_k |a(p, n, k)| = 1 \quad \text{uniformly in} \ n.
\]

Let us consider the following functionals defined on \( m \):

\[
l(x) = \lim \inf x, \quad L(x) = \lim \sup x, \quad q_\sigma(x) = \lim \sup \sup_t t_{pn}(x),
\]

\[
L^*(x) = \lim \sup \sup_p \frac{1}{p + 1} \sum_{i=0}^{p} x_{n+i}.
\]

In [11], the \( \sigma \)-core of a real bounded number sequence \( x \) has been defined as the closed interval \([-q_\sigma(-x), q_\sigma(x)]\) and also the inequalities \( q_\sigma(Ax) \leq L(x) \) (\( \sigma \)-core of \( Ax \subseteq K \)-core of \( x \)), \( q_\sigma(Ax) \leq q_\sigma(x) \) (\( \sigma \)-core of \( Ax \subseteq \sigma \)-core of \( x \)), for all \( x \in m \), have been studied. Here the \( K \)-core of \( x \) (or Knopp core of \( x \)) is the interval \([l(x), L(x)]\) (see [3]).

When \( \sigma(n) = n + 1 \), since \( q_\sigma(x) = L^*(x) \), the \( \sigma \)-core of \( x \) is reduced to the Banach core of \( x \) (B-core) defined by the interval \([-L^*(-x), L^*(x)]\) (see [13]).

The concepts of B-core and \( \sigma \)-core have been studied by many authors [4, 5, 6, 11, 13].

Recently, Fridy and Orhan [7] have introduced the notions of statistical boundedness, statistical limit superior (st-lim sup) and inferior (st-lim inf), defined the statistical core (or briefly st-core) of a statistically bounded sequence as the closed interval \([\text{st-lim inf} x, \text{st-lim sup} x]\) and also determined necessary and sufficient conditions for a matrix \( A \) to yield \( K \)-core\((Ax) \subseteq \text{st-core}(x) \) for all \( x \in m \).

After all these explanations, one can naturally ask: What are necessary and sufficient conditions on a matrix \( A \) so that the \( \sigma \)-core of \( Ax \) is contained in the st-core of \( x \) for all \( x \in m ? \) Our main purpose is to find an answer to that question. To do this we need to characterize the class of matrices \( A \) such that \( Ax \in V_\sigma \) and \( \sigma \)-lim \( Ax = \text{st-lim} x \) for all \( x \in S \cap m \), i.e., \( A \in (S \cap m, V_\sigma)_{\text{reg}} \).

**2. Main results**

**Theorem 2.1.** \( A \in (S \cap m, V_\sigma)_{\text{reg}} \) if and only if \( A \) is \( \sigma \)-regular and

\[
\lim_p \sum_{k \in E} |a(p, n, k)| = 0 \quad \text{uniformly in} \ n,
\]

for every \( E \subseteq \mathbb{N} \) with natural density zero.
Proof. First, suppose that \( A \in (S \cap m, V_\sigma)_{\text{reg}} \). The \( \sigma \)-regularity of \( A \) immediately follows from the fact that \( c \subset S \cap m \). Now, define a sequence \( z = (z_k) \) for \( x \in m \) as

\[
z_k = \begin{cases} x_k, & k \in E, \\ 0, & k \notin E, \end{cases}
\]

where \( E \) is any subset of \( \mathbb{N} \) with \( \delta(E) = 0 \). By our assumption, since \( z \in S_0 \), we have \( Az \in Z \). On the other hand, since \( Az = \sum_{k \in E} a_{nk} x_k \), the matrix \( B = (b_{nk}) \) defined by

\[
b_{nk} = \begin{cases} a_{nk}, & k \in E, \\ 0, & k \notin E, \end{cases}
\]

for all \( n \), must belong to the class \( (m, Z) \). Hence, the necessity of (2.1) follows from Lemma 1.1.

Conversely, suppose that \( A \) is \( \sigma \)-regular and (2.1) holds. Let \( x \) be any sequence in \( S \cap m \) with \( \text{st-lim} x = l \). Write \( E = \{ k : |x_k - l| \geq \varepsilon \} \) for any given \( \varepsilon > 0 \), so that \( \delta(E) = 0 \). Now, from (1.4) we have

\[
\sigma\text{-lim}(Ax) = \sigma\text{-lim} \left( \sum_k a_{nk} (x_k - l) + l \sum_k a_{nk} \right)
= \sigma\text{-lim} \sum_k a_{nk} (x_k - l) + l
= \lim_p \sum_k a(p, n, k)(x_k - l) + l.
\]

On the other hand, since

\[
\left| \sum_k a(p, n, k)(x_k - l) \right| \leq \|x\| \sum_{k \in E} |a(p, n, k)| + \varepsilon \|A\|,
\]

the condition (2.1) implies that

\[
\lim_p \sum_k a(p, n, k)(x_k - l) = 0 \quad \text{uniformly in } n.
\]

Hence, \( \sigma\text{-lim}(Ax) = \text{st-lim} x \); that is, \( A \in (S \cap m, V_\sigma)_{\text{reg}} \), which completes the proof. \( \blacksquare \)

In the special case \( \sigma(n) = n + 1 \), we also have the following theorem:

**Theorem 2.2.** \( A \in (S \cap m, f)_{\text{reg}} \) if and only if \( A \) is almost regular (see [8]) and

\[
\lim_p \sum_{k \in E} \frac{1}{p+1} \left| \sum_{i=0}^p a_{n+i,k} \right| = 0 \quad \text{uniformly in } n,
\]

for every \( E \subseteq \mathbb{N} \) with natural density zero.

**Theorem 2.3.** \( \sigma\text{-core}(Ax) \subseteq \text{st-core}(x) \) for all \( x \in m \) if and only if \( A \in (S \cap m, V_\sigma)_{\text{reg}} \) and \( A \) is \( \sigma \)-uniformly positive.
Proof. Assume that \( \sigma\)-core\((Ax) \subseteq \text{st-core}(x) \) for all \( x \in m \). Then \( q_\sigma(Ax) \leq \beta(x) \) for all \( x \in m \) where \( \beta(x) = \text{st-lim sup } x \). Hence, since \( \beta(x) \leq L(x) \) for all \( x \in m \) (see [7]), the \( \sigma \)-uniform positivity of \( A \) follows from Theorem 2 of [11]. One can also easily see that

\[
-\beta(-x) \leq -q_\sigma(-Ax) \leq q_\sigma(Ax) \leq \beta(x),
\]
i.e.,

\[
\text{st-lim inf } x \leq -q_\sigma(-Ax) \leq q_\sigma(Ax) \leq \text{st-lim sup } x.
\]

If \( x \in S \cap m \), then \( \text{st-lim inf } x = \text{st-lim sup } x = \text{st-lim } x \) (see [7]). Thus, the last inequality implies that \( \text{st-lim } x = -q_\sigma(-Ax) = q_\sigma(Ax) = \sigma\text{-lim } Ax \), that is, \( A \in (S \cap m, V_\sigma)_{\text{reg}} \).

Conversely, assume \( A \in (S \cap m, V_\sigma)_{\text{reg}} \) and \( A \) is \( \sigma \)-uniformly positive. If \( x \in m \), then \( \beta(x) \) is finite. Let \( E \) be a subset of \( \mathbb{N} \) defined by \( E = \{ k : x_k > \beta(x) + \varepsilon \} \) for a given \( \varepsilon > 0 \). Then it is obvious that \( \delta(E) = 0 \) and \( x_k \leq \beta(x) + \varepsilon \) if \( k \notin E \).

Now, we can write

\[
t_{pn}(Ax) = \sum_{k \in E} a(p, n, k)x_k + \sum_{k \notin E} a^+(p, n, k)x_k - \sum_{k \notin E} a^-(p, n, k)x_k
\]

\[
\leq ||x|| \sum_{k \in E} |a(p, n, k)| + (\beta(x) + \varepsilon) \sum_{k \notin E} |a(p, n, k)|
\]

\[+ ||x|| \sum_{k \notin E} a^-(p, n, k).
\]

Using (1.5), (2.1) and \( \sigma \)-uniform positivity of \( A \) we have

\[
\limsup_p \sup_n t_{pn}(Ax) \leq \beta(x) + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we conclude that \( q_\sigma(Ax) \leq \beta(x) \) for all \( x \in m \), that is, \( \sigma\text{-core}(Ax) \subseteq \text{st-core}(x) \) for all \( x \in m \) and the proof is complete. \( \blacksquare \)

Now, since \( q_\sigma(Ax) = L^*(Ax) \) whenever \( \sigma(n) = n + 1 \), we have the following result:

**Theorem 2.4.** \( B\text{-core}(Ax) \subseteq \text{st-core}(x) \) for all \( x \in m \) if and only if \( A \in (S \cap m, f)_{\text{reg}} \) and

\[
\lim_p \sum_k \frac{1}{p+1} \left| \sum_{i=0}^p a_{n+i,k} \right| = 1 \quad \text{uniformly in } n.
\]

The next theorem is a slight generalization of our main theorem as well as an analogue of Theorem 2 of [1]:

**Theorem 2.5.** Let \( B \) be a normal matrix and \( A \) be any matrix. In order that whenever \( Bx \) is bounded \( Ax \) should exist and be bounded and satisfy

\[
(2.2) \quad \sigma\text{-core}(Ax) \subseteq \text{st-core}(Bx),
\]
it is necessary and sufficient that
\begin{align}
(2.3) & \quad C = (c_{nk}) = AB^{-1} \text{ exists}, \\
(2.4) & \quad C \in (S \cap m, V_\sigma)_{\text{reg}}, \\
(2.5) & \quad C \text{ is } \sigma\text{-uniformly positive}, \\
(2.6) & \quad \text{for any fixed } n, \sum_{k=0}^{N} \sum_{j=N+1}^{\infty} a_{nj}\gamma_{jk} \to 0 \text{ as } N \to \infty,
\end{align}

where \( \gamma_{jk} \) are the entries of the matrix \( B^{-1} \).

Proof. Let (2.2) hold and suppose \( A_n(x) \) exists for every \( n \) whenever \( Bx \in m \). Then by Lemma 2 of Choudhary [1] it follows that conditions (2.3) and (2.6) hold. Further by the same lemma, we obtain \( Ax = Cy \), where \( y = Bx \). Since \( Ax \in m \), we have \( Cy \in m \). Therefore (2.2) implies that
\[ \sigma\text{-core}(Cy) \subseteq \text{st-core}(y). \]

Hence using Theorem 2.3, we see that conditions (2.4) and (2.5) hold.

Conversely, let conditions (2.3)–(2.6) hold. Then obviously the assumptions of Lemma 2 of [1] are satisfied and so \( Cy \in m \). Hence \( Ax \in m \) and by Theorem 2.3, we obtain
\[ \sigma\text{-core}(Cy) \subseteq \text{st-core}(y), \]

and consequently
\[ \sigma\text{-core}(Ax) \subseteq \text{st-core}(Bx), \]

since \( y = Bx \) and \( Cy = Ax \). This completes the proof. \( \blacksquare \)

Finally, from Theorem 2.5 we have the following result:

**Theorem 2.6.** Let \( B \) be a normal matrix and \( A \) be any matrix. In order that whenever \( Bx \) is bounded \( Ax \) should exist and be bounded and satisfy
\begin{align}
(2.7) & \quad B\text{-core}(Ax) \subseteq \text{st-core}(Bx),
(2.8) & \quad C \in (S \cap m, f)_{\text{reg}},
(2.9) & \quad \lim sup_{p} \sup_{n} \frac{1}{p+1} \left| \sum_{i=0}^{p} c_{n+i,k} \right| = 1.
\end{align}

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References


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