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On the statistical and σ-cores

by

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Abstract. In [11] and [7], the concepts of σ-core and statistical core of a bounded number sequence x have been introduced and also some inequalities which are analogues of Knopp's core theorem have been proved. In this paper, we characterize the matrices of the class $(S\cap m, V_\sigma)_{\mathrm{reg}}$ and determine necessary and sufficient conditions for a matrix A to satisfy σ-core$(Ax) \subseteq$ st-core(x) for all $x \in m$.

1. Introduction. Let K be a subset of $\mathbb{N}$, the set of positive integers. The *natural density* δ of K is defined by

$$\delta(K) = \lim_n \frac{1}{n}\, |\{k \le n : k \in K\}|,$$

where the vertical bars indicate the number of elements in the enclosed set. The number sequence $x = (x_k)$ is said to be *statistically convergent* to the number l if for every ε, $\delta\{k : |x_k - l| \ge \varepsilon\} = 0$ (see [7]). In this case, we write st-$\lim x = l$. We shall also write S and S_0 to denote the sets of all statistically convergent sequences and of all sequences statistically convergent to zero. The statistically convergent sequences were studied by several authors (see [2], [7] and others).

Let m and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual supremum norm. Let σ be a one-to-one mapping from $\mathbb{N}$ into itself. An element $\Phi \in m'$, the conjugate space of m, is called an *invariant mean* or a σ-*mean* if (i) $\Phi(x) \ge 0$ when the sequence $x = (x_k)$ has $x_k \ge 0$ for all k, (ii) $\Phi(e) = 1$, where $e = (1,1,1,\ldots)$, (iii) $\Phi((x_{\sigma(k)})) = \Phi(x)$ for all $x \in m$.

Throughout this paper we consider the mapping σ such that $\sigma^p(k) \ne k$ for all positive integers $k \ge 0$ and $p \ge 1$, where $\sigma^p(k)$ is the pth iterate of σ at k. Thus, a σ-mean extends the limit functional on c in the sense that $\Phi(x) = \lim x$ for all $x \in c$ (see [12]). Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-means are equal.

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In case $\sigma(k) = k+1$, a σ-mean is often called a *Banach limit* and V_σ is the set of almost convergent sequences, introduced by Lorentz [9]. If $x = (x_n)$, write $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown [15] that

$$V_\sigma = \{x \in m : \lim_p t_{pn}(x) = s \text{ uniformly in } n,\ s = \sigma\text{-}\lim x\}$$

where

$$t_{pn}(x) = (x_n + Tx_n + \ldots + T^p x_n)/(p+1), \qquad t_{-1,n}(x) = 0.$$

We say that a bounded sequence $x = (x_k)$ is *σ-convergent* if $x \in V_\sigma$. By Z, we denote the set of σ-convergent sequences with σ-limit zero. It is well known [14] that $x \in m$ if and only if $Tx - x \in Z$.

Let A be an infinite matrix of real entries a_{nk} and $x = (x_k)$ be a real number sequence. Then $Ax = ((Ax)_n) = \left(\sum_k a_{nk} x_k\right)$ denotes the transformed sequence of x. If X and Y are two non-empty sequence spaces, then we use (X, Y) to denote the set of all matrices A such that Ax exists and $Ax \in Y$ for all $x \in X$. Throughout, $\sum_k$ will denote summation from $k = 1$ to ∞.

A matrix A is called (i) *regular* if $A \in (c, c)$ and $\lim Ax = \lim x$ for all $x \in c$, (ii) *σ-regular* if $A \in (c, V_\sigma)$ and σ-$\lim Ax = \lim x$ for all $x \in c$, and (iii) *σ-coercive* if $A \in (m, V_\sigma)$. The regularity conditions for A are well known [10].

The following two lemmas which were established in [15] will enable us to prove our results:

Lemma 1.1 ([15, Th. 3]). *The matrix A is σ-coercive if and only if*

$$\|A\| = \sup_n \sum_k |a_{nk}| < \infty, \tag{1.1}$$

$$\sigma\text{-}\lim a_{nk} = \alpha_k \quad \textit{for each } k, \tag{1.2}$$

$$\lim_p \sum_k \frac{1}{p+1} \Big| \sum_{i=0}^{p} (a_{\sigma^i(n),k} - \alpha_k) \Big| = 0 \quad \textit{uniformly in } n. \tag{1.3}$$

Lemma 1.2 ([15, Th. 2]). *The matrix A is σ-regular if and only if the conditions* (1.1) *and* (1.2) *hold with $\alpha_k = 0$ for each k and*

$$\sigma\text{-}\lim \sum_k a_{nk} = 1. \tag{1.4}$$

A matrix A is called *normal* if $a_{nk} = 0$ $(k > n)$ and $a_{nn} \neq 0$ for all n. If A is normal, then it has its reciprocal.

For any real number λ we write $\lambda^- = \max\{-\lambda, 0\}$, $\lambda^+ = \max\{0, \lambda\}$. Then $\lambda = \lambda^+ - \lambda^-$. We recall (see [11]) that a matrix A is said to be *σ-uniformly positive* if

$$\lim_p \sum_k a^-(p, n, k) = 0 \quad \text{uniformly in } n$$

where

$$a(p,n,k) = \frac{1}{p+1}\sum_{i=0}^{p} a_{\sigma^i(n)}.$$

It is known [11] that a σ-regular matrix A is σ-uniformly positive if and only if

$$\lim_p \sum_k |a(p,n,k)| = 1 \quad \text{uniformly in } n. \tag{1.5}$$

Let us consider the following functionals defined on m:

$$l(x) = \liminf x, \qquad L(x) = \limsup x, \qquad q_\sigma(x) = \limsup_p \sup_n t_{pn}(x),$$

$$L^*(x) = \limsup_p \sup_n \frac{1}{p+1}\sum_{i=0}^{p} x_{n+i}.$$

In [11], the *σ-core* of a real bounded number sequence x has been defined as the closed interval $[-q_\sigma(-x), q_\sigma(x)]$ and also the inequalities $q_\sigma(Ax) \le L(x)$ (σ-core of $Ax \subseteq K$-core of x), $q_\sigma(Ax) \le q_\sigma(x)$ (σ-core of $Ax \subseteq \sigma$-core of x), for all $x \in m$, have been studied. Here the *K-core* of x (or *Knopp core* of x) is the interval $[l(x), L(x)]$ (see [3]).

When $\sigma(n) = n+1$, since $q_\sigma(x) = L^*(x)$, the σ-core of x is reduced to the Banach core of x (B-core) defined by the interval $[-L^*(-x), L^*(x)]$ (see [13]).

The concepts of B-core and σ-core have been studied by many authors [4, 5, 6, 11, 13].

Recently, Fridy and Orhan [7] have introduced the notions of statistical boundedness, statistical limit superior (st-lim sup) and inferior (st-lim inf), defined the *statistical core* (or briefly st-core) of a statistically bounded sequence as the closed interval [st-lim inf x, st-lim sup x] and also determined necessary and sufficient conditions for a matrix A to yield K-core(Ax) $\subseteq$ st-core(x) for all $x \in m$.

After all these explanations, one can naturally ask: What are necessary and sufficient conditions on a matrix A so that the σ-core of Ax is contained in the st-core of x for all $x \in m$? Our main purpose is to find an answer to that question. To do this we need to characterize the class of matrices A such that $Ax \in V_\sigma$ and σ-lim Ax = st-lim x for all $x \in S \cap m$, i.e., $A \in (S \cap m, V_\sigma)_{\text{reg}}$.

2. Main results

THEOREM 2.1. *$A \in (S \cap m, V_\sigma)_{\text{reg}}$ if and only if A is σ-regular and*

$$\lim_p \sum_{k \in E} |a(p,n,k)| = 0 \quad \textit{uniformly in } n, \tag{2.1}$$

for every $E \subseteq \mathbb{N}$ with natural density zero.

Proof. First, suppose that $A \in (S \cap m, V_\sigma)_{\text{reg}}$. The σ-regularity of A immediately follows from the fact that $c \subset S \cap m$. Now, define a sequence $z = (z_k)$ for $x \in m$ as

$$z_k = \begin{cases} x_k, & k \in E, \\ 0, & k \notin E, \end{cases}$$

where E is any subset of $\mathbb{N}$ with $\delta(E) = 0$. By our assumption, since $z \in S_0$, we have $Az \in Z$. On the other hand, since $Az = \sum_{k \in E} a_{nk} x_k$, the matrix $B = (b_{nk})$ defined by

$$b_{nk} = \begin{cases} a_{nk}, & k \in E, \\ 0, & k \notin E, \end{cases}$$

for all n, must belong to the class (m, Z). Hence, the necessity of (2.1) follows from Lemma 1.1.

Conversely, suppose that A is σ-regular and (2.1) holds. Let x be any sequence in $S \cap m$ with $\text{st-lim}\, x = l$. Write $E = \{k : |x_k - l| \geq \varepsilon\}$ for any given $\varepsilon > 0$, so that $\delta(E) = 0$. Now, from (1.4) we have

$$\begin{aligned} \sigma\text{-lim}(Ax) &= \sigma\text{-lim}\Big(\sum_k a_{nk}(x_k - l) + l \sum_k a_{nk}\Big) \\ &= \sigma\text{-lim} \sum_k a_{nk}(x_k - l) + l \\ &= \lim_p \sum_k a(p, n, k)(x_k - l) + l. \end{aligned}$$

On the other hand, since

$$\Big|\sum_k a(p, n, k)(x_k - l)\Big| \leq \|x\| \sum_{k \in E} |a(p, n, k)| + \varepsilon \|A\|,$$

the condition (2.1) implies that

$$\lim_p \sum_k a(p, n, k)(x_k - l) = 0 \quad \text{uniformly in } n.$$

Hence, $\sigma\text{-lim}(Ax) = \text{st-lim}\, x$; that is, $A \in (S \cap m, V_\sigma)_{\text{reg}}$, which completes the proof. ∎

In the special case $\sigma(n) = n + 1$, we also have the following theorem:

THEOREM 2.2. *$A \in (S \cap m, f)_{\text{reg}}$ if and only if A is almost regular* (see [8]) *and*

$$\lim_p \sum_{k \in E} \frac{1}{p+1} \Big|\sum_{i=0}^{p} a_{n+i,k}\Big| = 0 \quad \textit{uniformly in } n,$$

for every $E \subseteq \mathbb{N}$ with natural density zero.

THEOREM 2.3. *σ-core$(Ax) \subseteq$ st-core(x) for all $x \in m$ if and only if $A \in (S \cap m, V_\sigma)_{\text{reg}}$ and A is σ-uniformly positive.*

Proof. Assume that σ-core$(Ax) \subseteq$ st-core(x) for all $x \in m$. Then $q_\sigma(Ax) \leq \beta(x)$ for all $x \in m$ where $\beta(x) = \text{st-}\limsup x$. Hence, since $\beta(x) \leq L(x)$ for all $x \in m$ (see [7]), the σ-uniform positivity of A follows from Theorem 2 of [11]. One can also easily see that

$$-\beta(-x) \leq -q_\sigma(-Ax) \leq q_\sigma(Ax) \leq \beta(x),$$

i.e.,

$$\text{st-}\liminf x \leq -q_\sigma(-Ax) \leq q_\sigma(Ax) \leq \text{st-}\limsup x.$$

If $x \in S \cap m$, then $\text{st-}\liminf x = \text{st-}\limsup x = \text{st-}\lim x$ (see [7]). Thus, the last inequality implies that $\text{st-}\lim x = -q_\sigma(-Ax) = q_\sigma(Ax) = \sigma\text{-}\lim Ax$, that is, $A \in (S \cap m, V_\sigma)_{\text{reg}}$.

Conversely, assume $A \in (S \cap m, V_\sigma)_{\text{reg}}$ and A is σ-uniformly positive. If $x \in m$, then $\beta(x)$ is finite. Let E be a subset of $\mathbb{N}$ defined by $E = \{k : x_k > \beta(x) + \varepsilon\}$ for a given $\varepsilon > 0$. Then it is obvious that $\delta(E) = 0$ and $x_k \leq \beta(x) + \varepsilon$ if $k \notin E$.

Now, we can write

$$\begin{aligned} t_{pn}(Ax) &= \sum_{k \in E} a(p,n,k)x_k + \sum_{k \notin E} a^+(p,n,k)x_k - \sum_{k \notin E} a^-(p,n,k)x_k \\ &\leq \|x\| \sum_{k \in E} |a(p,n,k)| + (\beta(x) + \varepsilon) \sum_{k \notin E} |a(p,n,k)| \\ &\quad + \|x\| \sum_{k \notin E} a^-(p,n,k). \end{aligned}$$

Using (1.5), (2.1) and σ-uniform positivity of A we have

$$\limsup_p \sup_n t_{pn}(Ax) \leq \beta(x) + \varepsilon.$$

Since ε is arbitrary, we conclude that $q_\sigma(Ax) \leq \beta(x)$ for all $x \in m$, that is, σ-core$(Ax) \subseteq$ st-core(x) for all $x \in m$ and the proof is complete. ∎

Now, since $q_\sigma(Ax) = L^*(Ax)$ whenever $\sigma(n) = n + 1$, we have the following result:

THEOREM 2.4. B-core$(Ax) \subseteq$ st-core(x) *for all* $x \in m$ *if and only if* $A \in (S \cap m, f)_{\text{reg}}$ *and*

$$\lim_p \sum_k \frac{1}{p+1} \Big| \sum_{i=0}^{p} a_{n+i,k} \Big| = 1 \quad \textit{uniformly in } n.$$

The next theorem is a slight generalization of our main theorem as well as an analogue of Theorem 2 of [1]:

THEOREM 2.5. *Let B be a normal matrix and A be any matrix. In order that whenever Bx is bounded Ax should exist and be bounded and satisfy*

$$\sigma\text{-core}(Ax) \subseteq \text{st-core}(Bx), \tag{2.2}$$

it is necessary and sufficient that

$$C = (c_{nk}) = AB^{-1} \text{ exists,} \tag{2.3}$$

$$C \in (S \cap m, V_\sigma)_{\mathrm{reg}}, \tag{2.4}$$

$$C \text{ is } \sigma\text{-uniformly positive,} \tag{2.5}$$

$$\text{for any fixed } n, \quad \sum_{k=0}^{N} \Big| \sum_{j=N+1}^{\infty} a_{nj}\gamma_{jk} \Big| \to 0 \quad \text{as } N \to \infty, \tag{2.6}$$

where γ_{jk} *are the entries of the matrix* B^{-1}.

Proof. Let (2.2) hold and suppose $A_n(x)$ exists for every n whenever $Bx \in m$. Then by Lemma 2 of Choudhary [1] it follows that conditions (2.3) and (2.6) hold. Further by the same lemma, we obtain $Ax = Cy$, where $y = Bx$. Since $Ax \in m$, we have $Cy \in m$. Therefore (2.2) implies that

$$\sigma\text{-core}(Cy) \subseteq \text{st-core}(y).$$

Hence using Theorem 2.3, we see that conditions (2.4) and (2.5) hold.

Conversely, let conditions (2.3)–(2.6) hold. Then obviously the assumptions of Lemma 2 of [1] are satisfied and so $Cy \in m$. Hence $Ax \in m$ and by Theorem 2.3, we obtain

$$\sigma\text{-core}(Cy) \subseteq \text{st-core}(y),$$

and consequently

$$\sigma\text{-core}(Ax) \subseteq \text{st-core}(Bx),$$

since $y = Bx$ and $Cy = Ax$. This completes the proof. ∎

Finally, from Theorem 2.5 we have the following result:

THEOREM 2.6. *Let* B *be a normal matrix and* A *be any matrix. In order that whenever* Bx *is bounded* Ax *should exist and be bounded and satisfy*

$$B\text{-core}(Ax) \subseteq \text{st-core}(Bx), \tag{2.7}$$

it is necessary and sufficient that (2.3) *and* (2.6) *hold and*

$$C \in (S \cap m, f)_{\mathrm{reg}}, \tag{2.8}$$

$$\limsup_{p} \sup_{n} \sum_{k} \frac{1}{p+1} \Big| \sum_{i=0}^{p} c_{n+i,k} \Big| = 1. \tag{2.9}$$

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