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Littlewood–Paley–Stein functions on complete Riemannian manifolds for $1 \le p \le 2$

by

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Abstract. We study the weak type $(1,1)$ and the L^p-boundedness, $1 < p \le 2$, of the so-called vertical (i.e. involving space derivatives) Littlewood–Paley–Stein functions $\mathcal{G}$ and $\mathcal{H}$ respectively associated with the Poisson semigroup and the heat semigroup on a complete Riemannian manifold M. Without any assumption on M, we observe that $\mathcal{G}$ and $\mathcal{H}$ are bounded in L^p, $1 < p \le 2$. We also consider modified Littlewood–Paley–Stein functions that take into account the positivity of the bottom of the spectrum. Assuming that M satisfies the doubling volume property and an optimal on-diagonal heat kernel estimate, we prove that $\mathcal{G}$ and $\mathcal{H}$ (as well as the corresponding horizontal functions, i.e. involving time derivatives) are of weak type $(1,1)$. Finally, we apply our methods to divergence form operators on arbitrary domains of $\mathbb{R}^n$.

1. Introduction

1.1. *Background.* It is well known (cf. e.g. [37]) that the horizontal Littlewood–Paley g-function defined, for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, by

$$g(f)(x) = \left[\int_0^\infty t \left| \frac{\partial}{\partial t} e^{-t\sqrt{\Delta}} f(x) \right|^2 dt \right]^{1/2}$$

and the vertical Littlewood–Paley $\mathcal{G}$-function defined by

$$\mathcal{G}(f)(x) = \left[\int_0^\infty t |\nabla e^{-t\sqrt{\Delta}} f(x)|^2 \, dt \right]^{1/2}$$

are bounded in $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, i.e., for such p, there exists C_p such that

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$$\|g(f)\|_p + \|\mathcal{G}(f)\|_p \le C_p \|f\|_p, \quad \forall f \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

For $p = 1$, the operators g and $\mathcal{G}$ are of weak type $(1,1)$. In classical harmonic analysis, the Littlewood–Paley functions play an important role in the study of non-tangential convergence of Fatou type and the boundedness of Riesz transforms and multipliers (cf. [37]–[39]).

In [38], Stein extended the L^p-boundedness for $1 < p < \infty$ of the Littlewood–Paley $\mathcal{G}$-function to the context of compact Lie groups for $1 < p < \infty$, and the L^p-boundedness for $1 < p < \infty$ of the Littlewood–Paley g-function to a general setting of symmetric Markov semigroups. For the latter aspect, see also [31] and the references therein.

The above facts have been subsequently generalised further. One direction is the Littlewood–Paley theory on Coifman–Weiss's spaces of homogeneous type; for this, we refer to Han and Sawyer [26]. Another direction is the study of the Littlewood–Paley functions on (non-compact) complete Riemannian manifolds, in connection with the study of Riesz transforms: some results have been obtained by N. Lohoué ([29], [30]) for Cartan–Hadamard manifolds and non-amenable Lie groups, and by Chen Jie-Cheng [11] for Riemannian manifolds with non-negative Ricci curvature. Note that the work of Bakry ([7]–[9]) on Riesz transforms on manifolds whose Ricci curvature is non-negative or bounded below relies on a Littlewood–Paley theory of a slightly different kind. Let us also mention the related works [36], [41], [35].

The first two authors of the present article have proved in [14] that the Riesz transform is of weak type $(1,1)$ and bounded in L^p for $1 < p \le 2$ on any complete Riemannian manifold satisfying the doubling volume property and an on-diagonal optimal heat kernel estimate. One of the aims of this paper is to study the weak type $(1,1)$ of the Littlewood–Paley–Stein functions under the same assumptions. The L^p-boundedness of these functions for $1 < p \le 2$ can be treated directly, and without assumptions, via a classical argument due to Stein. These estimates can be improved substantially, as in [29], in the case where the bottom of the spectrum of the manifold is positive. In the case $p = 1$, we have to use, as in [14], the recent singular integral theory developed by Duong and McIntosh in [20] and Grigor'yan's weighted estimates of the space derivatives of the heat kernel on complete Riemannian manifolds ([24]). Finally, we apply our methods to the case of second order operators in divergence form on arbitrary domains of $\mathbb{R}^n$.

As was already observed in [14] for Riesz transforms, the case $p > 2$ is of a completely different nature and requires much stronger assumptions (see [16], [17], [28]).

1.2. *Notation, definitions.* Let M be a complete non-compact Riemannian manifold, d be the geodesic distance on M, and μ be the Riemannian measure. Denote by $B(x,r)$ the geodesic ball of center $x \in M$ and radius

$r > 0$ and by $V(x,r)$ its Riemannian volume $\mu(B(x,r))$. One says that (M,μ) has the *doubling volume property* if there exists $C > 0$ such that

$$V(x,2r) \leq CV(x,r), \qquad \forall x \in M,\ r > 0.$$

Let Δ be the (non-negative) Laplace–Beltrami operator on M, $e^{-t\Delta}$ be the heat semigroup, and $p_t(x,y)$ be the heat kernel on M. Let $e^{-t\sqrt{\Delta}}$ be the Poisson semigroup on M.

For $f \in C_0^\infty(M)$, define the (so-called *vertical*) *Littlewood–Paley–Stein $\mathcal{G}$-function* and *$\mathcal{H}$-function* by

$$\mathcal{G}(f)(x) = \Big(\int_0^\infty t|\nabla e^{-t\sqrt{\Delta}} f(x)|^2\,dt\Big)^{1/2},$$
$$\mathcal{H}(f)(x) = \Big(\int_0^\infty |\nabla e^{-t\Delta} f(x)|^2\,dt\Big)^{1/2},$$

as well as the (so-called *horizontal*) *Littlewood–Paley–Stein g-function* and *h-function* by

$$g(f)(x) = \Big\{\int_0^\infty t|\sqrt{\Delta} e^{-t\sqrt{\Delta}} f(x)|^2\,dt\Big\}^{1/2} = \Big\{\int_0^\infty t\Big|\frac{\partial}{\partial t} e^{-t\sqrt{\Delta}} f(x)\Big|^2\,dt\Big\}^{1/2},$$
$$h(f)(x) = \Big\{\int_0^\infty t|\Delta e^{-t\Delta} f(x)|^2\,dt\Big\}^{1/2} = \Big\{\int_0^\infty t\Big|\frac{\partial}{\partial t} e^{-t\Delta} f(x)\Big|^2\,dt\Big\}^{1/2}.$$

Let $\mathcal{R}$ be a sublinear operator, defined on $\mathcal{C}_0^\infty(M)$, with values in measurable functions on M. We shall say that $\mathcal{R}$ is *bounded* on $L^p(M,\mu)$, for some $p \in [1,\infty]$, if there exists C_p such that

$$\|\mathcal{R}(f)\|_p \leq C_p\|f\|_p, \qquad \forall f \in \mathcal{C}_0^\infty(M),$$

and that it is of *weak type* $(1,1)$ if there exists C such that

$$\mu\{x \in M;\ |\mathcal{R}(f)(x)| \geq \lambda\} \leq C\|f\|_1/\lambda$$

for every $f \in \mathcal{C}_0^\infty(M)$ and $\lambda > 0$.

1.3. *Some basic facts and remarks.* (i) As we already mentioned, g and h are always bounded on $L^p(M)$, $1 < p < \infty$; this even holds in a general symmetric Markov semigroup setting (see [38], [31]).

(ii) The function $\mathcal{G}$ is pointwise dominated by $\mathcal{H}$. Indeed, recall the subordination formula

$$e^{-t\sqrt{\Delta}} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\frac{t^2}{4u}\Delta} e^{-u} u^{-1/2}\,du. \tag{1.1}$$

One can write

$$\begin{aligned}\mathcal{G}^2(f)(x) &= \int_0^\infty t|\nabla e^{-t\sqrt{\Delta}} f(x)|^2\,dt \\ &\le \frac{1}{\pi}\int_0^\infty t\Big(\int_0^\infty |\nabla e^{-\frac{t^2}{4u}\Delta} f(x)| e^{-u}u^{-1/2}\,du\Big)^2\,dt,\end{aligned}$$

and, since

$$\int_0^\infty e^{-u}u^{-1/2}\,du < \infty,$$

we have

$$\begin{aligned}\mathcal{G}^2(f)(x) &\le C\int_0^\infty t\Big(\int_0^\infty |\nabla e^{-\frac{t^2}{4u}\Delta} f(x)|^2 e^{-u}u^{-1/2}\,du\Big)\,dt \\ &= C\int_0^\infty \Big(\int_0^\infty t|\nabla e^{-\frac{t^2}{4u}\Delta} f(x)|^2\,dt\Big) e^{-u}u^{-1/2}\,du \\ &= 2C\int_0^\infty \Big(\int_0^\infty |\nabla e^{-v\Delta} f(x)|^2\,dv\Big) e^{-u}u^{1/2}\,du = C'\mathcal{H}^2(f)(x).\end{aligned}$$

(iii) The L^2-boundedness of $\mathcal{H}$ is obvious. In fact, up to a multiplicative constant, $\mathcal{H}$ is an isometry of $L^2(M)$; the same is true for $\mathcal{G}$.

PROPOSITION 1.1.

$$\|\mathcal{H}(f)\|_2 = \frac{1}{\sqrt{2}}\|f\|_2, \qquad \forall f \in L^2(M).$$

Proof. Since $\|\nabla f\|_2 = \|\Delta^{1/2} f\|_2$, we have

$$\|\mathcal{H}(f)\|_2^2 = \int_0^\infty \|\nabla e^{-t\Delta} f\|_2^2\,dt = \int_0^\infty \|\Delta^{1/2}e^{-t\Delta} f\|_2^2\,dt.$$

On the other hand,

$$\begin{aligned}\int_0^\infty \|\Delta^{1/2}e^{-t\Delta} f\|_2^2\,dt &= \int_0^\infty \langle \Delta e^{-t\Delta} f, e^{-t\Delta} f\rangle\,dt = -\int_0^\infty \Big\langle \frac{\partial}{\partial t}e^{-t\Delta} f, e^{-t\Delta} f\Big\rangle\,dt \\ &= -\frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}\langle e^{-t\Delta} f, e^{-t\Delta} f\rangle\,dt = -\frac{1}{2}\|e^{-t\Delta} f\|_2^2\Big|_0^\infty = \frac{1}{2}\|f\|_2^2.\end{aligned}$$

This finishes the proof.

Thanks to the Marcinkiewicz interpolation theorem, it follows from Proposition 1.1 that the weak type $(1,1)$ of $\mathcal{G}$ or $\mathcal{H}$ implies the boundedness of the same operator on $L^p(M)$, $1 < p < 2$.

(iv) Let $p \in]1,\infty[$ and q its conjugate exponent. Then it is easy to see by duality that
$$\|\mathcal{H}(f)\|_p \le C\|f\|_p, \quad \forall f \in L^p(M),$$
implies
$$\|f\|_q \le C\|\mathcal{H}(f)\|_q, \quad \forall f \in L^q(M).$$
The same phenomenon holds for $\mathcal{G}$.

1.4. *Statement of the results on manifolds*

THEOREM 1.2. *Let M be a complete Riemannian manifold. Then $\mathcal{H}$ and $\mathcal{G}$ are bounded on $L^p(M)$ for every $1 < p \le 2$.*

One can make Theorem 1.2 more precise when the bottom of the spectrum of M is positive. Following [29], define, for $a \in \mathbb{R}$,
$$\mathcal{H}_a(f)(x) = \Big\{ \int_0^\infty e^{at} |\nabla e^{-t\Delta} f(x)|^2 \, dt \Big\}^{1/2}.$$

THEOREM 1.3. *Assume that the bottom of the spectrum of M is positive, i.e.*
$$\lambda_1 = \lambda_1(M) = \inf_{f \in \mathcal{C}_0^\infty(M)} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} > 0.$$
Then for every $1 < p \le 2$ and every $a < 2\lambda_1(p-1)$, $\mathcal{H}_a$ is bounded on $L^p(M)$; more precisely there exists $C_{p,\lambda_1,a}$ only depending on p, λ_1 and a such that
$$\|\mathcal{H}_a(f)\|_p \le C_{p,\lambda_1,a}\|f\|_p, \quad \forall f \in L^p(M).$$

THEOREM 1.4. *Let M be a complete Riemannian manifold satisfying the doubling volume property and such that*
$$p_t(x,x) \le \frac{C}{V(x,\sqrt{t})} \tag{1.2}$$
for some $C > 0$ and all $x \in M$, $t > 0$. Then $\mathcal{H}$, $\mathcal{G}$, h and g are of weak type $(1,1)$.

REMARK. The above two assumptions on M, namely the doubling volume property and the on-diagonal heat kernel upper estimate, are known to be together equivalent to a more geometric condition, the so-called relative Faber–Krahn inequality (see [23, Prop. 5.2]).

1.5. *Examples, comparison with known results.* We have not found Theorem 1.2 in the literature, but it is probably known, and implicit for instance in [29], that Stein's argument works on manifolds, at least for $\mathcal{G}$ (see also [8], where a version of this argument appears). Note that Stein's argument can also be used in an infinite-dimensional context (see [40], [13]).

If one replaces $\mathcal{G}$ by $\widetilde{\mathcal{G}}$, where

$$\widetilde{\mathcal{G}}(f)(x) = \Big\{ \int_0^\infty (e^{-t\sqrt{\Delta}}|\nabla e^{-t\sqrt{\Delta}} f|(x))^2 t\, dt \Big\}^{1/2},$$

then the L^p-boundedness of $\widetilde{\mathcal{G}}$, for $1 < p < \infty$, follows from probabilistic Littlewood–Paley theory (see [32], [33]; a slightly stronger inequality is even true when $p > 2$). Lucien Chevalier ([12]) explained to us that in the case $1 < p < 2$ this also follows from Stein's argument.

Theorem 1.3 should be considered as a partial generalisation of the results in [29] for Cartan–Hadamard manifolds with a positive bottom of the spectrum and of the corresponding ones in [30] for non-amenable Lie groups endowed with a family of Hörmander vector fields, since our method also works in this setting. Our assumptions are much weaker, and our proof much simpler. In [29], [30], the function under consideration is rather

$$\mathcal{G}_a(f)(x) = \Big\{ \int_0^\infty te^{at}|\nabla e^{-t\sqrt{\Delta}} f(x)|^2\, dt \Big\}^{1/2}.$$

Stein's argument also works for this function, and yields a statement analogous to Theorem 1.3, with the parameter range $a < 2\sqrt{\lambda_1}(p-1)$, which is slightly worse than the range in [29], [30]: $a < 4\sqrt{\lambda_1}(p-1)/p$. On the other hand, recall that, in their more restrictive settings, [29], [30] also yield some results for $p > 2$.

Theorem 1.4 is our main result. It contains the case of manifolds with non-negative Ricci curvature, which was treated in [11] (see also [8] for related results), but it covers a much larger class of manifolds: e.g. manifolds with doubling volume property and suitable Poincaré inequalities (including Lie groups with polynomial volume growth and manifolds that are roughly isometric to a manifold with non-negative Ricci curvature), cocompact covering manifolds with polynomial volume growth. For details we refer to [14]. On the other hand, recall that, in his particular case, [11] is also able to treat the case $p > 2$.

If one assumes in addition a pointwise upper bound for the gradient of the heat kernel, then the conclusion of Theorem 1.4 follows from [3]. If one assumes Poincaré inequalities, one can obtain an H^p theory, $0 < p < 1$ (see [34]). Under even stronger assumptions, one can also treat the case where $2 < p < \infty$ ([16], [17], [28]).

An analogue of Theorem 1.4 should hold when instead of (1.2), M satisfies so-called subgaussian estimates (see [10]).

Theorems 1.2–1.4 could be formulated in an abstract diffusion semigroup setting (see [14, p. 1153]).

2. The case $1 < p < 2$. In this section, we prove Theorems 1.2 and 1.3.

2.1. *Boundedness of LPS functions.* In this section we will follow Stein's approach in [38] to prove Theorem 1.2. We recall the Hardy–Littlewood maximal inequality for semigroups ([38, p. 73]).

LEMMA 2.1. *Let T_t be a symmetric submarkovian semigroup on some measure space (X,μ). For a suitable function f on X, define $f^*(x) = \sup_{t>0}|T_tf(x)|$, $x \in X$. Then for every $1 < p \leq \infty$, there exists C_p such that*
$$\|f^*\|_p \leq C_p\|f\|_p, \quad \forall f \in L^p(X,\mu).$$

Proof of Theorem 1.2. According to Section 1.3(ii), we need only consider $\mathcal{H}$. Let $f \in \mathcal{C}_0^\infty(M)$, and set $u(x,t) = e^{-t\Delta}f(x)$. One can assume that f is non-negative and not identically zero, and then by standard estimates u is smooth and positive everywhere. For any $1 < p \leq 2$, we have
$$\begin{aligned}\left(\frac{\partial}{\partial t}+\Delta\right)u^p(x,t) &= p\,u^{p-1}(x,t)\left(\frac{\partial}{\partial t}+\Delta\right)u(x,t)\\ &\qquad - p(p-1)u^{p-2}(x,t)|\nabla u(x,t)|^2\\ &= -p(p-1)u^{p-2}(x,t)|\nabla u(x,t)|^2,\end{aligned}$$
which yields
$$|\nabla u(x,t)|^2 = -\frac{1}{p(p-1)}\,u^{2-p}(x,t)\left(\frac{\partial}{\partial t}+\Delta\right)u^p(x,t),$$
therefore
$$\begin{aligned}\mathcal{H}^2(f)(x) &= \int_0^\infty |\nabla u(x,t)|^2\,dt = -C_p\int_0^\infty u^{2-p}(x,t)\left(\frac{\partial}{\partial t}+\Delta\right)u^p(x,t)\,dt\\ &\leq C_p\sup_{t>0}u^{2-p}(x,t)J(x),\end{aligned}$$
where
$$J(x) = -\int_0^\infty\left(\frac{\partial}{\partial t}+\Delta\right)u^p(x,t)\,dt.$$
One may write, applying the Hölder inequality with exponents $2/(2-p)$ and $2/p$,
$$\begin{aligned}\int_M \mathcal{H}^p(f)(x)\,d\mu(x) &\leq C_p\int_M\sup_{t>0}u^{(2-p)p/2}(x,t)J^{p/2}(x)\,d\mu(x)\\ &\leq C_p\Big[\int_M\sup_{t>0}u^p(x,t)\,d\mu(x)\Big]^{(2-p)/2}\Big[\int_M J(x)\,d\mu(x)\Big]^{p/2}\\ &\leq C_p'\|f\|_p^{(2-p)p/2}\Big[\int_M J(x)\,d\mu(x)\Big]^{p/2},\end{aligned}$$
where in the last step we have used Lemma 2.1.

On the other hand,

$$\begin{aligned}\int_M J(x)\,d\mu(x) &= -\int_M \int_0^\infty \left(\frac{\partial}{\partial t}+\Delta\right) u^p(x,t)\,dt\,d\mu(x)\\ &= -\int_M u^p(x,t)|_0^\infty\,d\mu(x) + \int_0^\infty \int_M \Delta u^p(x,t)\,d\mu(x)\,dt\\ &= \int_M u^p(x,0)\,d\mu(x) = \|f\|_p^p,\end{aligned}$$

since $\int_M \Delta u^p(x,t)\,d\mu(x)=0$. Hence

$$\|\mathcal{H}(f)\|_p \le C_p' \|f\|_p^{(2-p)/2}\|f\|_p^{p/2} \le C_p'\|f\|_p.$$

This finishes the proof.

2.2. *Manifolds with a positive* λ_1. This section is inspired by [29], where N. Lohoué introduced a new class of Littlewood–Paley functions on Cartan–Hadamard manifolds (i.e., complete and simply connected Riemannian manifolds with non-positive curvature) that took into account the positivity of the bottom of the spectrum.

Let $\lambda_1>0$ be the bottom of the spectrum of Δ on $L^2(M)$, i.e., the norm of $e^{-t\Delta}$ on $L^2(M)$ is $e^{-\lambda_1 t}$. Recall the definition

$$\mathcal{H}_a(f)(x) = \Big\{\int_0^\infty e^{at}|\nabla e^{-t\Delta}f(x)|^2\,dt\Big\}^{1/2}.$$

The case $p=2$ in Theorem 1.3 is nothing but a variation on Section 1.3(iii).

Proposition 2.2. *For any* $a<2\lambda_1$, *there exists* C *such that*

$$\|\mathcal{H}_a(f)\|_2 \le C\|f\|_2, \qquad \forall f\in L^2(M).$$

Proof. Since $\|\nabla f\|_2 = \|\Delta^{1/2}f\|_2$,

$$\begin{aligned}\|\mathcal{H}_a(f)\|_2^2 &= \int_0^\infty e^{at}\|\nabla e^{-t\Delta}f\|_2^2\,dt\\ &\le \int_0^\infty e^{at}\|e^{-t(1-\eta)\Delta}f\|_{2\to 2}^2\|\Delta^{1/2}e^{-t\eta\Delta}f\|_2^2\,dt\\ &\le \int_0^\infty e^{(a-2\lambda_1(1-\eta))t}\|\Delta^{1/2}e^{-t\eta\Delta}f\|_2^2\,dt\\ &\le C_{a,\lambda_1,\eta}\|f\|_2^2,\end{aligned}$$

as soon as $\eta\in\,]0,1[$ is chosen so that

$$a<2\lambda_1(1-\eta),$$

which is possible if $a<2\lambda_1$. The proof is complete.

Proof of Theorem 1.3. Let $f \in \mathcal{C}_0^\infty(M)$ be non-negative and not identically zero, and $u(x,t) = e^{-t\Delta} f(x)$. Then, as above,

$$|\nabla u(x,t)|^2 = -\frac{1}{p(p-1)}\, u^{2-p}(x,t)\left(\frac{\partial}{\partial t} + \Delta\right) u(x,t)^p.$$

Let

$$J_a(x) = -\int_0^\infty e^{at}\left(\frac{\partial}{\partial t} + \Delta\right) u^p(x,t)\, dt.$$

One has

$$\mathcal{H}_a^2(f)(x) \le C_p \sup_{t>0} u^{2-p}(x,t) J_a(x).$$

It follows that

$$\int_M \mathcal{H}_a^p(f)(x)\, d\mu(x) \le C_p' \|f\|_p^{(2-p)p/2} \Big[\int_M J_a(x)\, d\mu(x)\Big]^{p/2}.$$

Now write

$$\begin{aligned}
\int_M J_a(x)\, d\mu(x) &= -\int_M \int_0^\infty e^{at}\left(\frac{\partial}{\partial t} + \Delta\right) u^p(x,t)\, dt\, d\mu(x) \\
&= -\int_M \int_0^\infty e^{at} \frac{\partial}{\partial t} u^p(x,t)\, dt\, d\mu(x) \\
&= -\int_M e^{at} u^p(x,t)\Big|_0^\infty d\mu(x) + a\int_M \int_0^\infty e^{at} u^p(x,t)\, d\mu(x)\, dt \\
&\le \int_M u^p(x,0)\, d\mu(x) + a\int_M \int_0^\infty e^{at} u^p(x,t)\, d\mu(x)\, dt \\
&= \|f\|_p^p + a\int_0^\infty e^{at} \|e^{-t\Delta} f\|_p^p\, dt \\
&\le \|f\|_p^p \Big(1 + a\int_0^\infty e^{at} \|e^{-t\Delta}\|_{p\to p}^p\, dt\Big).
\end{aligned}$$

Now by definition $\|e^{-t\Delta}\|_{2\to 2} = e^{-\lambda_1 t}$. On the other hand, $e^{-t\Delta}$ is a symmetric submarkovian semigroup, therefore $\|e^{-t\Delta}\|_{1\to 1} \le 1$. By the Riesz–Thorin interpolation theorem, for any $1 < p < 2$, we have

$$\|e^{-t\Delta}\|_{p\to p} \le \|e^{-t\Delta}\|_{1\to 1}^{1-\theta} \|e^{-t\Delta}\|_{2\to 2}^{\theta},$$

where $\theta \in (0,1)$ is taken so that $1/p = \theta/2 + (1-\theta)/1$, i.e., $\theta = 2(p-1)/p$. Hence $\|e^{-t\Delta}\|_{p\to p}^p \le e^{-2\lambda_1(p-1)t}$. This implies that, if $a < 2(p-1)\lambda_1$,

$$\int_0^\infty e^{at} \|e^{-t\Delta}\|_{p\to p}^p\, dt \le \int_0^\infty e^{(a-2(p-1)\lambda_1)t}\, dt = \frac{1}{2(p-1)\lambda_1 - a}$$

is finite. The claim follows.

3. The case $p = 1$. In this section, we prove Theorem 1.4.

3.1. *A criterion of weak type* $(1,1)$. The following statement is the main technical tool in [14]. It will be instrumental in the proof of Theorem 1.4.

PROPOSITION 3.1. *Let M be a complete Riemannian manifold, with Riemannian measure μ, satisfying the doubling volume property and the heat kernel upper bound*

$$p_t(x,x) \le \frac{C}{V(x,\sqrt{t})} \tag{3.3}$$

for some $C > 0$ and all $x \in M$, $t > 0$. Let $\mathcal{R} : L^2(M) \to L^2(M)$ be a bounded sublinear operator. Assume that there exists a kernel $k_t(x,y) \ge 0$ such that

$$|\mathcal{R}[(I - e^{-t\Delta})f](x)| \le \int_M k_t(x,y)|f(y)|\,d\mu(y), \qquad \forall t > 0,\ f \in \mathcal{C}_0^\infty(M),\ \text{for a.e. } x \in M.$$

If

$$\sup_{y\in M,\, t>0} \int_{d(x,y)\ge\sqrt{t}} k_t(x,y)\,d\mu(x) < \infty, \tag{3.4}$$

then $\mathcal{R}$ is of weak type $(1,1)$.

3.2. *Weighted estimates of derivatives of the heat kernel.* In the next two sections, we shall work under the assumptions of Theorem 1.4. Our first lemma is standard; for a proof, see [14, Lemma 2.1].

LEMMA 3.2. *For all $\gamma > 0$,*

$$\int_{d(x,y)\ge\sqrt{t}} e^{-2\gamma d^2(x,y)/s}\,d\mu(x) \le C_\gamma V(y,\sqrt{s})e^{-\gamma t/s}, \qquad \forall y \in M,\ s,t > 0.$$

Our next lemma is a simple consequence of [24, Lemma 3.2] and our assumption on the heat kernel. Recall that assumption (3.3), together with the doubling volume property, self-improves to

$$p_s(x,y) \le C_\alpha \frac{e^{-\alpha d^2(x,y)/s}}{V(y,\sqrt{s})}, \qquad \forall x \in M,\ s > 0, \tag{3.5}$$

for any $\alpha \in \,]0, 1/4[$ (cf. [25, Thm. 1.1]).

LEMMA 3.3. *For $\varepsilon > 0$ small enough,*

$$\int_{d(x,y)\ge\sqrt{t}} |\nabla_x \Delta_x p_s(x,y)|^2 e^{\varepsilon d^2(x,y)/s}\,d\mu(x) \le \frac{C_\varepsilon}{s^3 V(y,\sqrt{s})}\, e^{-\varepsilon t/s}, \qquad \forall y \in M,\ s,t > 0.$$

Sketch of proof. Set

$$E_0(y,s) = \int_M |p_s(x,y)|^2 e^{2\varepsilon d^2(x,y)/s}\, d\mu(x),$$

$$E_3(y,s) = \int_M |\nabla_x \Delta_x p_s(x,y)|^2 e^{2\varepsilon d^2(x,y)/s}\, d\mu(x).$$

It follows from [24, Corollary 1.3] that, for $\varepsilon < 1/4$,

$$E_3(y,s) \le \frac{C_\varepsilon}{s^3} E_0(y,s/2). \tag{3.6}$$

Now (3.5) together with Lemma 3.2 easily implies that

$$E_0(y,s) \le \frac{C_\varepsilon}{V(y,\sqrt{s})}, \qquad \forall y \in M,\ s>0$$

(see [14, Lemma 2.2]). It then follows from (3.6) and doubling that

$$E_3(y,s) \le \frac{C_\varepsilon}{s^3 V(y,\sqrt{s})}, \qquad \forall y \in M,\ s>0.$$

Now write

$$\int_{d(x,y)\ge\sqrt{t}} |\nabla_x \Delta_x p_s(x,y)|^2 e^{\varepsilon d^2(x,y)/s}\, d\mu(x) \le E_3(y,s) e^{-\varepsilon t/s}.$$

The lemma follows.

3.3. *Weak type* $(1,1)$ *of* $\mathcal{H}$ *and* $\mathcal{G}$. One has

$$\begin{aligned}\mathcal{H}[(I-e^{-t\Delta})f](x) &= \Big(\int_0^\infty |\nabla(e^{-s\Delta} - e^{-(s+t)\Delta})f(x)|^2\, ds\Big)^{1/2}\\ &= \Big(\int_0^\infty \Big|\int_M (\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)) f(y)\, d\mu(y)\Big|^2 ds\Big)^{1/2}\\ &\le \int_M \Big(\int_0^\infty |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2\, ds\Big)^{1/2} |f(y)|\, d\mu(y).\end{aligned}$$

According to Proposition 3.1, it is enough to prove the existence of $C>0$ such that

$$\int_{d(x,y)\ge\sqrt{t}} \Big(\int_0^\infty |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2\, ds\Big)^{1/2} d\mu(x) \le C \tag{3.7}$$

for all $y \in M$, $t>0$.

Set

$$A = A(y,t) = \int_{d(x,y)\ge\sqrt{t}} \Big(\int_0^\infty |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2\, ds\Big)^{1/2} d\mu(x),$$

and, for $k \in \mathbb{N}$,

$$A_k = A_k(y,t) = \int\limits_{d(x,y)\geq\sqrt{t}} \Big(\int\limits_{kt}^{(k+1)t} |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2 \, ds\Big)^{1/2} d\mu(x).$$

For $k \geq 1$, write

$$\begin{aligned} A_k = &\int\limits_{d(x,y)\geq\sqrt{t}} \Big(\int\limits_{kt}^{(k+1)t} |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2 e^{2\varepsilon d^2(x,y)/kt} \, ds\Big)^{1/2} \\ &\times e^{-\varepsilon d^2(x,y)/kt} \, d\mu(x) \\ \leq &\Big(\int\limits_M \int\limits_{kt}^{(k+1)t} |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2 e^{2\varepsilon d^2(x,y)/kt} \, ds \, d\mu(x)\Big)^{1/2} \\ &\times \Big(\int\limits_{d(x,y)\geq\sqrt{t}} e^{-2\varepsilon d^2(x,y)/kt} \, d\mu(x)\Big)^{1/2}. \end{aligned}$$

According to Lemma 3.2, we have

$$\int\limits_{d(x,y)\geq\sqrt{t}} e^{-2\varepsilon d^2(x,y)/kt} \, d\mu(x) \leq C_\varepsilon V(y, \sqrt{kt}),$$

therefore

$$A_k \leq C\sqrt{B_k V(y,\sqrt{kt})} \tag{3.8}$$

with

$$B_k = B_k(y,t) = \int\limits_M \int\limits_{kt}^{(k+1)t} |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2 e^{2\varepsilon d^2(x,y)/kt} \, ds \, d\mu(x).$$

Since

$$\frac{\partial}{\partial u} \nabla_x p_u(x,y) = \nabla_x \frac{\partial}{\partial u} p_u(x,y) = -\nabla_x \Delta_x p_u(x,y),$$

one may write

$$\begin{aligned} B_k &= \int\limits_M \int\limits_{kt}^{(k+1)t} \Big| \int\limits_s^{s+t} \nabla_x \Delta_x p_u(x,y) \, du \Big|^2 e^{2\varepsilon d^2(x,y)/kt} \, ds \, d\mu(x) \\ &\leq \int\limits_M \int\limits_{kt}^{(k+1)t} \Big(t \int\limits_s^{s+t} |\nabla_x \Delta_x p_u(x,y)|^2 \, du\Big) e^{2\varepsilon d^2(x,y)/kt} \, ds \, d\mu(x) \\ &= t \int\limits_{kt}^{(k+1)t} \int\limits_s^{s+t} \Big(\int\limits_M |\nabla_x \Delta_x p_u(x,y)|^2 e^{2\varepsilon d^2(x,y)/kt} \, d\mu(x)\Big) \, du \, ds. \end{aligned}$$

In the above expression, $u \in [s, s+t]$, and $s \in [kt, (k+1)t]$, thus $u \in [kt, (k+2)t]$, and

$$e^{2\varepsilon d^2(x,y)/kt} \le e^{2\varepsilon(k+2)d^2(x,y)/ku} \le e^{6\varepsilon d^2(x,y)/u}.$$

Hence

$$B_k \le t \int_{kt}^{(k+1)t} \int_s^{s+t} \Big(\int_M |\nabla_x \Delta_x p_u(x,y)|^2 e^{6\varepsilon d^2(x,y)/u}\, d\mu(x)\Big)\, du\, ds.$$

Now, according to Lemma 3.3, for $\varepsilon > 0$ small enough,

$$\int_M |\nabla_x \Delta_x p_u(x,y)|^2 e^{6\varepsilon d^2(x,y)/u}\, d\mu(x) \le \frac{C}{u^3 V(y, \sqrt{u})}.$$

This yields

$$\begin{aligned} B_k &\le Ct \int_{kt}^{(k+1)t} \int_s^{s+t} \frac{du}{u^3 V(y,\sqrt{u})}\, ds \le Ct^2 \int_{kt}^{(k+1)t} \frac{ds}{s^3 V(y,\sqrt{s})} \\ &\le C' t^3 \frac{1}{(kt)^3 V(y, \sqrt{kt})}, \end{aligned}$$

therefore, by (3.8),

$$A_k \le Ck^{-3/2}, \qquad k \ge 1.$$

Let us now turn to the case $k = 0$. Write

$$\begin{aligned} A_0 \le \Big(&\int_{d(x,y)\ge\sqrt{t}} \int_0^t |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2 e^{2\varepsilon d^2(x,y)/t}\, ds\, d\mu(x)\Big)^{1/2} \\ &\times \Big(\int_{d(x,y)\ge\sqrt{t}} e^{-2\varepsilon d^2(x,y)/t}\, d\mu(x)\Big)^{1/2}. \end{aligned}$$

Lemma 3.2 yields

$$\int_{d(x,y)\ge\sqrt{t}} e^{-2\varepsilon d^2(x,y)/t}\, d\mu(x) \le C_\varepsilon V(y, \sqrt{t}),$$

therefore

$$A_0 \le C\sqrt{B_0' V(y, \sqrt{t})}, \tag{3.9}$$

with

$$B_0' = \int_{d(x,y)\ge\sqrt{t}} \int_0^t |\nabla_x p_s(x,y) - \nabla_x p_{s+t}(x,y)|^2 e^{2\varepsilon d^2(x,y)/t}\, ds\, d\mu(x).$$

Following the above argument, one obtains

$$B_0' \le t \int_0^t \int_s^{s+t} \Big(\int_{d(x,y)\ge\sqrt{t}} |\nabla_x \Delta_x p_u(x,y)|^2 e^{4\varepsilon d^2(x,y)/u}\, d\mu(x) \Big)\, du\, ds.$$

Lemma 3.3 now gives, for ε small enough,

$$\int_{d(x,y)\ge\sqrt{t}} |\nabla_x \Delta_x p_u(x,y)|^2 e^{4\varepsilon d^2(x,y)/u}\, d\mu(x) \le \frac{C}{u^3 V(y,\sqrt{u})}\, e^{-ct/u},$$

hence

$$\begin{aligned} B_0' &\le Ct \int_0^t \int_s^{s+t} \frac{e^{-ct/u}}{u^3 V(y,\sqrt{u})}\, du\, ds \\ &= \frac{C}{t^2 V(y,\sqrt{t})} \int_0^t \int_s^{s+t} \Big(\frac{t}{u}\Big)^3 \frac{V(y,\sqrt{t})}{V(y,\sqrt{u})}\, e^{-ct/u}\, du\, ds. \end{aligned}$$

Using doubling, one sees easily that the quantity

$$\Big(\frac{t}{u}\Big)^3 \frac{V(y,\sqrt{t})}{V(y,\sqrt{u})}\, e^{-ct/u}$$

is uniformly bounded from above. Thus

$$B_0' \le \frac{C'}{V(y,\sqrt{t})},$$

and it follows from (3.9) that A_0 is uniformly bounded.

Finally, since

$$A \le \sum_{k\ge 0} A_k,$$

the estimate (3.7), and therefore the weak type $(1,1)$ of $\mathcal{H}$, is proved. According to Section 1.3(ii), $\mathcal{G}$ also has weak type $(1,1)$.

3.4. *Weak type* $(1,1)$ *of* h *and* g. It is well known ([19, Thm. 4], [25, Cor. 3.3]) that (3.5), together with the doubling volume property, implies, for every $m \in \mathbb{N}^*$ and $\alpha \in\,]0,1/4[$,

$$\Big| \frac{\partial^m}{\partial s^m} p_s(x,y) \Big| = |\Delta_x^m p_s(x,y)| \le C_{m,\alpha} \frac{e^{-\alpha d^2(x,y)/s}}{s^m V(y,\sqrt{s})}, \quad \forall x,y \in M,\ s>0. \tag{3.10}$$

By applying Lemma 3.2, one then easily obtains the following.

LEMMA 3.4. *For* $m \in \mathbb{N}^*$ *and* $\varepsilon > 0$ *small enough*,

$$\int_{d(x,y)\ge\sqrt{t}} |\Delta_x^m p_s(x,y)|^2\, e^{\varepsilon d^2(x,y)/s}\, d\mu(x) \le \frac{C_{\varepsilon,m}}{s^{2m} V(y,\sqrt{s})}\, e^{-\varepsilon t/s}, \quad \forall y \in M,\ s,t>0.$$

Let us start by studying h. Write

$$h[(I-e^{-t\Delta})f](x) = \Big(\int_0^\infty s\Big|\frac{\partial}{\partial s}(e^{-s\Delta}-e^{-(s+t)\Delta})f(x)\Big|^2\,ds\Big)^{1/2}.$$

According to Proposition 3.1, it is enough to prove

$$\int_{d(x,y)\geq\sqrt{t}} \Big(\int_0^\infty s|\Delta_x p_s(x,y)-\Delta_x p_{s+t}(x,y)|^2\,ds\Big)^{1/2}\,d\mu(x)\leq C. \tag{3.11}$$

Set, for $k\in\mathbb{N}$,

$$a_k = a_k(y,t) = \int_{d(x,y)\geq\sqrt{t}} \Big(\int_{kt}^{(k+1)t} s|\Delta_x p_s(x,y)-\Delta_x p_{s+t}(x,y)|^2\,ds\Big)^{1/2} d\mu(x).$$

The same calculations as in Section 3.3 show that, for $k\geq 1$,

$$a_k\leq C\sqrt{b_k V(y,\sqrt{kt})}, \tag{3.12}$$

where

$$\begin{aligned}
b_k = b_k(y,t) &= \int_M\int_{kt}^{(k+1)t} s|\Delta_x p_s(x,y)-\Delta_x p_{s+t}(x,y)|^2 e^{2\varepsilon d^2(x,y)/kt}\,ds\,d\mu(x)\\
&= \int_M\int_{kt}^{(k+1)t} s\Big|\int_s^{s+t}\Delta_x^2 p_u(x,y)\,du\Big|^2 e^{2\varepsilon d^2(x,y)/kt}\,ds\,d\mu(x)\\
&\leq \int_M\int_{kt}^{(k+1)t} s\Big(t\int_s^{s+t}|\Delta_x^2 p_u(x,y)|^2\,du\Big) e^{2\varepsilon d^2(x,y)/kt}\,ds\,d\mu(x)\\
&= t\int_{kt}^{(k+1)t} s\int_s^{s+t}\Big(\int_M|\Delta_x^2 p_u(x,y)|^2 e^{2\varepsilon d^2(x,y)/kt}\,d\mu(x)\Big)\,du\,ds\\
&\leq t\int_{kt}^{(k+1)t} s\int_s^{s+t}\Big(\int_M|\Delta_x^2 p_u(x,y)|^2 e^{6\varepsilon d^2(x,y)/u}\,d\mu(x)\Big)\,du\,ds.
\end{aligned}$$

Thus, using Lemma 3.4 for $m=2$ and $t=0$, we get

$$\begin{aligned}
b_k &\leq Ct\int_{kt}^{(k+1)t} s\int_s^{s+t}\frac{du}{u^4V(y,\sqrt{u})}\,ds\leq Ct^2\int_{kt}^{(k+1)t}\frac{ds}{s^3V(y,\sqrt{s})}\\
&\leq C't^3\frac{1}{(kt)^3V(y,\sqrt{kt})},
\end{aligned}$$

and

$$a_k\leq Ck^{-3/2},\qquad k\geq 1.$$

One estimates a_0 in a similar way, using Lemma 3.4 for $t > 0$, and one can conclude that h has weak type $(1,1)$.

Now for g. Consider

$$g_2^2(f)(x) = \int_0^\infty t^3 |\Delta e^{-t\sqrt{\Delta}} f(x)|^2 \, dt.$$

Using (1.1), one can write

$$\begin{aligned} g_2^2(f)(x) &\le \frac{1}{\pi} \int_0^\infty t^3 \Big(\int_0^\infty |\Delta e^{-\frac{t^2}{4u}\Delta} f(x)| e^{-u} u^{-1/2} \, du \Big)^2 dt, \\ &\le C \int_0^\infty t^3 \Big(\int_0^\infty |\Delta e^{-\frac{t^2}{4u}\Delta} f(x)|^2 e^{-u} u^{-1/2} \, du \Big) \, dt \\ &= C \int_0^\infty \Big(\int_0^\infty t^3 |\Delta e^{-\frac{t^2}{4u}\Delta} f(x)|^2 \, dt \Big) e^{-u} u^{-1/2} \, du \\ &= 8C \int_0^\infty \Big(\int_0^\infty v |\Delta e^{-v\Delta} f(x)|^2 \, dv \Big) e^{-u} u^{3/2} \, du \\ &= C' \, h^2(f)(x). \end{aligned}$$

Finally it is well known that $g \le g_2$ (see [38, p. 59]), thus the proof of Theorem 1.4 is complete.

4. Divergence form operators on bad domains of $\mathbb{R}^n$. In this section, we consider a divergence form operator acting on a bad domain as in [21].

Let Ω be the Euclidean space $\mathbb{R}^n$ or a domain of $\mathbb{R}^n$. In the latter case, no smoothness condition is assumed on the boundary of Ω unless it is implied by other assumptions. Consequently, Ω may not satisfy the doubling volume property, hence it is not necessarily a space of homogeneous type. However, we can always find a subset X of $\mathbb{R}^n$ such that X contains Ω and X satisfies the doubling volume property. One such space X is $\mathbb{R}^n$ itself, but for our purpose, we will keep X as small as possible (see assumption (i) below).

Let Q be the sesquilinear form on the product space $V \times V$, where V is a dense subspace of the Sobolev space $H_1 = W_2^1(\Omega)$, given by

$$Q(f,g) = \int_\Omega \sum_{i,j} a_{ij}(x) \, \frac{\partial f}{\partial x_i} \, \frac{\partial \overline{g}}{\partial x_j} \, dx$$

for $f, g \in V$, and a_{ij} are bounded, measurable, complex-valued coefficients which satisfy

$$|\Im m \sum_{i,j} a_{ij}(x) \zeta_i \overline{\zeta}_j| \le C \Re e \sum_{i,j} a_{ij}(x) \zeta_i \overline{\zeta}_j$$

for some constant C, for all $\zeta \in \mathbb{C}^n$, $x \in \Omega$. We also assume the uniformly elliptic conditions

$$\delta|\zeta|^2 \le \Re e \sum_{i,j} a_{ij}(x)\zeta_i\overline{\zeta}_j \le \kappa|\zeta|^2$$

for $x \in \Omega$ and $\zeta \in \mathbb{C}^n$, where δ and κ are positive constants.

Let A be the divergence form operator associated with the form Q in the sense that A is the operator in $L^2(\Omega)$ with largest domain $\mathcal{D}(A)$ which satisfies

$$\langle Af, g\rangle = Q(f,g)$$

for all $f \in \mathcal{D}(A)$ and all test functions $g \in V$. Different choices of the space V give the operator A different corresponding boundary value conditions when Ω is a domain in $\mathbb{R}^n$. For example, when V is $W_0^{1,2}(\Omega)$ and $W^{1,2}(\Omega)$, this corresponds to the Dirichlet boundary conditions and Neumann boundary conditions, respectively.

The operator A generates a bounded holomorphic semigroup e^{-zA}, $|\arg z| < \mu$ for some $\mu < \pi/2$. See [27, Chapter 9].

We aim to prove the L^p-boundedness of the Littlewood–Paley–Stein function associated with A for $1 < p < 2$ under the assumptions that A has heat kernel bounds and the LPS function is bounded on L^2. More specifically, we assume the following:

(i) The analytic semigroup e^{-tA} generated by A has kernel $p_t(x,y)$ with Gaussian upper bounds, that is,

$$|p_t(x,y)| \le Ch_t(x,y)$$

for all $t > 0$, and all $x, y \in \Omega$, with $h_t(x,y)$ defined on $X \times X$ by

$$h_t(x,y) = \frac{1}{|B_X(x,\sqrt{t})|}\, e^{-\alpha|x-y|^2/t},$$

where C and α are positive constants.

(ii) The LPS function associated with A is bounded on $L^2(\Omega)$, i.e. the operator

$$\mathcal{H}(f) = \Big\{ \int_0^\infty |\nabla e^{-tA} f|^2\, dt \Big\}^{1/2}$$

satisfies

$$\|\mathcal{H}(f)\|_2 \le C\|f\|_2, \quad \forall f \in \mathcal{D}(A).$$

(iii) The space V, the domain of the sesquilinear form, is invariant under multiplication by bounded functions with bounded, continuous first derivatives. This condition is satisfied by Dirichlet, Neumann and mixed boundary conditions.

Remark. (a) In assumption (i), B_X means the ball in X where $\Omega \subseteq X \subseteq \mathbb{R}^n$ as explained above, and $|B_X|$ its Lebesgue measure. It is also sufficient to assume that A satisfies a Poisson type heat kernel upper bound instead of (i), i.e. the exponential decay is replaced by a fast enough polynomial decay. The proof of the main result of this section then needs only a minor modification.

(b) Assumption (ii) is satisfied if the generalised Riesz transform associated with A is bounded on $L^2(\Omega)$, i.e. $\|\nabla A^{-1/2}\|_{2\to 2} \le C$. This holds if and only if the domain $\mathcal{D}(A^{1/2}) \subseteq V$ with $\|\nabla f\|_2 \le C\|A^{1/2}f\|_2$ for every $f \in \mathcal{D}(A^{1/2})$. In this case, ∇ in the LPS function can be replaced by $A^{1/2}$, and the resulting square function norm is equivalent to the L^2-norm on Ω as a consequence of the fact that the operator A is maximal accretive, hence has a bounded holomorphic functional calculus on $L^2(\Omega)$.

(c) The main problem of this case is that Ω no longer satisfies the doubling volume property, hence it is not a space of homogeneous type, and the usual Calderón–Zygmund operator theory is not directly applicable. We overcome this problem by using the results in [20].

Condition (i) implies that the semigroup e^{-tA} has estimates on time derivatives of its kernels. More specifically, we have the following lemma.

Lemma 4.1. *Let T_t be a uniformly bounded analytic semigroup on $L^2(\Omega)$ and assume that T_t, $t > 0$, has a kernel $p_t(x,y)$ satisfying*

$$|p_t(x,y)| \le \frac{C}{|B_X(x,\sqrt{t})|}\, e^{-\alpha|x-y|^2/t}, \qquad \forall x,y \in \Omega,\ t>0.$$

Then the time derivatives $\frac{d^k}{dt^k}T_t$, $k \in \mathbb{N}$, have kernels $\frac{\partial^k}{\partial t^k}p_t$ which satisfy

$$\left|\frac{\partial^k}{\partial t^k}p_t(x,y)\right| \le \frac{C_k}{t^k|B_X(x,\sqrt{t})|}\, e^{-\alpha_k|x-y|^2/t}, \qquad \forall x,y \in \Omega,\ t>0.$$

For a proof, see [21]. See also [15, Section 2.1], [22] and their references for details.

It follows from boundedness of the operators $\frac{d^k}{dt^k}e^{-tA} = A^k e^{-tA}$ on the L^2-space that the heat kernels $p_t(\cdot,y)$ and their time derivatives $\frac{\partial}{\partial t}p_t(\cdot,y)$ belong to the domain of the operator A.

Lemma 2 in [21] shows that the space derivative of $p_t(x,y)$ satisfies a weighted L^2-estimate. In that proof, if we replace the heat kernel $p_t(x,y)$ by its time derivative $\frac{\partial}{\partial t}p_t(\cdot,y)$ and replace the weight $w_t(x,y)$ by the weight $e^{\varepsilon|x-y|^2}$ for sufficiently small ε, then we obtain the estimate in Lemma 3.3 above with the divergence form operator A in place of the Laplace–Beltrami operator. By repeating the proof of Theorem 1.4, we obtain the following.

THEOREM 4.2. *Under the above assumptions* (i)–(iii), *the operator*

$$\mathcal{H}(f)=\Big\{\int_0^\infty |\nabla e^{-tA}f|^2\,dt\Big\}^{1/2}$$

is of weak type $(1,1)$. *Hence it can be extended to a bounded operator on* $L^p(\Omega)$ *for* $1<p\le 2$.

NOTES. (a) Assumption (i) on heat kernel bounds is satisfied by large classes of divergence form operators on $\mathbb{R}^n$ or a domain of $\mathbb{R}^n$. In the case of Dirichlet boundary conditions, we can have Gaussian heat kernel bounds without any conditions on smoothness of the boundary of Ω. In the case of Neumann boundary conditions, one needs the domain Ω to have the extension property to ensure heat kernel bounds even with the Laplacian. For example, see [18], [1] for divergence form operators with real coefficients, and [5], [2] for certain operators with complex coefficients.

(b) Let A be a second order divergence form elliptic operator with bounded, complex coefficients. It has been shown recently in [4] for A acting on the Euclidean space $\mathbb{R}^n$, and in [6] for A acting on a strongly Lipschitz domain, that

$$\|\nabla f\|_2\le C\|A^{1/2}f\|_2,\qquad \forall f\in\mathcal{D}(A).$$

As in Remark (b) of this section, this implies that assumption (ii) is satisfied for these operators.

(c) Under assumptions (i) and (iii), the (horizontal) operator

$$h(f)=\Big\{\int_0^\infty t\Big|\frac{\partial}{\partial t}e^{-tA}f\Big|^2\,dt\Big\}^{1/2}$$

is of weak type $(1,1)$. In this case, the analogue of assumption (ii) is satisfied as a consequence of A being maximal accretive (see Remark (b) of this section). The proof is then along the lines of Section 3.4.

(d) As in the case of manifolds, one can also consider vertical and horizontal Littlewood–Paley–Stein functions defined with the help of the Poisson semigroup $e^{-t\sqrt{A}}$; the results are similar.

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