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Some seminorms on quasi *-algebras

by

CAMILLO TRAPANI (Palermo)

Abstract. Different types of seminorms on a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ are constructed from a suitable family $\mathcal{F}$ of sesquilinear forms on $\mathfrak{A}$. Two particular classes, extended C^*-seminorms and CQ^*-seminorms, are studied in some detail. A necessary and sufficient condition for the admissibility of a sesquilinear form in terms of extended C^*-seminorms on $(\mathfrak{A}, \mathfrak{A}_0)$ is given.

1. Introduction and basic definitions. Let $\mathfrak{A}_0$ be a *-algebra. A seminorm p_0 on $\mathfrak{A}_0$ is called a C^*-*seminorm* if

$$p_0(X^*X) = p_0(X)^2, \quad \forall X \in \mathfrak{A}_0.$$

This notion, first considered by Fell [8], has been extensively studied in the literature [6] from several different points of view.

In [12] Yood studied in particular C^*-seminorms on a *-algebra $\mathfrak{A}_0$ that can be defined via a family $\mathcal{F}$ of positive linear functionals on $\mathfrak{A}_0$ and gave a characterization of those defined by *admissible* positive linear functionals. The importance of admissibility relies on the fact that the Gelfand–Naimark–Segal construction based on an admissible form produces a bounded representation.

Further generalizations have led Bhatt, Inoue and Ogi [5] to consider *unbounded* C^*-seminorms on a *-algebra $\mathfrak{A}_0$, i.e. C^*-seminorms p defined only on a *-subalgebra $D(p)$ of $\mathfrak{A}_0$.

In [2], unbounded C^*-seminorms on *partial* *-algebras [1] have also been studied with the aim of extending some results of representation theory already known for the case of *-algebras.

The main aim of this paper is to extend Yood's approach also to the partial algebraic situation. Instead of general partial *-algebras, we confine ourselves to quasi *-algebras whose partial algebraic structure is simpler. For the reader's convenience we recall the definition (originally due to Lassner [9, 10]).

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Let $\mathfrak{A}$ be a linear space and $\mathfrak{A}_0$ a *-algebra contained in $\mathfrak{A}$. We say that $\mathfrak{A}$ is a *quasi *-algebra* with distinguished *-algebra $\mathfrak{A}_0$ (or, simply, over $\mathfrak{A}_0$) if:

(i) the right and left multiplications of an element of $\mathfrak{A}$ by an element of $\mathfrak{A}_0$ are always defined and linear;
(ii) an involution * (which extends the involution of $\mathfrak{A}_0$) is defined in $\mathfrak{A}$ with the property $(AB)^* = B^*A^*$ whenever the multiplication is defined.

A quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said to have a *unit* $\mathbb{I}$ if there exists an element $\mathbb{I} \in \mathfrak{A}_0$ such that $A\mathbb{I} = \mathbb{I}A = A$, $\forall A \in \mathfrak{A}$. A quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ is said to be *topological* if a locally convex topology ξ is defined in $\mathfrak{A}$ such that (a) the involution is continuous and the multiplications are separately continuous; and (b) $\mathfrak{A}_0$ is dense in $\mathfrak{A}[\xi]$.

For the purposes of this paper we need to define certain particular types of seminorms.

Definition 1.1. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit $\mathbb{I}$ and p a seminorm on $\mathfrak{A}$. We say that p is a *Q^*-seminorm* on $(\mathfrak{A}, \mathfrak{A}_0)$ if

(Q^*1) $p(A) = p(A^*)$, $\forall A \in \mathfrak{A}$;
(Q^*2) $p(\mathbb{I}) = 1$;
(Q^*3) for each $X \in \mathfrak{A}_0$ there exists $K_X > 0$ such that

$$p(AX) \leq K_X\, p(A), \quad \forall A \in \mathfrak{A}.$$

If p is a Q^*-seminorm, we can define

$$p_0(X) := \max\{ \sup_{p(A)=1} p(AX), \sup_{p(A)=1} p(XA)\}; \tag{1}$$

then $p(X) \leq p_0(X)$ for every $X \in \mathfrak{A}_0$ and

$$p(AX) \leq p(A)p_0(X), \quad \forall A \in \mathfrak{A},\ X \in \mathfrak{A}_0.$$

The seminorm p_0 on $\mathfrak{A}_0$ satisfies: (a) $p_0(X^*) = p_0(X)$ for every $X \in \mathfrak{A}_0$ and (b) $p_0(XY) \leq p_0(X)p_0(Y)$ for every $X, Y \in \mathfrak{A}_0$, and so it is an m^*-seminorm on $\mathfrak{A}_0$.

Definition 1.2. A Q^*-seminorm p is called a *CQ^*-seminorm* if p_0 is a C^*-seminorm on $\mathfrak{A}_0$.

If p itself satisfies the C^*-condition when restricted to $\mathfrak{A}_0$, then we call it an *extended* C^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$. More precisely,

Definition 1.3. A Q^*-seminorm p is called an *extended C^*-seminorm* if

(C^*1) $p(X^*X) = p(X)^2$, $\forall X \in \mathfrak{A}_0$.

We notice that if p is an extended C^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$ and p is submultiplicative, i.e.

$$p(AX) \leq p(A)p(X), \quad \forall A \in \mathfrak{A},\ X \in \mathfrak{A}_0,$$

then $p_0(X) = p(X)$ for every $X \in \mathfrak{A}_0$.

The paper is organized as follows.

In Section 2, we show that certain families $\mathcal{F}$ of sesquilinear forms on a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ always define a quasi *-algebra $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$ and a Q^*-seminorm $p_{\mathcal{F}}$ to which a C^*-seminorm on $\mathfrak{A}_0(\mathcal{F}^0)$ is closely linked. We also give conditions for $p_{\mathcal{F}}$ to be an extended C^*-seminorm or a CQ^*-seminorm. Moreover, we consider some examples constructed with L^p-spaces or with (partial) *-algebras of operators in Hilbert space.

In Section 3 we study the admissibility of families of sesquilinear forms on $(\mathfrak{A}, \mathfrak{A}_0)$ and we characterize them in terms of the extended C^*-seminorms they generate. Furthermore, we examine the C^*-seminorms generated by the family of all sesquilinear forms that are continuous with respect to a given seminorm q on $\mathfrak{A}$. In particular we give a counterexample that shows that the starting family $\mathcal{F}$ of sesquilinear forms used to define these extended C^*-seminorms does not exhaust the set of all $p_{\mathcal{F}}$-continuous sesquilinear forms even in the case of a *-algebra.

Finally, in Section 4, we examine, with analogous methods, the construction of CQ^*-seminorms starting once more from a family $\mathcal{F}$ of sesquilinear forms. This study is of interest for the investigation of auxiliary norms in CQ^*-algebras [3].

2. Seminorms defined by sesquilinear forms on quasi *-algebras

Definition 2.1. Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra with unit $\mathbb{I}$. A positive sesquilinear form Ω on $\mathfrak{A} \times \mathfrak{A}$ is called *left-invariant* if

$$\Omega(XA, B) = \Omega(A, X^*B), \quad \forall A, B \in \mathfrak{A},\ X \in \mathfrak{A}_0, \tag{2}$$

and *right-invariant* if

$$\Omega(AX, B) = \Omega(A, BX^*), \quad \forall A, B \in \mathfrak{A},\ X \in \mathfrak{A}_0. \tag{3}$$

The set of all positive, left-invariant (resp. right-invariant) sesquilinear forms is denoted by $\mathcal{P}_l$ (resp. $\mathcal{P}_r$).

Due to positivity, any $\Omega \in \mathcal{P}_l$ is hermitian, i.e. $\Omega(B, A) = \overline{\Omega(A, B)}$ for any $A, B \in \mathfrak{A}$, and satisfies the Cauchy–Schwarz inequality:

$$|\Omega(A, B)|^2 \le \Omega(A, A)\Omega(B, B), \quad \forall A, B \in \mathfrak{A}. \tag{4}$$

We notice that (2) and (4) imply that the set

$$\mathfrak{N}(\Omega) = \{A \in \mathfrak{A} : \Omega(A, A) = 0\}$$

is a left *quasi-ideal* of $\mathfrak{A}$, in the sense that if $A \in \mathfrak{N}(\Omega)$ and $X \in \mathfrak{A}_0$ then $XA \in \mathfrak{N}(\Omega)$.

If $\Omega \in \mathcal{P}_l$, the sesquilinear form Ω^* defined by

$$\Omega^*(A, B) = \Omega(B^*, A^*), \quad A, B \in \mathfrak{A},$$

is positive and right-invariant, i.e. $\Omega^* \in \mathcal{P}_r$. For this reason, we will only work with $\mathcal{P}_l$, since the properties of $\mathcal{P}_r$ can be easily derived by taking *.

Definition 2.2. An element $\Omega \in \mathcal{P}_l$ is called *admissible* if for each $A \in \mathfrak{A}$ there exists $K_A > 0$ such that

$$\Omega(AX, AX) \leq K_A \Omega(X, X), \quad \forall X \in \mathfrak{A}_0.$$

The set of all admissible sesquilinear forms in $\mathcal{P}_l$ will be denoted with $\mathcal{P}_l^a$.

If $\Omega \in \mathcal{P}_l$ and $A \in \mathfrak{A}$, we can define a *linear* functional ω_Ω^A on $\mathfrak{A}_0$ by

$$\omega_\Omega^A(X) = \Omega(XA, A), \quad X \in \mathfrak{A}_0.$$

The equality

$$\Omega(X^*XA, A) = \Omega(XA, XA)$$

implies that each ω_Ω^A is positive on $\mathfrak{A}_0$, and it is a state if, and only if, $\Omega(A, A) = 1$.

We put

$$\mathcal{F}^0 = \{\omega_\Omega^A : \Omega \in \mathcal{F},\, A \in \mathfrak{A}\}.$$

Following [12], we say that a family $\mathcal{K}$ of positive linear functionals on $\mathfrak{A}_0$ is *balanced* if for each $\omega \in \mathcal{K}$ and for each $Y \in \mathfrak{A}_0$, the positive linear functional ω^Y defined by

$$\omega^Y(X) = \omega(Y^*XY), \quad X \in \mathfrak{A}_0,$$

is still an element of $\mathcal{K}$.

Then it is easy to prove that $\mathcal{F}^0$ is balanced. Indeed, one has $(\omega_\Omega^A)_Y = \omega_\Omega^{YA}$ for each $Y \in \mathfrak{A}_0$. Then following Yood's construction, the set

$$\mathfrak{A}_0(\mathcal{F}^0) = \{X \in \mathfrak{A}_0 : \sup\{\omega_\Omega^A(X^*X) : \Omega \in \mathcal{F},\, A \in \mathfrak{A},\, \Omega(A, A) = 1\} < \infty\}$$

is a *-subalgebra of $\mathfrak{A}_0$ and

$$|X|_{\mathcal{F}^0} = (\sup\{\omega_\Omega^A(X^*X) : \Omega \in \mathcal{F},\, A \in \mathfrak{A},\, \Omega(A, A) = 1\})^{1/2}$$

defines a C^*-seminorm on $\mathfrak{A}_0(\mathcal{F}^0)$.

Let $\mathcal{F} \subseteq \mathcal{P}_l$. For $A \in \mathfrak{A}$, we put

$$p_\mathcal{F}^R(A) = \sup_{\Omega \in \mathcal{F}_s} \Omega(A, A)^{1/2}, \quad p_\mathcal{F}^L(A) = \sup_{\Omega \in \mathcal{F}_s} \Omega(A^*, A^*)^{1/2},$$

where $\mathcal{F}_s = \{\Omega \in \mathcal{F} : \Omega(\mathbb{I}, \mathbb{I}) = 1\}$. One, or even both, of these numbers may be ∞. Then we set

$$\mathfrak{A}_R(\mathcal{F}) = \{A \in \mathfrak{A} : p_\mathcal{F}^R(A) < \infty\}, \quad \mathfrak{A}_L(\mathcal{F}) = \{A \in \mathfrak{A} : p_\mathcal{F}^L(A) < \infty\}.$$

Clearly, $p_\mathcal{F}^R(A) = p_\mathcal{F}^L(A^*)$ for each $A \in \mathfrak{A}$.

Proposition 2.3. *Let $\mathcal{F} \subseteq \mathcal{P}_l$ be a family of sesquilinear forms. Then*

(i) $\mathfrak{A}_R(\mathcal{F})$ *is a left module over* $\mathfrak{A}_0(\mathcal{F}^0)$ *and*

$$p_\mathcal{F}^R(XA) \leq |X|_{\mathcal{F}^0}\, p_\mathcal{F}^R(A), \quad \forall A \in \mathfrak{A}_R(\mathcal{F}),\, X \in \mathfrak{A}_0(\mathcal{F}^0).$$

(ii) $\mathfrak{A}_L(\mathcal{F})$ *is a right module over* $\mathfrak{A}_0(\mathcal{F}^0)$ *and*

$$p^L_{\mathcal{F}}(AX) \le |X|_{\mathcal{F}^0}\, p^L_{\mathcal{F}}(A), \qquad \forall A \in \mathfrak{A}_L(\mathcal{F}),\ X \in \mathfrak{A}_0(\mathcal{F}^0).$$

Proof. It is easy to check that $\mathfrak{A}_R(\mathcal{F})$ is a vector space and that $A^* \in \mathfrak{A}_L(\mathcal{F})$ if, and only if, $A \in \mathfrak{A}_R(\mathcal{F})$. Now let $X \in \mathfrak{A}_0(\mathcal{F}^0)$ and $A \in \mathfrak{A}_R(\mathcal{F})$; we first prove that $XA \in \mathfrak{A}_R(\mathcal{F})$. If $\Omega(A,A) = 0$, then $\Omega(XA,XA) = 0$ for each $\Omega \in \mathcal{F}$, and so $p_{\mathcal{F}}(XA) = 0$. Now let $\Omega(A,A) > 0$. Put $B = A/\Omega(A,A)^{1/2}$. Then ω^B_Ω is a state on $\mathfrak{A}_0$ and belongs to $\mathcal{F}_0$. Since

$$\Omega(XA,XA) = \omega^B_\Omega(X^*X)\Omega(A,A),$$

it follows that $\Omega(XA,XA)^{1/2} \le |X|_{\mathcal{F}^0}\, p^R_{\mathcal{F}}(A)$ and so

$$p^R_{\mathcal{F}}(XA) \le |X|_{\mathcal{F}^0}\, p^R_{\mathcal{F}}(A). \tag{5}$$

In conclusion, $XA \in \mathfrak{A}_R(\mathcal{F})$. Taking adjoints it also follows that if $A \in \mathfrak{A}_L(\mathcal{F})$ and $X \in \mathfrak{A}_0(\mathcal{F})$, then $AX \in \mathfrak{A}_L(\mathcal{F})$ and

$$p^L_{\mathcal{F}}(AX) \le |X|_{\mathcal{F}^0}\, p^L_{\mathcal{F}}(A). \ ■ \tag{6}$$

Now put

$$\mathfrak{A}(\mathcal{F}) = \mathfrak{A}_R(\mathcal{F}) \cap \mathfrak{A}_L(\mathcal{F}), \qquad p_{\mathcal{F}}(A) = \max\{p^R_{\mathcal{F}}(A), p^L_{\mathcal{F}}(A)\}.$$

Then $\mathfrak{A}(\mathcal{F})$ is a *-invariant subspace of $\mathfrak{A}$ but, in general, need not be a quasi *-algebra over $\mathfrak{A}_0(\mathcal{F}^0)$. Clearly, $p_{\mathcal{F}}$ is a seminorm on $\mathfrak{A}(\mathcal{F})$ satisfying $p_{\mathcal{F}}(A^*) = p_{\mathcal{F}}(A)$ for each $A \in \mathfrak{A}(\mathcal{F})$ (we call it a **-invariant seminorm*) but an inequality like (5) or (6) does not hold for $p_{\mathcal{F}}$, in general.

There is, however, some special situation. If $\Omega \in \mathcal{P}_l$ and $X \in \mathfrak{A}_0$, we put $\Omega_X(A,B) := \Omega(AX,BX)$. It is easily seen that $\Omega_X \in \mathcal{P}_l$.

Definition 2.4. Let $\mathcal{F} \subseteq \mathcal{P}_l$. We say that $\mathcal{F}$ is *strongly balanced* if for each $\Omega \in \mathcal{F}$ and each $X \in \mathfrak{A}_0$, the following conditions are satisfied:

(i) $\Omega_X \in \mathcal{F}$.

(ii) If $\Omega(X,X) = 0$ for some $X \in \mathfrak{A}_0$, then $\Omega(AX,AX) = 0$ for any $A \in \mathfrak{A}$.

Proposition 2.5. *Let $\mathcal{F}$ be strongly balanced. Then*

$$p^R_{\mathcal{F}}(AX) \le |X|_{\mathcal{F}^0}\, p^R_{\mathcal{F}}(A), \qquad \forall A \in \mathfrak{A}(\mathcal{F}),\ X \in \mathfrak{A}_0(\mathcal{F}^0),$$
$$p^L_{\mathcal{F}}(XA) \le |X|_{\mathcal{F}^0}\, p^L_{\mathcal{F}}(A), \qquad \forall A \in \mathfrak{A}(\mathcal{F}),\ X \in \mathfrak{A}_0(\mathcal{F}^0).$$

Therefore, in this case, $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$ *is a quasi *-algebra and* $p_{\mathcal{F}}$ *is a* Q^**-seminorm on* $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$ *and*

$$p_{\mathcal{F}}(AX) \le |X|_{\mathcal{F}^0}\, p_{\mathcal{F}}(A), \qquad \forall A \in \mathfrak{A}(\mathcal{F}),\ X \in \mathfrak{A}_0(\mathcal{F}^0).$$

Proof. If $\Omega(X,X) = 0$ for every $\Omega \in \mathcal{F}$ (in particular, if $|X|_{\mathcal{F}^0} = 0$), then by (ii) of Definition 2.4, for every $\Omega \in \mathcal{F}$ and $A \in \mathfrak{A}$, $\Omega(AX,AX) = 0$. Hence $p^R_{\mathcal{F}}(AX) = 0$.

Let now $A \in \mathfrak{A}(\mathcal{F})$, $X \in \mathfrak{A}_0(\mathcal{F}^0)$. For each $\Omega \in \mathcal{F}$ such that $\Omega(X,X) > 0$ we have

$$\Omega(AX,AX) = \Omega(AY,AY)\Omega(X,X)$$

with $Y = X/\Omega(X,X)^{1/2}$. Taking the sup over $\mathcal{F}_s$ and making use of (i) of Definition 2.4, we get

$$p^R_{\mathcal{F}}(AX) \le p^R_{\mathcal{F}}(X)\, p^R_{\mathcal{F}}(A) \le |X|_{\mathcal{F}^0}\, p^R_{\mathcal{F}}(A). \blacksquare$$

DEFINITION 2.6. We say that a family $\mathcal{F} \subseteq \mathcal{P}_l$ is *well-behaved* if

$$\mathfrak{A}(\mathcal{F}) = \mathfrak{A}_R(\mathcal{F}) = \mathfrak{A}_L(\mathcal{F}), \qquad p_{\mathcal{F}}(A) = p^R_{\mathcal{F}}(A) = p^L_{\mathcal{F}}(A).$$

For well-behaved families $\mathcal{F}$ Proposition 2.3 gives:

PROPOSITION 2.7. *If $\mathcal{F}$ is well-behaved, then $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$ is a quasi *-algebra and*

$$p_{\mathcal{F}}(AX) \le |X|_{\mathcal{F}^0}\, p_{\mathcal{F}}(A), \tag{7}$$

i.e. $p_{\mathcal{F}}$ is a Q^*-seminorm on $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$.

Notice that $p_{\mathcal{F}}(X) \le |X|_{\mathcal{F}^0}$ for every $X \in \mathfrak{A}_0$.

We can summarize the previous discussion in the following:

THEOREM 2.8. *Let $(\mathfrak{A}, \mathfrak{A}_0)$ be a quasi *-algebra and $\mathcal{F} \subset \mathcal{P}_l$ a family of sesquilinear forms on $\mathfrak{A} \times \mathfrak{A}$. Then:*

(i) *There exists a quasi *-algebra $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$ contained in $(\mathfrak{A}, \mathfrak{A}_0)$ such that $p_{\mathcal{F}}$ is a *-invariant seminorm on $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$.*

(ii) *If $\mathcal{F}$ is well-behaved, then $p_{\mathcal{F}}$ is a Q^*-seminorm on $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$.*

(iii) *If $\mathcal{F}$ is strongly balanced, then $p_{\mathcal{F}}$ is an extended C^*-seminorm on $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$.*

Proof. We need only prove (iii). Let $\mathcal{F}$ be strongly balanced. Then the set

$$\mathcal{F}^u = \{\omega_\Omega : \Omega \in \mathcal{F}\}$$

is a balanced family of positive linear functionals on $\mathfrak{A}_0$ in the sense of [12]. Indeed, let $\omega_\Omega \in \mathcal{F}^u$ and $Y \in \mathfrak{A}_0$. Then, for each $X \in \mathfrak{A}_0$,

$$\omega^Y_\Omega(X) = \omega_\Omega(Y^*XY) = \Omega(Y^*XY, \mathbb{I}) = \Omega(XY,Y) = \Omega_Y(X, \mathbb{I})$$

and $\Omega^Y \in \mathcal{F}$ by assumption. Since $\mathcal{F}^u \subset \mathcal{F}^0$, we get

$$\sup_{\Omega \in \mathcal{F}_s} \omega_\Omega(X^*X) < \infty, \qquad \forall X \in \mathfrak{A}_0(\mathcal{F}^0).$$

Then it turns out [12, Lemma 2.1] that

$$p^u_{\mathcal{F}}(X) = \sup_{\Omega \in \mathcal{F}_s} \omega_\Omega(X^*X)^{1/2}, \qquad X \in \mathfrak{A}_0(\mathcal{F}^0),$$

is a C^*-seminorm on $\mathfrak{A}_0(\mathcal{F}^0)$. But clearly $p^u_{\mathcal{F}}(X) = p_{\mathcal{F}}(X)$ for every $X \in \mathfrak{A}_0(\mathcal{F}^0)$. Therefore $p_{\mathcal{F}}$ is an extended C^*-seminorm on $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$. $\blacksquare$

A natural question arises: when $|\ |_{\mathcal{F}^0} = (p_{\mathcal{F}})_0$, where $(p_{\mathcal{F}})_0$ is constructed as in (1) starting from $p_{\mathcal{F}}$? We first notice that (7) implies $(p_{\mathcal{F}})_0(X) \le |X|_{\mathcal{F}_0}$ for every $X \in \mathfrak{A}_0(\mathcal{F}^0)$. Moreover, we have the following

PROPOSITION 2.9. *Let $\mathcal{F} \subset \mathcal{P}_l$ be a well-behaved family of sesquilinear forms and $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$ the quasi *-algebra constructed as above. The following statements are equivalent*:

(i) $|X|_{\mathcal{F}^0} = (p_{\mathcal{F}})_0(X),\ \forall X \in \mathfrak{A}_0(\mathcal{F}^0)$.
(ii) $\Omega(XA, XA) \le (p_{\mathcal{F}})_0(X)^2 \Omega(A, A),\ \forall \Omega \in \mathcal{F}, X \in \mathfrak{A}_0, A \in \mathfrak{A}$.
(iii) *For each $\Omega \in \mathcal{F}$ and $A \in \mathfrak{A}$, ω_Ω^A is $(p_{\mathcal{F}})_0$-continuous.*

If any of the previous statements holds then $p_{\mathcal{F}}$ is a CQ^*-seminorm on $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$.

Proof. (i)⇒(ii). By the definition of $|\ |_{\mathcal{F}^0}$ itself it follows that

$$\Omega(XA, XA) \le (p_{\mathcal{F}})_0(X)^2 \Omega(A, A), \quad \forall \Omega \in \mathcal{F},\ X \in \mathfrak{A}_0,\ A \in \mathfrak{A}.$$

(ii)⇒(iii). We have

$$|\omega_\Omega^A(X)| = |\Omega(XA, A)| \le \Omega(XA, XA)^{1/2} \Omega(A, A)^{1/2} \le (p_{\mathcal{F}})_0(X) \Omega(A, A).$$

Thus each ω_Ω^A is $(p_{\mathcal{F}})_0$-continuous.

(iii)⇒(i). Assume that each ω_Ω^A is $(p_{\mathcal{F}})_0$-continuous. Then

$$|\omega_\Omega^A(X)| \le \|\omega_\Omega^A\|_{(p_{\mathcal{F}})_0} (p_{\mathcal{F}})_0(X), \quad \forall X \in \mathfrak{A}_0,$$

where $\|\omega_\Omega^A\|_{(p_{\mathcal{F}})_0} = \sup\{|\omega_\Omega^A(X)| : (p_{\mathcal{F}})_0(X) = 1\}$.

By Kaplansky's inequality, we get

$$\begin{aligned}\omega_\Omega^A(X^*X) &\le \Omega(A, A)^{1-2^{-n}} (\omega_\Omega^A((X^*X)^{2^n}))^{2^{-n}} \\ &\le \Omega(A, A)^{1-2^{-n}} (\|\omega_\Omega^A\|_{(p_{\mathcal{F}})_0} (p_{\mathcal{F}})_0((X^*X)^{2^n}))^{2^{-n}}.\end{aligned}$$

For $n \to \infty$, we have

$$\omega_\Omega^A(X^*X) \le \Omega(A, A)(p_{\mathcal{F}})_0(X^*X).$$

This implies that $(p_{\mathcal{F}})_0(X^*X) = |X^*X|_{\mathcal{F}^0} = |X|^2_{\mathcal{F}^0}$ for each $X \in \mathfrak{A}_0(\mathcal{F}^0)$. Finally, for each $X \in \mathfrak{A}_0(\mathcal{F}^0)$ we have

$$(p_{\mathcal{F}})_0(X)^2 \ge (p_{\mathcal{F}})_0(X^*X) = |X^*X|_{\mathcal{F}^0} = |X|^2_{\mathcal{F}^0} \ge (p_{\mathcal{F}})_0(X)^2$$

and the statement is proved. ■

Before proceeding we give some examples.

EXAMPLE 2.10. Let I be a compact interval on the real line and consider the quasi *-algebra $(L^p(I), C(I))$ where $C(I)$ stands for the *-algebra of all continuous functions on I and $L^p(I)$ is the usual L^p-space on I. We assume that $p \ge 2$. Let $w \in L^{p/(p-2)}(I)$ (we take $1/0 = \infty$) and $w \ge 0$. Then

$$\Omega^{(w)}(f, g) = \int_I f(x)\overline{g(x)} w(x)\, dx, \quad f, g \in L^p(I),$$

defines a left (and right) invariant positive sesquilinear form on $L^p(I)$. If $w \in L^\infty(I)$, then $\Omega^{(w)}$ is admissible.

We put

$$\mathcal{F} = \{\Omega^{(w)} : w \in L^{p/(p-2)}(I),\, w \geq 0\}.$$

It is easy to see that $\mathcal{F}$ is strongly balanced and that $\Omega^{(w)} \in \mathcal{F}_s$ if, and only if, $\|w\|_1 = 1$. Now, for each $w \in L^{p/(p-2)}(I)$ such that $w \geq 0$, $\|w\|_1 = 1$ and for each $\phi \in C(I)$ and $f \in L^p(I)$ we have

$$\Omega^{(w)}(\phi f, \phi f) = \int_I |\phi(x)|^2 |f(x)|^2 w(x)\, dx \leq \|\phi\|_\infty^2 \int_I |f(x)|^2 w(x)\, dx.$$

Thus $\mathfrak{A}_0(\mathcal{F}^0) = C(I)$ and $|\phi|_{\mathcal{F}^0} \leq \|\phi\|_\infty$.

It is easy to see that the C^*-seminorm $|\ |_{\mathcal{F}^0}$ is a norm that makes $C(I)$ a normed algebra [7, Ch. VII]. Therefore [11, Theorem 1.2.4], $|\phi|_{\mathcal{F}^0} = \|\phi\|_\infty$ for every $\phi \in C(I)$.

On the other hand,

$$\mathfrak{A}(\mathcal{F}) = \Big\{ f \in L^p(I) : \sup_{\|w\|_1=1} \int_I |f(x)|^2 w(x)\, dx < \infty \Big\} = L^\infty(I).$$

Therefore, the extended C^*-seminorm $p_{\mathcal{F}}$ coincides with the L^∞-norm.

Example 2.11. Let $\mathcal{D}$ be a dense domain in $\mathcal{H}$. As usual, $\mathcal{L}^+(\mathcal{D}, \mathcal{H})$ (see e.g. [1]) denotes the space of all closable operators A in $\mathcal{H}$ such that $D(A) = \mathcal{D}$ and $D(A^*) \supseteq \mathcal{D}$. The equality $A^+ = A^* \upharpoonright_{\mathcal{D}}$ defines an involution in $\mathcal{L}^+(\mathcal{D}, \mathcal{H})$. Let

$$\mathcal{B}(\mathcal{D}) = \{X \in \mathcal{L}^+(\mathcal{D}, \mathcal{H}) : X \text{ is bounded and } X : \mathcal{D} \to \mathcal{D},\, X^+ : \mathcal{D} \to \mathcal{D}\}.$$

Then $\mathcal{B}(\mathcal{D})$ is a *-algebra and $\big(\mathcal{L}^+(\mathcal{D}, \mathcal{H}), \mathcal{B}(\mathcal{D})\big)$ is a quasi *-algebra under the *weak* multiplication:

$$A \,\square\, Bf = ABf, \qquad B \,\square\, Af = \overline{B}Af, \qquad A \in \mathcal{L}^+(\mathcal{D}, \mathcal{H}),\ B \in \mathcal{B}(\mathcal{D}).$$

If $f \in \mathcal{D}$, we put

$$\Omega^f(A, B) = \langle Af, Bf \rangle.$$

Then each Ω^f is left-invariant.

Let now $\mathcal{M}$ be a subspace of $\mathcal{D}$ and let

$$\mathcal{F}_{\mathcal{M}} = \{\Omega^f : f \in \mathcal{M}\}.$$

Then $\mathfrak{A}_0(\mathcal{F}^0) = \mathcal{B}(\mathcal{D})$ and

$$p^R_{\mathcal{F}_{\mathcal{M}}}(A) = \sup_{\Omega \in (\mathcal{F}_{\mathcal{M}})_s} \Omega(A, A)^{1/2} = \|A \upharpoonright_{\mathcal{M}}\|,$$

$$p^L_{\mathcal{F}_{\mathcal{M}}}(A) = \sup_{\Omega \in (\mathcal{F}_{\mathcal{M}})_s} \Omega(A^*, A^*)^{1/2} = \|A^* \upharpoonright_{\mathcal{M}}\|.$$

Thus

$$\mathfrak{A}(\mathcal{F}) = \{A \in \mathcal{L}^+(\mathcal{D}, \mathcal{H}) : A \upharpoonright_{\mathcal{M}} \text{ and } A^* \upharpoonright_{\mathcal{M}} \text{ are bounded}\}.$$

The corresponding seminorm $p_{\mathcal{F}_{\mathcal{M}}}$ is a *-invariant seminorm but not, in general, a Q^*-seminorm. If $\mathcal{M} = \mathcal{D}$, then $\mathfrak{A}(\mathcal{F}) = \mathfrak{A}_0(\mathcal{F}^0) = \mathcal{B}(\mathcal{D})$ and $p_{\mathcal{F}}$ is a C^*-seminorm.

As in Example 2.10, when $\mathfrak{A}_0$ is a Banach *-algebra, the situation simplifies:

Proposition 2.12. *Assume $\mathfrak{A}_0$ is a Banach *-algebra with unit. Then $\mathfrak{A}_0(\mathcal{F}^0) = \mathfrak{A}_0$ for every family $\mathcal{F} \subseteq \mathcal{P}_l$.*

Proof. Let $\Omega \in \mathcal{F}$. For each $A \in \mathfrak{A}$, the functional $\omega_\Omega^A(X) = \Omega(XA, A)$ is positive on $\mathfrak{A}_0$ and so it is continuous. Then

$$\omega_\Omega^A(X^*X) \le \omega_\Omega^A(\mathbb{I})\|X^*X\| \le \Omega(A,A)\|X\|^2, \quad \forall X \in \mathfrak{A}_0.$$

Therefore, $\mathfrak{A}_0 = \mathfrak{A}(\mathcal{F}_0)$. ∎

The following lemma, already proven in [2] for unbounded C^*-seminorms on partial *-algebras, allows the construction of a C^*-algebra starting from an extended C^*-seminorm p on a quasi *-algebra $(\mathfrak{A}, \mathfrak{A}_0)$ at least when $\mathfrak{A}_0$ is p-dense in $\mathfrak{A}$, by which we mean that for each $A \in \mathfrak{A}$ there exists a sequence $\{A_n\} \subset \mathfrak{A}_0$ such that $p(A - A_n) \to 0$. The proof is similar to that given in [2] and we omit it.

Proposition 2.13. *Let p be an extended C^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$ and assume that $\mathfrak{A}_0$ is p-dense in $\mathfrak{A}$. Denote by $\widehat{\mathfrak{A}}$ the set of all Cauchy sequences in $\mathfrak{A}$ with respect to the seminorm p and define an equivalence relation in $\widehat{\mathfrak{A}}$ by $\{A_n\} \sim \{B_n\}$ iff $\lim_{n\to\infty} p(A_n - B_n) = 0$. Then*:

(1) *The quotient space $\widehat{\mathfrak{A}}/\sim$ is a *-algebra under the following operations, involution and norm*:

$$\{A_n\}^\sim + \{B_n\}^\sim \equiv \{A_n + B_n\}^\sim, \quad \lambda\{A_n\}^\sim \equiv \{\lambda A_n\}^\sim,$$
$$\{A_n\}^\sim\{B_n\}^\sim \equiv \{X_nY_n\}^\sim,$$

where $\{X_n\}$ and $\{Y_n\}$ in $\mathfrak{A}_0$ are such that $\{X_n\}^\sim \equiv \{A_n\}^\sim$ and $\{Y_n\}^\sim \equiv \{B_n\}^\sim$, and

$$\{A_n\}^{\sim *} \equiv \{A_n^*\}^\sim, \quad \|\{A_n\}^\sim\|_p \equiv \lim_{n\to\infty} p(A_n).$$

(2) *If for each $A \in \mathfrak{A}$ we put*

$$\widetilde{A} = \{A_n\}^\sim \quad (A_n = A,\ n \in \mathbb{N}), \quad \widetilde{\mathfrak{A}} = \{\widetilde{A} : A \in \mathfrak{A}\},$$

*then $\widetilde{\mathfrak{A}}$ is a dense *-invariant subspace of $\widehat{\mathfrak{A}}/\sim$ satisfying $\widetilde{A}\widetilde{B} = (AB)^\sim$ whenever $A \in \mathfrak{A}_0$.*

3. Admissibility and continuity. In this section we try to characterize positive sesquilinear forms that are admissible. Admissibility is an important concept since it implies the boundedness of certain *-representations of $\mathfrak{A}$.

PROPOSITION 3.1. *Let $\mathcal{F} \subseteq \mathcal{P}_l$ be strongly balanced. If $\mathfrak{A}(\mathcal{F}) = \mathfrak{A}$, then each $\Omega \in \mathcal{F}$ is admissible.*

Proof. Without loss of generality we may assume that $\Omega(\mathbb{I}, \mathbb{I}) = 1$. Now, suppose that the statement is false. Then there would be $A \in \mathfrak{A}$ such that for each $n \in \mathbb{N}$ we could find $X_n \in \mathfrak{A}_0$ with

$$\Omega(AX_n, AX_n) > n\Omega(X_n, X_n).$$

We have $\Omega(X_n, X_n) > 0$, since otherwise $\Omega(AX_n, AX_n) = 0$ by (ii) of Definition 2.4. Thus we can define $Y_n = X_n/\Omega(X_n, X_n)^{1/2}$. Then $\Omega_{Y_n} \in \mathcal{F}_s$ (because $\mathcal{F}$ is strongly balanced and $\Omega_{Y_n}(\mathbb{I}, \mathbb{I}) = 1$). But $\Omega_{Y_n}(A, A) > n$ so that $A \notin \mathfrak{A}(\mathcal{F}) = \mathfrak{A}$ and this is a contradiction. ∎

Let q be a seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$. A sesquilinear form Φ is said to be continuous with respect to q, or simply q-*continuous*, if there exists $K > 0$ such that

$$|\Phi(A, B)| \le Kq(A)q(B), \quad \forall A, B \in \mathfrak{A}, \tag{8}$$

or, equivalently,

$$\Phi(A, A) \le Kq(A)^2, \quad \forall A \in \mathfrak{A}.$$

The infimum of all positive constants for which (8) holds will be denoted by $\|\Phi\|_q$.

In what follows we assume that q is a Q^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$ and we denote by $\mathcal{C}(q)$ the set of all q-continuous elements of $\mathcal{P}_l$. It is easy to see that for each Φ is in $\mathcal{C}(q)$, the form Φ_X is in $\mathcal{C}(q)$ for every $X \in \mathfrak{A}_0$ (but $\mathcal{C}(q)$ is not necessarily strongly balanced, since (i) of Definition 2.4 may fail). Moreover $\|\Phi_X\|_q \le \|\Phi\|_q\, q(X)^2$ for each $\Phi \in \mathcal{C}(q)$ and $X \in \mathfrak{A}_0$.

By Theorem 2.8, $\mathfrak{A}(\mathcal{C}(q))$ is a quasi *-algebra over $\mathfrak{A}_0(\mathcal{C}(q)^0)$. Since q is a Q^*-seminorm, we have $\mathfrak{A}_0(\mathcal{C}(q)^0) = \mathfrak{A}_0$. Indeed, in this case, for each $\Omega \in \mathcal{C}(q)$ and $A \in \mathfrak{A}$, the linear form ω^A_Ω is q_0-continuous (q_0 being defined as in (1)), since

$$|\omega^A_\Omega(X)| = |\Omega(XA, A)| \le Kq_0(X)q(A)^2, \quad \forall X \in \mathfrak{A}_0.$$

This implies that

$$|\omega^A_\Omega(X)| \le \omega^A_\Omega(\mathbb{I})q_0(X), \quad \forall X \in \mathfrak{A}_0.$$

Therefore

$$\sup\{\omega^A_\Omega(X^*X) : A \in \mathfrak{A},\ \Omega(A, A) = 1\} \le q_0(X^*X), \quad \forall X \in \mathfrak{A}_0,$$

and this implies the statement.

In order to describe admissibility, we consider some special subsets of $\mathcal{C}(q)$. Let

$$\mathcal{C}^0(q) = \{\Omega \in \mathcal{C}(q) : \|\Omega\|_q = \Omega(\mathbb{I}, \mathbb{I})\},$$
$$\mathcal{C}^e(q) = \{\Omega \in \mathcal{C}^0(q) : \Omega_X \in \mathcal{C}^0(q),\ \forall X \in \mathfrak{A}_0\}.$$

Let $\mathcal{F} \subset \mathcal{C}^0(q)$. In this case we have

$$p^L_{\mathcal{F}}(A) = \sup_{\Omega \in \mathcal{F}_s} \Omega(A,A)^{1/2} \le q(A), \quad \forall A \in \mathfrak{A}.$$

So $\mathfrak{A}(\mathcal{F}) = \mathfrak{A}$ and therefore $p_{\mathcal{F}}$ is a *-invariant seminorm on $\mathfrak{A}$ (but not necessarily a Q^*-seminorm).

If $\mathcal{G}$ is another subset of $\mathcal{C}^0(q)$ with $\mathcal{F} \subset \mathcal{G}$, we have

$$p_{\mathcal{F}}(A) \le p_{\mathcal{G}}(A), \quad \forall A \in \mathfrak{A}.$$

In conclusion, for each $\mathcal{F} \subset \mathcal{C}^0(q)$ we get

$$p_{\mathcal{F}}(A) \le p_{\mathcal{C}^0(q)}(A) \le q(A), \quad \forall A \in \mathfrak{A}. \tag{9}$$

LEMMA 3.2. *If $\Omega \in \mathcal{C}^e(q)$, then Ω is admissible, i.e. $\mathcal{C}^e(q) \subset \mathcal{P}^a_l$.*

Proof. If $\Omega \in \mathcal{C}^e(q)$, then $\Omega_X \in \mathcal{C}^0(q)$ for each $X \in \mathfrak{A}_0$. Thus $\|\Omega_X\| = \Omega_X(\mathbb{I},\mathbb{I})$. This implies that

$$\Omega(AX,AX) \le \Omega(X,X)q(A)^2, \quad A \in \mathfrak{A},$$

and therefore Ω is admissible. ■

From this lemma it follows that $\mathcal{C}^e(q)$ is strongly balanced and therefore, by Theorem 2.8(iii), $p_{\mathcal{C}^e(q)}$ is an extended C^*-seminorm. More generally, if $\mathcal{F} \subset \mathcal{C}^0(q)$ is strongly balanced then $\mathcal{F} \subseteq \mathcal{C}^e(q)$, $p_{\mathcal{F}}$ is an extended C^*-seminorm and

$$p_{\mathcal{F}}(A) \le p_{\mathcal{C}^e(q)}(A) \le p_{\mathcal{C}^0(q)}(A) \le q(A), \quad \forall A \in \mathfrak{A}. \tag{10}$$

PROPOSITION 3.3. *Let $\Omega \in \mathcal{P}_l$. A necessary and sufficient condition for Ω to be admissible is that there exists an* (*everywhere defined*) *extended C^*-seminorm q on* $(\mathfrak{A},\mathfrak{A}_0)$ *such that $\Omega \in \mathcal{C}^e(q)$.*

Proof. We only need to prove the necessity. Let Ω be admissible, i.e. $\Omega \in \mathcal{P}^a_l$. Put $\mathcal{F} = \{\Omega_X : X \in \mathfrak{A}_0\}$. Then $\mathcal{F}$ is strongly balanced. The admissibility of Ω implies that

$$p^L_{\mathcal{F}}(A)^2 = \sup_{\Omega \in \mathcal{F}_s} \Omega(A,A)$$

is finite for each $A \in \mathfrak{A}$. Thus $\mathfrak{A}(\mathcal{F}) = \mathfrak{A}$. If we define, as before,

$$p_{\mathcal{F}}(A) = \max\{p^L_{\mathcal{F}}(A), p^R_{\mathcal{F}}(A)\}, \quad A \in \mathfrak{A},$$

then $p_{\mathcal{F}}$ is an extended C^*-seminorm on $(\mathfrak{A},\mathfrak{A}_0)$. Clearly Ω is $p_{\mathcal{F}}$-continuous and $\|\Omega\|_{p_{\mathcal{F}}} = \Omega(\mathbb{I},\mathbb{I})$. Thus $\Omega \in \mathcal{C}^e(p_{\mathcal{F}})$. ■

The discussion so far applies to the particular case where $q = p_{\mathcal{F}}$ for some subset $\mathcal{F}$ of $\mathcal{P}_l$. In this case, each $\Omega \in \mathcal{F}$ is $p_{\mathcal{F}}$-continuous on $(\mathfrak{A}(\mathcal{F}),\mathfrak{A}_0(\mathcal{F}^0))$; i.e., $\mathcal{F} \subseteq \mathcal{C}(p_{\mathcal{F}})$. Moreover, for each $\Omega \in \mathcal{F}_s$, we have $\|\Omega\|_{p_{\mathcal{F}}} = 1$. Indeed, by the Cauchy–Schwarz inequality, for any $A, B \in \mathfrak{A}(\mathcal{F})$ we have

$$|\Omega(A,B)|^2 \le \Omega(A,A)\Omega(B,B) \le p_{\mathcal{F}}(A)^2 p_{\mathcal{F}}(B)^2.$$

Thus $\|\Omega\|_{p_{\mathcal{F}}} \leq 1$; but since $\Omega(\mathbb{I}, \mathbb{I}) = 1$, the equality follows. This also implies that $\|\Omega\|_{p_{\mathcal{F}}} = \Omega(\mathbb{I}, \mathbb{I})$ for each $\Omega \in \mathcal{F}$. Thus $\mathcal{F} \subset \mathcal{C}^0(p_{\mathcal{F}})$.

In particular, if $\mathcal{F}$ is strongly balanced, then by (10) we get

$$p_{\mathcal{F}}(A) = p_{\mathcal{C}^e(p_{\mathcal{F}})}(A) = p_{\mathcal{C}^0(p_{\mathcal{F}})}(A), \qquad \forall A \in \mathfrak{A},$$

and by Proposition 3.3, each $\Omega \in \mathcal{F}$ is admissible on $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$. In Example 3.11 it is shown that $\mathcal{F} \neq \mathcal{C}^e(p_{\mathcal{F}})$, in general.

Summarizing the previous results we have:

PROPOSITION 3.4. *Let $\mathcal{F}$ be a strongly balanced subset of $\mathcal{P}_l$. Then each $\Omega \in \mathcal{F}$ is admissible on* $(\mathfrak{A}(\mathcal{F}), \mathfrak{A}_0(\mathcal{F}^0))$.

We now return to the general situation and look for conditions under which $\mathcal{C}(q) = \mathcal{C}^0(q)$, as it happens for the sets of positive linear forms on a *-algebra defined in an analogous way.

PROPOSITION 3.5. *Let q be an extended C^*-seminorm on* $(\mathfrak{A}, \mathfrak{A}_0)$ *and Φ be a q-continuous sesquilinear form on* $(\mathfrak{A}, \mathfrak{A}_0)$. *Assume that $\mathfrak{A}_0$ is q-dense in $\mathfrak{A}$. If $\Phi(\mathbb{I}, \mathbb{I}) = 1$, then*

$$|\Phi(A, B)| \leq q(A)q(B), \qquad \forall A, B \in \mathfrak{A}. \tag{11}$$

Proof. We begin by proving (11) for $A, B \in \mathfrak{A}_0$. Indeed, put

$$\omega(A) = \Phi(A, \mathbb{I}), \qquad A \in \mathfrak{A}_0.$$

Thus, as shown in [12],

$$|\omega(A)| \leq q(A), \qquad \forall A \in \mathfrak{A}_0.$$

Then, for any $A, B \in \mathfrak{A}_0$, we have

$$|\Phi(A, B)| = |\omega(B^*A)| \leq q(B^*A) \leq q(A)q(B).$$

Now, let $A, B \in \mathfrak{A}$. Then there exist sequences $(A_n), (B_n)$ in $\mathfrak{A}_0$ such that $q(A - A_n) \to 0$ and $q(B - B_n) \to 0$. By some simple estimates and making use of the q-continuity of Φ, one can easily prove that

$$\Phi(A, B) = \lim_{n \to \infty} \Phi(A_n, B_n).$$

So finally

$$|\Phi(A, B)| = \lim_{n \to \infty} |\Phi(A_n, B_n)| \leq \lim_{n \to \infty} q(A_n)q(B_n) = q(A)q(B). \blacksquare$$

COROLLARY 3.6. *Assume that* $(\mathfrak{A}, \mathfrak{A}_0)$ *is a normed quasi *-algebra* (*i.e. a topological quasi *-algebra whose topology is defined by a norm $\|\cdot\|$*) *and q an extended C^*-seminorm on* $(\mathfrak{A}, \mathfrak{A}_0)$ *such that for some* $C > 0$,

$$q(A) \leq C\|A\|, \qquad \forall A \in \mathfrak{A}.$$

Then the conclusion of Proposition 3.5 *holds.*

Proof. It is sufficient to observe that, in this case, $\mathfrak{A}_0$ is q-dense in $\mathfrak{A}$. $\blacksquare$

REMARK 3.7. We notice that the q-density of $\mathfrak{A}_0$ in $\mathfrak{A}$ is quite a strong condition. For instance, if $\{A \in \mathfrak{A} : q(A) = 0\} = \{0\}$, then q is a C^*-norm on $\mathfrak{A}_0$. The q-density of $\mathfrak{A}_0$ then forces $\mathfrak{A}$ to be embedded in a C^*-algebra (its completion).

We conclude our discussion with the following

THEOREM 3.8. *Let q be an extended C^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$ and assume that $\mathfrak{A}_0$ is q-dense in $\mathfrak{A}$. Then*:

(i) $\mathcal{C}(q) = \mathcal{C}^e(q)$.

(ii) $\sup_{\Omega \in \mathcal{C}(q)_s} \Omega(X, X) = \sup_{\Omega \in \mathcal{C}(q)_s} \Omega(X^*, X^*),\ \forall X \in \mathfrak{A}$.

(iii) *If $\mathcal{F} = \mathcal{C}(q)$, then $\mathfrak{A}(\mathcal{F}) = \mathfrak{A}$, $\mathfrak{A}_0(\mathcal{F}^0) = \mathfrak{A}_0$ and*

$$q(X)^2 = \sup_{\Omega \in \mathcal{C}(q)_s} \Omega(X, X), \quad \forall X \in \mathfrak{A}. \tag{12}$$

Proof. (i) From Proposition 3.5, it follows that for each $\Omega \in \mathcal{C}(q)$, $\|\Omega\|_q = \Omega(\mathbb{I}, \mathbb{I})$. Thus $\mathcal{C}(q) = \mathcal{C}^0(q)$. But $\mathcal{C}(q)$ satisfies (ii) of Definition 2.4, so finally $\mathcal{C}(q) = \mathcal{C}^0(q) = \mathcal{C}^e(q)$ and each $\Omega \in \mathcal{C}(q)$ is admissible.

(ii) We first show that the equality holds for $X \in \mathfrak{A}_0$. To do this we consider the family of linear functionals $\{\omega_\Omega : \Omega \in \mathcal{C}(q)\}$. Then the desired equality follows immediately from the corresponding equality for balanced families of linear functionals, shown in [12].

To extend the equality to $X \in \mathfrak{A}$, we proceed as follows: the q-density of $\mathfrak{A}_0$ in $\mathfrak{A}$ implies that for each $X \in \mathfrak{A}$ there exists a sequence $\{X_n\} \subset \mathfrak{A}_0$ such that $q(X - X_n) \to 0$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n > n_\varepsilon$,

$$|\Omega(X, X) - \Omega(X_n, X_n)| \le 2(q(X) + \varepsilon) q(X - X_n), \quad \forall \Omega \in \mathcal{C}(q)_s,$$

where we have also made use of (i). This implies that

$$\sup_{\Omega \in \mathcal{C}(q)_s} \Omega(X, X) = \lim_{n \to \infty} \sup_{\Omega \in \mathcal{C}(q)_s} \Omega(X_n, X_n).$$

This easily yields the equality

$$\sup_{\Omega \in \mathcal{C}(q)_s} \Omega(X, X) = \sup_{\Omega \in \mathcal{C}(q)_s} \Omega(X^*, X^*), \quad \forall X \in \mathfrak{A}.$$

(iii) The equalities $\mathfrak{A}(\mathcal{F}) = \mathfrak{A}$, $\mathfrak{A}_0(\mathcal{F}^0) = \mathfrak{A}_0$ are obvious. In order to prove (12), we observe that if $\phi = \{W \in \mathfrak{A} : q(W) = 0\}$, then $\mathfrak{A}/\phi$ is a normed space under the norm $\|X + W\| = q(X)$ for all $X \in \mathfrak{A}$ and $W \in \phi$. Its completion is a C^*-algebra, due to the q-density of $\mathfrak{A}_0$ in $\mathfrak{A}$. Then an argument completely analogous to that of [12, Theorem 2.5] can be applied. ∎

EXAMPLE 3.9. In this example we will consider the special case $\mathfrak{A} = \mathfrak{A}_0$, i.e. $\mathfrak{A}$ is a *-algebra with unit and q is a C^*-seminorm on $\mathfrak{A}$. In this case there is one-to-one correspondence between left-invariant positive sesquilinear forms and positive linear functionals on $\mathfrak{A}$. Indeed, if Ω is a left-invariant positive sesquilinear form then

$$\omega_\Omega(A) = \Omega(A, \mathbb{I}), \qquad A \in \mathfrak{A},$$

is a positive linear functional on $\mathfrak{A}$. Conversely, if ω is a positive linear functional on $\mathfrak{A}$, then

$$\Omega(A, B) = \omega(B^* A), \qquad A, B \in \mathfrak{A},$$

defines a left-invariant positive sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$.

If Ω is q-continuous, then also ω_Ω is q-continuous, in the sense that there exists a $K > 0$ such that

$$|\omega_\Omega(A)| \leq K q(A), \qquad \forall A \in \mathfrak{A}.$$

If $\|\omega_\Omega\|_q^0$ denotes the infimum of these K's, then $\|\omega_\Omega\|_q^0 = \|\Omega\|_q = \Omega(\mathbb{I}, \mathbb{I})$. Therefore $\mathcal{C}(q) = \mathcal{C}^0(q) = \mathcal{C}^e(q)$ (the last equality is due to the fact that $\mathcal{C}(q)$ is strongly balanced).

Example 3.10. Let $\mathcal{H}$ be a Hilbert space and let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be any *-algebra of bounded operators with unit. If $f \in \mathcal{H}$, we put

$$\Omega^f(A, B) = \langle Af, Bf \rangle, \qquad A, B \in \mathfrak{A}.$$

Then each Ω^f is left-invariant and the set $\mathcal{F} = \{\Omega^f : f \in \mathcal{H}\}$ is strongly balanced. In this case $\mathfrak{A}(\mathcal{F}) = \mathfrak{A}_0(\mathcal{F}^0) = \mathfrak{A}$ and $p_\mathcal{F}(A) = \|A\|$ for $A \in \mathfrak{A}$. The set $\mathcal{C}(p_\mathcal{F})$ consists of all sesquilinear forms Φ for which there exists $K > 0$ such that

$$\Phi(A, B) \leq K \|A\| \, \|B\|, \qquad \forall A, B \in \mathfrak{A}.$$

This set properly contains $\mathcal{F}$, in general. As seen in Example 3.9, in this case $\mathcal{C}(p_\mathcal{F}) = \mathcal{C}^0(p_\mathcal{F}) = \mathcal{C}^e(p_\mathcal{F})$ and

$$p_\mathcal{F}(A) = p_\mathcal{C}(A) = \|A\|, \qquad \forall A \in \mathfrak{A}.$$

Example 3.11. Let A be a bounded self-adjoint operator with continuous spectrum $\sigma \subset \mathbb{R}$ and let $C(\sigma)$ denote the *-algebra of all continuous functions on the compact set σ with its usual sup norm $\| \; \|_\infty$. Let

$$\mathfrak{M} = \{f(A) : f \in C(\sigma)\},$$

where $f(A)$ is defined via the functional calculus. As is known, each $f(A)$ is bounded and $\|f(A)\| = \|f\|_\infty$. Then $\mathfrak{M}$ is a C^*-algebra of operators. We take as $\mathcal{F}$ the family of sesquilinear forms considered in Example 3.10.

Fix $\lambda_0 \in \sigma$. We define a sesquilinear form Ω_{λ_0} by

$$\Omega_{\lambda_0}(f(A), g(A)) = f(\lambda_0)\overline{g(\lambda_0)}, \qquad f, g \in C(\sigma).$$

Then Ω_{λ_0} is positive, left-invariant and bounded. Indeed,

$$\begin{aligned} |\Omega_{\lambda_0}(f(A), g(A))| &= |f(\lambda_0)\overline{g(\lambda_0)}| \leq \|f\|_\infty \|g\|_\infty \\ &= \|f(A)\| \, \|g(A)\|, \qquad f, g \in C(\sigma). \end{aligned}$$

This implies that $\|\Omega_{\lambda_0}\|_{p_{\mathcal{F}}} \leq 1$. In fact, it is easy to realize that equality holds true. We now show that it is not possible to find $\eta \in \mathcal{H}$ such that

$$\Omega_{\lambda_0}(f(A), g(A)) = \langle f(A)\eta, g(A)\eta\rangle, \quad \forall f, g \in C(\sigma).$$

Indeed, if $E(\cdot)$ denotes the spectral measure of A, we would have

$$\begin{aligned}\Omega_{\lambda_0}(f(A), g(A)) &= \langle f(A)\eta, g(A)\eta\rangle \\ &= \int_\sigma f(\lambda)\overline{g(\lambda)}\, d\langle E(\lambda)\eta, \eta\rangle = f(\lambda_0)\overline{g(\lambda_0)}, \quad \forall f, g \in C(\sigma),\end{aligned}$$

and this is possible only if λ_0 is an eigenvalue of A. This example shows that, in general, $\mathcal{F} \neq \mathcal{C}^e(p_{\mathcal{F}})$.

4. CQ^*-seminorms on quasi *-algebras. Let q be a Q^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$ and $\mathcal{C}(q)$ the set of all q-continuous sesquilinear forms in $\mathcal{P}_l$. If $\mathcal{F} \subset \mathcal{C}(q)$, we put $\mathcal{F}_u = \{\Omega \in \mathcal{F} : \|\Omega\|_q = 1\}$. Define

$$\pi_{\mathcal{F}}(A) = \max\{\sup_{\Omega\in\mathcal{F}_u} \Omega(A, A)^{1/2}, \sup_{\Omega\in\mathcal{F}_u} \Omega(A^*, A^*)^{1/2}\}.$$

By the q-continuity of each $\Omega \in \mathcal{F}$, we have $\pi_{\mathcal{F}}(A) \leq q(A)$ for every $A \in \mathfrak{A}$; therefore $\pi_{\mathcal{F}}$ is an everywhere defined seminorm on $\mathfrak{A}$. Obviously, $\pi_{\mathcal{F}}(A^*) = \pi_{\mathcal{F}}(A)$ for each $A \in \mathfrak{A}$; hence $\pi_{\mathcal{F}}$ is *-invariant. For simplicity, in what follows we will assume that

$$\pi_{\mathcal{F}}(A) = \sup_{\Omega\in\mathcal{F}_u} \Omega(A, A)^{1/2}, \quad \forall A \in \mathfrak{A}.$$

Since $\Omega(\mathbb{I}, \mathbb{I}) \leq \|\Omega\|_q$ for every $\Omega \in \mathcal{F}$, it also follows that

$$\pi_{\mathcal{F}}(A) \leq p_{\mathcal{F}}(A), \quad \forall A \in \mathfrak{A}.$$

Analogously, for each $\Omega \in \mathcal{F}$ and $A \in \mathfrak{A}$, the positive linear functional ω_Ω^A is q_0-continuous, since

$$|\omega_\Omega^A(X)| \leq \|\Omega\|_q q_0(X) q(A)^2, \quad \forall X \in \mathfrak{A}_0.$$

Therefore $\mathfrak{A}_0(\mathcal{F}^0) = \mathfrak{A}_0$ and $|X|_{\mathcal{F}^0} \leq q_0(X)$ for every $X \in \mathfrak{A}_0$.

PROPOSITION 4.1. *The following inequalities hold*:

$$\pi_{\mathcal{F}}(XA) \leq |X|_{\mathcal{F}^0}^2 \pi_{\mathcal{F}}(A), \quad \pi_{\mathcal{F}}(AX) \leq |X|_{\mathcal{F}^0}^2 \pi_{\mathcal{F}}(A), \quad \forall A \in \mathfrak{A},\ X \in \mathfrak{A}_0.$$

Thus $\pi_{\mathcal{F}}$ is a Q^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$. Moreover $(\pi_{\mathcal{F}})_0(X) = |X|_{\mathcal{F}^0}$ for each $X \in \mathfrak{A}_0$. Therefore $\pi_{\mathcal{F}}$ is a CQ^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$.

Proof. As in the proof of Proposition 2.8, taking into account that for each $\Omega \in \mathcal{F}$, $A \in \mathfrak{A}$ the form ω_Ω^A is $\|\ \|_{\mathcal{F}^0}$-continuous we get, for each $n \in \mathbb{N}$,

$$\Omega(XA, XA) \leq \Omega(A, A)^{1-2^{-n}} (\|\omega_\Omega^A\|_{\mathcal{F}^0} |(X^*X)^{2^n}|_{\mathcal{F}^0}^{2^{-n}}), \quad \forall X \in \mathfrak{A}_0.$$

Letting $n \to \infty$, we have

$$\Omega(XA, XA) \leq |X^*X|_{\mathcal{F}^0}\, \Omega(A, A).$$

This in turn implies that

$$\pi_{\mathcal{F}}(XA) \leq |X|^2_{\mathcal{F}^0}\pi_{\mathcal{F}}(A), \qquad \forall A \in \mathfrak{A},\ X \in \mathfrak{A}_0.$$

From this estimate it also follows that $(\pi_{\mathcal{F}})_0(X) \leq |X|_{\mathcal{F}^0}$ for each $X \in \mathfrak{A}_0$.

To complete the proof we only need to prove the opposite inequality. For this we observe that for $\Omega \in \mathcal{F}$, one has

$$|\Omega(XA, A)| \leq \|\Omega\|_q \pi_{\mathcal{F}}(XA)\pi_{\mathcal{F}}(A) \leq \|\Omega\|_q (\pi_{\mathcal{F}})_0(X)\pi_{\mathcal{F}}(A)^2$$

for each $X \in \mathfrak{A}_0(\mathcal{F}^0)$ and $A \in \mathfrak{A}$; therefore each ω^A_Ω is $(\pi_{\mathcal{F}})_0$-continuous. Then proceeding as before we can prove

$$\omega^A_\Omega(X^*X) = \Omega(XA, XA) \leq (\pi_{\mathcal{F}})_0(X^*X)\Omega(A, A),$$

and this implies that

$$|X|_{\mathcal{F}^0} \leq (\pi_{\mathcal{F}})_0(X), \quad \forall X \in \mathfrak{A}_0. \ \blacksquare$$

In conclusion, $\pi_{\mathcal{F}}$ is a CQ^*-seminorm on $(\mathfrak{A}, \mathfrak{A}_0)$, for any $\mathcal{F} \subset \mathcal{C}(q)$. The maximal such seminorm is obtained, of course, for $\mathcal{F} = \mathcal{C}(q)$. So the question arises whether $\pi_{\mathcal{C}(q)} = q$. We do not have a complete answer to this question. We only mention that also in simple cases very extreme situations may occur, as the next example shows.

Example 4.2. We consider once more the situation of Example 2.10, i.e. the quasi *-algebra $(L^p(I), C(I))$ with $p \geq 2$. Each sesquilinear form $\Omega^{(w)} \in \mathcal{F}$ is continuous with respect to the L^p-norm $\|\ \|_p$ (which we take as the q of the previous discussion). In this case, $\|\ \|_p$ is a CQ^*-norm and one finds that $\pi_{\mathcal{F}}(f) = \|f\|_p$ for every $f \in L^p(I)$ and $(\pi_{\mathcal{F}})_0(\phi) = \|\phi\|_\infty$ for every $\phi \in C(I)$.

If $1 \leq p < 2$, the L^p-norm is still a CQ^*-norm. But $C(q) = \{0\}$ and so $\pi_{\mathcal{F}}(f) = 0$ for each $f \in L^p(I)$. For the details we refer to [4].

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Dipartimento di Matematica e Applicazioni
Università di Palermo
I-90123 Palermo, Italy
E-mail: trapani@unipa.it

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