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Diffusion phenomenon for second order linear evolution equations

by

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Abstract. We present an abstract theory of the diffusion phenomenon for second order linear evolution equations in a Hilbert space. To derive the diffusion phenomenon, a new device developed in Ikehata–Matsuyama [5] is applied. Several applications to damped linear wave equations in unbounded domains are also given.

1. Introduction. Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$, and let $A : D(A) \subset H \to H$ be a nonnegative self-adjoint operator in H with dense domain $V = D(A)$. Then it is well known that the fractional power $A^{1/2} : D(A^{1/2}) \to H$ is well defined with dense domain $W = D(A^{1/2})$, and $A^{1/2}$ is also a nonnegative self-adjoint operator in H. In this article we are concerned with the following abstract Cauchy problems in H:

$$u''(t) + Au(t) + u'(t) = 0, \qquad t > 0, \qquad \text{in } H, \tag{1.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{1.2}$$

and

$$v'(t) + Av(t) = 0, \qquad t > 0, \qquad \text{in } H, \tag{1.3}$$

$$v(0) = u_0 + u_1, \tag{1.4}$$

where $u'(t) = \frac{d}{dt}u(t)$ and so on.

As solution spaces, we set

$$X_2(0,\infty) = C([0,\infty); V) \cap C^1([0,\infty); W) \cap C^2([0,\infty); H),$$
$$Y_1(0,\infty) = C([0,\infty); W) \cap C^1((0,\infty); H) \cap C((0,\infty); V),$$

and also

$$X_3(0,\infty) = C([0,\infty); W_3) \cap C^1([0,\infty); V) \cap C^2([0,\infty); W) \cap C^3([0,\infty); H),$$

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where $W_3 = D(A^{3/2})$ is a real Hilbert space with the usual graph norm. We denote by $|u|_V$ and $|v|_W$ the V-graph norm of u and W-graph norm of v, respectively. The space $X_3(0,\infty)$ will be used in the proof of estimates for higher order derivatives (see Lemma 2.2 below).

In 1997 Nishihara [8] described the so-called diffusion phenomenon for quasilinear damped wave equations on 1-dimensional Euclidian space $\mathbb{R}$ in a concrete context, and Han–Milani [2] extended Nishihara's results to the case of N-dimensional Euclidian space $\mathbb{R}^N$ for any quasilinear damped wave equation (see also Milani–Han [7] for another type of diffusion phenomenon). Furthermore, in [6] Karch has discovered the asymptotic self-similarity as $t \to \infty$ of solutions to the equation (1.1) with $A = -\Delta$ in $\mathbb{R}^N$ (in fact, he deals with more general dissipative wave equations). These results imply that the solution of a damped wave equation is asymptotically equal to that of the corresponding heat equation as $t \to \infty$. Recently, Nishihara [9] has succeeded in deriving L^p-L^q estimates for the difference $u(t,x) - v(t,x)$, where $u(t,x)$ and $v(t,x)$ represent the solutions of the Cauchy problem in $\mathbb{R}^3$ for a linear damped wave equation and the corresponding heat equation with initial data like (1.4), respectively. On the other hand, quite recently, Ikehata [4] has studied the diffusion phenomenon for the "exterior" mixed problem for linear damped wave equations through a new device, which has its origin in Ikehata–Matsuyama [5]. Unfortunately, the decay rate of $u(t,x) - v(t,x)$ obtained in [4] is not optimal.

In this paper, our purpose is to derive the "diffusion phenomenon" for the "abstract" Cauchy problem (1.1)–(1.2) by using the device of [5], and to consider the optimal rate of decay for $u(t) - v(t)$ which implies the diffusion phenomenon in the abstract framework. We emphasize that the results in [5] cannot be applied in the abstract setting.

Our main result reads as follows.

Theorem 1.1. *Let* $[u_0, u_1] \in V \times W$. *Then the solutions* $u \in X_2(0,\infty)$ *to the problem* (1.1)–(1.2) *and* $v \in Y_1(0,\infty)$ *to the problem* (1.3)–(1.4) *satisfy*

$$|u(t) - v(t)| \leq C I_0 (1+t)^{-1} (\log(2+t))^{(1+\varepsilon)/2}$$

for any $\varepsilon > 0$, *where*

$$I_0 = |u_0|_V + |u_1|_W.$$

Remark 1.1. For the initial data $[u_0, u_1]$ in Theorem 1.1, both $u(t)$ and $v(t)$ are expected to be only bounded. However, basing on Nishihara's work [9, Theorem 1.1], we conjecture that the optimal estimate which implies the diffusion phenomenon for (1.1)–(1.2) is

$$|u(t) - v(t)| \leq C(1+t)^{-1}.$$

Hence the estimate we obtained is almost optimal.

In order to illustrate our results, let us take

$$H = L^2(\Omega), \quad A = -\Delta \quad \text{with} \quad V = H^2(\Omega) \cap H_0^1(\Omega),$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain with smooth boundary $\partial\Omega$, or $\Omega = \mathbb{R}^N$. Then the problems (1.1)–(1.4) are the following mixed problems:

$$u_{tt} - \Delta u + u_t = 0 \quad \text{in } (0,\infty) \times \Omega, \tag{1.5}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \quad \text{in } \Omega, \tag{1.6}$$

$$u|_{\partial\Omega} = 0 \quad \text{if } \partial\Omega \neq \emptyset, \tag{1.7}$$

$$v_t - \Delta v = 0 \quad \text{in } (0,\infty) \times \Omega, \tag{1.8}$$

$$v(0,x) = u_0(x) + u_1(x) \quad \text{in } \Omega, \tag{1.9}$$

$$v|_{\partial\Omega} = 0 \quad \text{if } \partial\Omega \neq \emptyset. \tag{1.10}$$

Furthermore, if we take

$$H = L^2(\Omega), \quad A = -\Delta \quad \text{with} \quad V = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},$$

where $\nu(x)$ represents the usual unit outward normal vector at $x \in \partial\Omega$, then the problems (1.1)–(1.4) are the initial-value problems with homogeneous Neumann boundary condition corresponding to (1.5)–(1.6) and (1.8)–(1.9).

2. Proof of Theorem 1.1. We shall prove Theorem 1.1 using a new device, which has its origin in [5]. Our argument is based on the following well-posedness result (cf. Ikawa [3] and Cazenave–Haraux [1]).

PROPOSITION 2.1. *For each* $(u_0, u_1) \in V \times W$, *there exists a unique solution* $u \in X_2(0,\infty)$ *to the problem* (1.1)–(1.2) *satisfying*

$$E_u(t) + \int_0^t |u'(\tau)|^2 \, d\tau = E_u(0), \tag{2.1}$$

where

$$E_u(t) = \frac{1}{2}\left(|u'(t)|^2 + |A^{1/2}u(t)|^2\right).$$

If, *in particular*, $(u_0, u_1) \in W_3 \times V$, *then we have the additional property*: $u \in X_3(0,\infty)$.

Furthermore, *for each* $v_0 = u_0 + u_1 \in W$, *there exists a unique solution* $v \in Y_1(0,\infty)$ *to the problem* (1.3)–(1.4).

To prove Theorem 1.1, we set

$$w(t) = u(t) - v(t).$$

Then w becomes the solution to the problem

$$\begin{aligned} &w'(t) + Aw(t) = -u''(t), \quad t \in (0,\infty), \quad \text{in } H, \\ &w(0) = -u_1. \end{aligned} \tag{2.2}$$

Set furthermore

$$Z(t) = \int_0^t w(s)\, ds,$$

following [5]. Then $Z = Z(t)$ satisfies

$$Z'(t) + AZ(t) = -u'(t), \quad t \in (0, \infty), \quad \text{in } H, \tag{2.3}$$

$$Z(0) = 0, \tag{2.4}$$

where we have used the special form (1.4) of the initial data.

To analyse (2.3) and (2.2) we need the information on $u'(t)$ and $u''(t)$, which is summed up in $\int_0^t (1+\tau)|u'(\tau)|^2\, d\tau \le C$ and $\int_0^t (1+\tau)^3 |u''(\tau)|^2\, d\tau \le C$. So, we shall prepare several facts concerning (1.1)–(1.4).

Lemma 2.1. *Let $u \in X_2(0,\infty)$ be a solution to the problem* (1.1)–(1.2) *and $v \in Y_1(0,\infty)$ be a solution to* (1.3)–(1.4). *Then*

$$(1+t)E_u(t) + \int_0^t (1+\tau)|u'(\tau)|^2\, dt \le C(|u_0|_W^2 + |u_1|^2), \tag{2.5}$$

$$\int_0^t (1+\tau)|v'(\tau)|^2\, d\tau \le C|v_0|_W^2, \tag{2.6}$$

with some constant $C > 0$, where $v_0 = u_0 + u_1$.

Proof. First, we shall prove (2.5). It follows from Proposition 2.1 that

$$\frac{d}{dt} E_u(t) + |u'(t)|^2 = 0. \tag{2.7}$$

Taking the inner product of both sides of (1.1) with $u'(t) + \frac{1}{2}u(t)$, we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\left(|u'(t)|^2 + (u'(t), u(t)) + \frac{1}{2}|u(t)|^2 + |A^{1/2}u(t)|^2 \right) \\ &\quad + \frac{1}{2}(|u'(t)|^2 + |A^{1/2}u(t)|^2) \\ &=: \frac{d}{dt} J_u(t) + E_u(t). \end{aligned} \tag{2.8}$$

We note that $J_u(t)$ is equivalent to $E_u(t) + |u(t)|^2$, that is,

$$C^{-1}(E_u(t) + |u(t)|^2) \le J_u(t) \le C(E_u(t) + |u(t)|^2). \tag{2.9}$$

Integrating (2.8) over $[0, t]$, we have

$$J_u(t) + \int_0^t E_u(\tau)\, d\tau \le J_u(0) \le C(|u_0|_W^2 + |u_1|^2). \tag{2.10}$$

Hence, multiplying (2.7) by $1+t$ and integrating the resulting equation over $[0,t]$, we get

$$(1+t)E_u(t)+\int_0^t(1+\tau)|u'(\tau)|^2\,d\tau \le E_u(0)+\int_0^t E_u(\tau)\,d\tau \le C(|u_0|_W^2+|u_1|^2),$$

which shows (2.5).

Next, we shall prove (2.6). Taking the inner product of both sides of (1.3) with $v'(t)$ and integrating it over $[0,t]$ we obtain

$$|v'(t)|^2=-\frac{1}{2}\frac{d}{dt}|A^{1/2}v(t)|^2.$$

Thus, we see that

$$\int_0^t(1+\tau)|v'(\tau)|^2d\tau=-\frac{1}{2}\int_0^t(1+\tau)\frac{d}{d\tau}|A^{1/2}v(\tau)|^2\,d\tau \tag{2.11}$$
$$=-\frac{1}{2}(1+t)|A^{1/2}v(t)|^2+\frac{1}{2}|A^{1/2}v_0|^2+\frac{1}{2}\int_0^t|A^{1/2}v(\tau)|^2\,d\tau.$$

On the other hand, taking the inner product of both sides of (1.3) with $v(t)$ and integrating it over $[0,t]$ we see that

$$\frac{1}{2}|v(t)|^2+\int_0^t|A^{1/2}v(\tau)|^2\,d\tau=\frac{1}{2}|v_0|^2. \tag{2.12}$$

Therefore, (2.11) and (2.12) imply the desired estimate (2.6). ■

LEMMA 2.2. *The solution $u\in X_2(0,\infty)$ of* (1.1)–(1.2) *satisfies*

$$\int_0^t(1+\tau)^3|u''(\tau)|^2\,d\tau\le CI_0^2. \tag{2.13}$$

Proof. We may assume that $u(t)$ is sufficiently smooth, say $(u_0,u_1)\in W_3\times V$, because it can be approximated by smooth solutions $\{v_n(t)\}\subset X_3(0,\infty)$ $(n=1,2,\ldots)$ to the problem (1.1)–(1.2) in the $X_2(0,\infty)$ topology.

Now for the solution $u\in X_3(0,\infty)$, we set $a(t)=u'(t)$. Then $a(t)$ becomes the strong solution to

$$a''(t)+Aa(t)+a'(t)=0,\qquad t>0,\qquad \text{in } H, \tag{2.14}$$
$$a(0)=u_1,\qquad a'(0)=-Au_0-u_1. \tag{2.15}$$

By applying Proposition 2.1 to the problem (2.14)–(2.15) we have

$$\frac{d}{dt}E_{u'}(t)+|u''(t)|^2=0 \tag{2.16}$$

and also

$$\frac{d}{dt}J_{u'}(t)+E_{u'}(t)=0. \tag{2.17}$$

Noting (2.5) and (2.9) and multiplying (2.17) by $(1+t)^k$, $k=0,1,2$, we iteratively have

$$J_{u'}(t)+\int_0^t E_{u'}(\tau)\,d\tau \le J_{u'}(0)\le CI_0^2,$$

$$\begin{aligned}(1+t)J_{u'}(t)+\int_0^t(1+\tau)E_{u'}(\tau)\,d\tau &\le J_{u'}(0)+\int_0^t J_{u'}(\tau)\,d\tau\\ &\le C\Big(I_0^2+\int_0^t(E_{u'}(\tau)+|u'(\tau)|^2)\,d\tau\Big)\le CI_0^2,\end{aligned}$$

and

$$\begin{aligned}(1+t)^2J_{u'}(t)+\int_0^t(1+\tau)^2E_{u'}(\tau)\,d\tau &\le J_{u'}(0)+2\int_0^t(1+\tau)J_{u'}(\tau)\,d\tau\\ &\le C\Big(I_0^2+2\int_0^t(1+\tau)(E_{u'}(\tau)+|u'(\tau)|^2)\,d\tau\Big)\le CI_0^2.\end{aligned} \tag{2.18}$$

Using (2.18) we multiply (2.16) by $(1+t)^3$ and integrate the resulting equation to obtain

$$(1+t)^3E_{u'}(t)+\int_0^t(1+\tau)^3|u''(\tau)|^2\,d\tau \le E_{u'}(0)+3\int_0^t(1+\tau)^2E_{u'}(\tau)\,d\tau\le CI_0^2,$$

which shows (2.13). ■

The following lemmas can also be shown by the device of [5] basing on Lemmas 2.1–2.2.

LEMMA 2.3. *Under the assumptions of Theorem* 1.1,

$$\begin{aligned}(\log(e+t))^{-1-\varepsilon}|Z(t)|^2+\int_0^t(\log(e+\tau))^{-1-\varepsilon}|A^{1/2}Z(\tau)|^2\,d\tau&\\ \le C(|u_0|_W^2+|u_1|^2)&,\end{aligned} \tag{2.19}$$

where $Z(t)$ is the function defined in (2.3).

Proof. Taking the inner product of both sides of (2.3) with $Z(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|Z(t)|^2+|A^{1/2}Z(t)|^2=-(u'(t),Z(t))\le|u'(t)|\,|Z(t)|. \tag{2.20}$$

Multiplying (2.20) by $(\log(e+t))^{-1-\varepsilon}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\{(\log(e+t))^{-1-\varepsilon}|Z(t)|^2\} + \frac{1}{2}(1+\varepsilon)(\log(e+t))^{-2-\varepsilon}\frac{1}{e+t}|Z(t)|^2 + (\log(e+t))^{-1-\varepsilon}|A^{1/2}Z(t)|^2$$
$$\le (1+t)^{1/2}|u'(t)|(1+t)^{-1/2}(\log(e+t))^{-(1+\varepsilon)/2-(1+\varepsilon)/2}|Z(t)|$$
$$\le \frac{1}{2}(1+t)|u'(t)|^2 + \frac{1}{2}(1+t)^{-1}(\log(e+t))^{-1-\varepsilon}(\log(e+t))^{-1-\varepsilon}|Z(t)|^2,$$

which implies, by Lemma 2.1,

$$\frac{1}{2}(\log(e+t))^{-1-\varepsilon}|Z(t)|^2 + \int_0^t (\log(e+\tau))^{-1-\varepsilon}|A^{1/2}Z(\tau)|^2\,d\tau$$
$$\le C(|u_0|_W^2 + |u_1|^2) + \int_0^t (1+\tau)^{-1}(\log(e+\tau))^{-1-\varepsilon}\cdot\frac{1}{2}(\log(e+\tau))^{-1-\varepsilon}|Z(\tau)|^2\,d\tau.$$

The desired estimate follows from the Gronwall inequality, because

$$(1+t)^{-1}(\log(e+t))^{-1-\varepsilon} \in L^1(0,\infty). \tag{2.21}$$ ■

LEMMA 2.4. *Under the assumptions of Theorem* 1.1,

$$\int_0^t \frac{e+\tau}{(\log(e+\tau))^{1+\varepsilon}}|Z'(\tau)|^2\,d\tau + \frac{e+t}{(\log(e+t))^{1+\varepsilon}}|A^{1/2}Z(t)|^2 \le CI_0^2. \tag{2.22}$$

Proof. Taking the inner product of both sides of (2.3) with $Z'(t)$, we have

$$|Z'(t)|^2 + \frac{1}{2}\frac{d}{dt}|A^{1/2}Z(t)|^2 = -(u'(t), Z'(t)).$$

This implies

$$|Z'(t)|^2 + \frac{d}{dt}|A^{1/2}Z(t)|^2 \le |u'(t)|^2. \tag{2.23}$$

Next, multiplying both sides of (2.23) by $(e+t)(\log(e+t))^{-1-\varepsilon}$ we see that

$$(e+t)(\log(e+t))^{-1-\varepsilon}|Z'(t)|^2 + \frac{d}{dt}\{(e+t)(\log(e+t))^{-1-\varepsilon}|A^{1/2}Z(t)|^2\}$$
$$\le (\log(e+t))^{-1-\varepsilon}\left(1 - \frac{1+\varepsilon}{\log(e+t)}\right)|A^{1/2}Z(t)|^2 + (e+t)(\log(e+t))^{-1-\varepsilon}|u'(t)|^2.$$

By integrating over $[0,t]$ and using (2.5) and Lemma 2.3, we obtain the desired estimate. ■

Since $Z'(t) = w(t)$, as a corollary we have

COROLLARY 2.1. *Under the assumptions of Lemma* 2.4,

$$\int_0^t (e+\tau)(\log(e+\tau))^{-1-\varepsilon}|w(\tau)|^2\,d\tau \le CI_0^2.$$

Now let us prove Theorem 1.1.

Proof of Theorem 1.1. Similarly to the proof of Lemma 2.3, taking the inner product of both sides of (2.2) with $(e+t)^2(\log(e+t))^{-1-\varepsilon}w(t)$, we see that

$$\frac{(e+t)^2(\log(e+t))^{-1-\varepsilon}}{2}\frac{d}{dt}|w(t)|^2 + (e+t)^2(\log(e+t))^{-1-\varepsilon}|A^{1/2}w(t)|^2 = -(e+t)^2(\log(e+t))^{-1-\varepsilon}(u''(t),w(t)),$$

so that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{(e+t)^2(\log(e+t))^{-1-\varepsilon}|w(t)|^2\} + (e+t)^2(\log(e+t))^{-1-\varepsilon}|A^{1/2}w(t)|^2 \\ = \left\{(e+t)(\log(e+t))^{-1-\varepsilon} - \frac{1+\varepsilon}{2}(e+t)(\log(e+t))^{-2-\varepsilon}\right\}|w(t)|^2 \\ + (e+t)^{3/2}|u''(t)|\cdot(e+t)^{-1/2}(\log(e+t))^{-(1+\varepsilon)/2} \\ \times (e+t)(\log(e+t))^{-(1+\varepsilon)/2}|w(t)| \\ \le (e+t)(\log(e+t))^{-1-\varepsilon}|w(t)|^2 + \frac{1}{2}(e+t)^3|u''(t)|^2 \\ + (e+t)^{-1}(\log(e+t))^{-(1+\varepsilon)}\cdot\frac{1}{2}(e+t)^2(\log(e+t))^{-(1+\varepsilon)}|w(t)|^2.\end{aligned}$$

Integrating over $[0,t]$, and using Lemma 2.2 and Corollary 2.1, we get

$$\begin{aligned}\frac{(e+t)^2}{2(\log(e+t))^{1+\varepsilon}}|w(t)|^2 + \int_0^t \frac{(e+\tau)^2}{2(\log(e+\tau))^{1+\varepsilon}}|A^{1/2}w(\tau)|^2\,d\tau \\ \le CI_0^2 + \int_0^t (e+\tau)^{-1}(\log(e+\tau))^{-1-\varepsilon}\cdot\frac{(e+\tau)^2}{2(\log(e+\tau))^{1+\varepsilon}}|w(\tau)|^2\,d\tau.\end{aligned}$$

By (2.21) the Gronwall inequality yields the desired estimate. ∎

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