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A Morita equivalence for Hilbert C^*-modules

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Abstract. We introduce a notion of Morita equivalence for Hilbert C^*-modules in terms of the Morita equivalence of the algebras of compact operators on Hilbert C^*-modules. We investigate the properties of the new Morita equivalence. We apply our results to study continuous actions of locally compact groups on full Hilbert C^*-modules. We also present an extension of Green's theorem in the context of Hilbert C^*-modules.

1. Introduction. The notion of a Hilbert C^*-module is a generalization of that of a Hilbert space in which the inner product takes values in a C^*-algebra rather than in the field of complex numbers. Hilbert C^*-modules are useful tools in AW^*-algebra theory, theory of operator algebras, operator K-theory, Morita equivalence of C^*-algebras, group representation theory and theory of operator spaces. The theory of Hilbert C^*-modules is very interesting in its own right. If E is a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$, then we denote by $\mathcal{L}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{A}}(E)$, respectively, the C^*-algebra of all adjointable maps on E and the C^*-algebra of all "compact" operators on E. The linking algebra associated to a Hilbert C^*-module E over a C^*-algebra $\mathscr{A}$ is denoted by $\mathcal{L}(E)$. If $\mathscr{H}$ is an infinite-dimensional separable Hilbert space, then $\mathbb{K}$ denotes the C^*-algebra of all compact operators on $\mathscr{H}$.

The notion of (strong) Morita equivalence of C^*-algebras was first introduced by Rieffel [15]. Two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ are said to be *Morita equivalent* if there exists an $\mathscr{A}$-$\mathscr{B}$-imprimitivity bimodule, or equivalently, there exists a full Hilbert $\mathscr{B}$-module E such that $\mathscr{A}$ is isomorphic to the C^*-algebra $\mathcal{K}_{\mathscr{B}}(E)$. There are other equivalent definitions of Morita equivalence in the literature (see [14]). This concept is weaker than the notion of C^*-isomorphism. It is an interesting problem to study the properties of C^*-algebras preserved under Morita equivalence. It is known that Morita equivalence preserves K-theory and K-homology and several properties of C^*-algebras such as type I-ness [15], nuclearity [3] and simplicity [14]. Two

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unital C^*-algebras are Morita equivalent if and only if they are Morita equivalent as rings (cf. [3]). Also two C^*-algebras are Morita equivalent if and only if their minimal dense ideals are Morita equivalent (cf. [1]). Foundations of Morita equivalence theory for operator algebras are established in [4]. Muhly and Solel [12] defined a notion of Morita equivalence for C^*-correspondences.

A Hilbert $\mathscr{A}$-module E and a Hilbert $\mathscr{B}$-module F are *Morita equivalent in the sense of Skeide* [16] if there exists a Morita equivalence M from $\mathscr{A}$ to $\mathscr{B}$ such that $E \otimes M = F$ (or $E = F \otimes M^*$). In [17, Definition 3.4], two Hilbert C^*-module E and F are called *stably Morita equivalent* if $E \otimes \mathscr{H}$ and $F \otimes \mathscr{H}$ are Morita equivalent, where $\mathscr{H}$ denotes any infinite-dimensional separable Hilbert space. Two full Hilbert C^*-modules E and F are Morita equivalent in the sense of Skeide if and only if the C^*-algebras $\mathcal{L}_{\mathscr{A}}(E)$ and $\mathcal{L}_{\mathscr{B}}(F)$ are bistrictly isomorphic, and this occurs if and only if the C^*-algebras $\mathcal{K}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{B}}(F)$ are isomorphic [17, Corollaries 2.13, 2.14 and 2.16].

If two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent as Hilbert C^*-modules, then $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent as C^*-algebras (in fact, $\mathscr{A}$ and $\mathscr{B}$ are isomorphic). The converse is not true. So the notion of Morita equivalence introduced by Skeide is stronger than Rieffel's.

In this paper we introduce a notion of Morita equivalence for Hilbert C^*-modules. It is defined as Morita equivalence of the algebras of compact operators on Hilbert C^*-modules. This notion is weaker than that of Skeide but under some countability hypotheses (σ-unital C^*-algebras and countably generated modules) our definition coincides with Skeide's definition of stable Morita equivalence. We investigate some properties of the new version of Morita equivalence. We apply our results to study continuous actions of locally compact groups on full Hilbert C^*-modules. We also present an extension of Green's theorem in the context of Hilbert C^*-modules.

2. Results. We start this section with the following essential definition.

Definition 1. Two Hilbert C^*-modules E and F over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, are *Morita equivalent*, denoted by $E \backsim_M F$, if the C^*-algebras $\mathcal{K}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{B}}(F)$ are Morita equivalent.

It is well known that any C^*-algebra $\mathscr{A}$ is a Hilbert $\mathscr{A}$-module in a natural way and the C^*-algebras $\mathscr{A}$ and $\mathcal{K}_{\mathscr{A}}(\mathscr{A})$ are isomorphic, hence Morita equivalent. Therefore, two C^*-algebras $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent as Hilbert C^*-modules if and only if they are Morita equivalent as C^*-algebras.

Since the Morita equivalence of C^*-algebras is an equivalence relation, so is the Morita equivalence $\backsim_M$ of Hilbert C^*-modules.

Example 2. Let $\mathscr{H}$ be a separable infinite-dimensional Hilbert space. Then $\mathscr{H} \backsim_M \mathbb{C} \backsim_M \mathbb{K}$ as Hilbert C^*-modules, since the C^*-algebras $\mathbb{C}$ and $\mathbb{K}$ are Morita equivalent.

A *morphism of Hilbert C^*-modules* from a Hilbert C^*-module E over $\mathscr{A}$ to a Hilbert C^*-module F over $\mathscr{B}$ is a map $\Phi : E \to F$ with the property that there is a C^*-morphism $\varphi : \mathscr{A} \to \mathscr{B}$ such that

$$\langle \Phi(\xi_1), \Phi(\xi_2)\rangle = \varphi(\langle \xi_1, \xi_2\rangle)$$

for all $\xi_1, \xi_2 \in E$. Two Hilbert C^*-modules E and F over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, are *isomorphic* if there is a bijective map $\Phi : E \to F$ such that Φ and Φ^{-1} are morphisms of Hilbert C^*-modules.

PROPOSITION 3. *Let E and F be Hilbert C^*-modules over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. If E and F are isomorphic, then $E \backsim_M F$.*

Proof. Since the Hilbert C^*-modules E and F are isomorphic, the C^*-algebras $\mathcal{K}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{B}}(F)$ are isomorphic [2, Proposition 2.11], and so Morita equivalent. Therefore $E \backsim_M F$. ∎

Given two Hilbert C^*-modules E and F over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, recall that their *exterior tensor product* $E \otimes F$ is a Hilbert C^*-module over the injective tensor product $\mathscr{A} \otimes \mathscr{B}$ (see [11]).

PROPOSITION 4. *Let E_1, E_2, F_1, F_2 be Hilbert C^*-modules over C^*-algebras $\mathscr{A}_1, \mathscr{A}_2, \mathscr{B}_1, \mathscr{B}_2$, respectively. If $E_1 \backsim_M F_1$ and $E_2 \backsim_M F_2$, then $E_1 \otimes E_2 \backsim_M F_1 \otimes F_2$.*

Proof. From $E_i \backsim_M F_i$, $i = 1, 2$, we have $\mathcal{K}_{\mathscr{A}_i}(E_i) \backsim_M \mathcal{K}_{\mathscr{B}_i}(F_i)$ $(i = 1, 2)$, and so $\mathcal{K}_{\mathscr{A}_1}(E_1) \otimes \mathcal{K}_{\mathscr{A}_2}(E_2) \backsim_M \mathcal{K}_{\mathscr{B}_1}(F_1) \otimes \mathcal{K}_{\mathscr{B}_2}(F_2)$. But the C^*-algebras $\mathcal{K}_{\mathscr{A}_1}(E_1) \otimes \mathcal{K}_{\mathscr{A}_2}(E_2)$ and $\mathcal{K}_{\mathscr{A}_1 \otimes \mathscr{A}_2}(E_1 \otimes E_2)$ are isomorphic, as also are the C^*-algebras $\mathcal{K}_{\mathscr{B}_1}(F_1) \otimes \mathcal{K}_{\mathscr{B}_2}(F_2)$ and $\mathcal{K}_{\mathscr{B}_1 \otimes \mathscr{B}_2}(F_1 \otimes F_2)$ (see, for example, [11, p. 57]). Therefore $\mathcal{K}_{\mathscr{A}_1 \otimes \mathscr{A}_2}(E_1 \otimes E_2) \backsim_M \mathcal{K}_{\mathscr{B}_1 \otimes \mathscr{B}_2}(F_1 \otimes F_2)$. ∎

COROLLARY 5. *Let E be a Hilbert C^*-module. Then $E \backsim_M E \otimes \mathscr{H} \backsim_M E \otimes \mathbb{K}$.*

Proof. From Example 2 and Proposition 4, we have $E \otimes \mathbb{C} \backsim_M E \otimes \mathscr{H} \backsim_M E \otimes \mathbb{K}$. Since the Hilbert C^*-modules E and $E \otimes \mathbb{C}$ are isomorphic we have $E \backsim_M E \otimes \mathbb{C}$. ∎

COROLLARY 6. *Let E and F be Hilbert C^*-modules over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. If the Hilbert C^*-modules $E \otimes \mathscr{H}$ and $F \otimes \mathscr{H}$ are isomorphic for some separable Hilbert space $\mathscr{H}$, then $E \backsim_M F$.*

Proof. If the Hilbert C^*-modules $E \otimes \mathscr{H}$ and $F \otimes \mathscr{H}$ are isomorphic, then, by Proposition 3, $E \otimes \mathscr{H} \backsim_M F \otimes \mathscr{H}$. By Corollary 5, $E \backsim_M E \otimes \mathscr{H}$ and $F \backsim_M F \otimes \mathscr{H}$. Hence $E \backsim_M F$. ∎

Let E and F be Hilbert C^*-modules over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, and let $\Phi : \mathscr{A} \to \mathcal{K}_{\mathscr{B}}(F)$ be a C^*-morphism. Recall that the *inner tensor product* $E \otimes_{\Phi} F$ of E and F corresponding to Φ is a Hilbert C^*-module over $\mathscr{B}$ (see [11]).

Proposition 7. *Let E_1, E_2, F_1, F_2 be Hilbert C^*-modules over C^*-algebras $\mathscr{A}_1, \mathscr{A}_2, \mathscr{B}_1, \mathscr{B}_2$, respectively. If $\Phi_i : \mathscr{A}_i \to \mathcal{K}_{\mathscr{B}_i}(F_i)$ $(i = 1, 2)$ are C^*-isomorphisms and $E_1 \backsim_M E_2$, then $E_1 \otimes_{\Phi_1} F_1 \backsim_M E_2 \otimes_{\Phi_2} F_2$.*

Proof. By [11, Proposition 4.7], we know that the C^*-algebras $\mathcal{K}_{\mathscr{A}_i}(E_i)$ and $\mathcal{K}_{\mathscr{B}_i}(E_i \otimes_{\Phi_i} F_i)$ are isomorphic for $i = 1, 2$, and so $E_i \backsim_M E_i \otimes_{\Phi_i} F_i$ for $i = 1, 2$. It follows from $E_1 \backsim_M E_2$ that $E_1 \otimes_{\Phi_1} F_1 \backsim_M E_2 \otimes_{\Phi_2} F_2$. ∎

Proposition 8. *Let E and F be full Hilbert C^*-modules over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Then the following assertions are equivalent:*

(1) $E \backsim_M F$;
(2) $\mathcal{L}(E) \backsim_M \mathcal{L}(F)$;
(3) $\mathscr{A} \backsim_M \mathscr{B}$.

Proof. (1)⇔(2). Since E and F are full, $\mathcal{K}_{\mathscr{A}}(E) \backsim_M \mathcal{L}(E)$ and $\mathcal{K}_{\mathscr{B}}(F) \backsim_M \mathcal{L}(F)$. Therefore, $E \backsim_M F$ if and only if $\mathcal{L}(E) \backsim_M \mathcal{L}(F)$.

For (2)⇔(3) see [5, Theorem 1.1]. ∎

Corollary 9. *Let E be a full Hilbert $\mathscr{A}$-module. Then $E \backsim_M E \oplus \mathscr{A} \backsim_M \mathscr{A} \otimes \mathscr{H} \backsim_M E \otimes \mathscr{H}$.*

Given a Hilbert C^*-module E over a C^*-algebra $\mathscr{A}$, the vector space $\mathcal{L}_{\mathscr{A}}(\mathscr{A}, E)$ of all adjointable module morphisms from $\mathscr{A}$ into E has a canonical Hilbert C^*-module structure over the multiplier algebra $M(\mathscr{A})$ of $\mathscr{A}$, which is called the *multiplier module* of E (see [13]). The vector space $\mathcal{K}_{\mathscr{A}}(E, \mathscr{A})$ of all compact operators from E into $\mathscr{A}$ has a natural Hilbert C^*-module structure over $\mathcal{K}_{\mathscr{A}}(E)$, which is denoted by E^*.

Theorem 10. *Let E and F be Hilbert C^*-modules over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively.*

(1) *If the Hilbert C^*-modules $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are isomorphic, then $E \backsim_M F$.*
(2) *If E and F are full and countably generated, E^* and F^* are countably generated in their corresponding multiplier modules and $E \backsim_M F$, then the Hilbert C^*-modules $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are isomorphic*

Proof. (1) This follows from Proposition 3 and Corollary 5.

(2) If $E \backsim_M F$, then $\mathcal{K}_{\mathscr{A}}(E) \backsim_M \mathcal{K}_{\mathscr{B}}(F)$. Since E and F are countably generated, the C^*-algebras $\mathcal{K}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{B}}(F)$ are *σ-unital*, i.e. have countable approximate units (see [11, Proposition 6.7]), and by [5, Theorem 1.2], the C^*-algebras $\mathcal{K}_{\mathscr{A}}(E) \otimes \mathbb{K}$ and $\mathcal{K}_{\mathscr{B}}(F) \otimes \mathbb{K}$ are isomorphic.

On the other hand, since E and E^* are countably generated in their corresponding multiplier modules, the Hilbert C^*-modules $\mathscr{A} \otimes \mathbb{K}$ and $E \otimes \mathbb{K}$ are unitarily equivalent (see [9, Proposition 3.1]) and hence the C^*-algebras $\mathscr{A} \otimes \mathbb{K}$ and $\mathcal{K}_{\mathscr{A}}(E) \otimes \mathbb{K}$ are isomorphic. In the same manner, we deduce that the C^*-algebras $\mathscr{B}\otimes\mathbb{K}$ and $\mathcal{K}_{\mathscr{B}}(F)\otimes\mathbb{K}$ are isomorphic. Therefore, the C^*-algebras $\mathscr{A}\otimes\mathbb{K}$ and $\mathscr{B}\otimes\mathbb{K}$ are isomorphic and so they are isomorphic as Hilbert C^*-modules. From these facts, we conclude that the Hilbert C^*-modules $E\otimes\mathbb{K}$ and $F\otimes\mathbb{K}$ are isomorphic. ■

Remark 11. If $\mathscr{A}$ is a σ-unital C^*-algebra, then $\mathscr{A}$ is a full countably generated Hilbert C^*-module over $\mathscr{A}$, and since $\mathcal{K}_{\mathscr{A}}(\mathscr{A},\mathscr{A})$ is isomorphic to $\mathscr{A}$, $\mathscr{A}^*$ is countably generated in its multiplier module. Therefore, Theorem 10 extends [5, Theorem 1.2].

The next result is another extension of [5, Theorem 1.2].

Corollary 12. *Let E and F be countably generated full Hilbert C^*-modules over commutative C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Then $E\backsim_M F$ if and only if the Hilbert C^*-modules $E\otimes\mathbb{K}$ and $F\otimes\mathbb{K}$ are isomorphic.*

Proof. This follows from Theorem 10 and [9, Corollary 3.7]. ■

Corollary 13. *Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras. Then $\mathscr{A}\backsim_M\mathscr{B}$ if and only if there are countably generated full Hilbert C^*-modules E and F over $\mathscr{A}$ and $\mathscr{B}$, respectively, such that E^* and F^* are countably generated in their corresponding multiplier modules and the Hilbert C^*-modules $E\otimes\mathbb{K}$ and $F\otimes\mathbb{K}$ are isomorphic.*

Theorem 14. *Let E and F be countably generated full Hilbert C^*-modules over σ-unital C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Then the following statements are equivalent:*

(1) *E and F are Morita equivalent;*
(2) *$E\otimes\mathbb{K}$ and $F\otimes\mathbb{K}$ are isomorphic;*
(3) *$\mathcal{K}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{B}}(F)$ are stably isomorphic;*
(4) *$\mathcal{K}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{B}}(F)$ are Morita equivalent;*
(5) *$\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent;*
(6) *$\mathscr{A}$ and $\mathscr{B}$ are stably isomorphic.*

Proof. Since E and F are full Hilbert C^*-modules over σ-unital C^*-algebras, the Hilbert C^*-modules E^* and F^* are countably generated and so they are countably generated in their corresponding multiplier modules. Therefore the equivalence of (1) and (2) is nothing other than Theorem 10. The equivalence of (5) and (6) is [5, Theorem 1.2], and the equivalence of (1) and (5) is Proposition 8. Since E and F are countably generated, the C^*-algebras $\mathcal{K}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{B}}(F)$ are σ-unital, and hence the equivalence of (3) and (4) is [5, Theorem 1.2]. The equivalence of (1) and (4) is directly deduced from Definition 1. ■

Remark 15. By [17, Theorem 3.5], two countably generated full Hilbert C^*-modules E and F over σ-unital C^*-algebras are stably Morita equivalent in the sense of Skeide if and only if they are modules over Morita equivalent

C^*-algebras. So, in this case, the notion of Morita equivalence introduced in this note coincides with the notion of stable Morita equivalence introduced by Skeide [16].

COROLLARY 16. *Let E and F be countably generated full Hilbert C^*-modules over σ-unital C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Then E and F are stably Morita equivalent in the sense of Skeide* [17] *if and only if the Hilbert C^*-modules $E \otimes \mathbb{K}$ and $F \otimes \mathbb{K}$ are isomorphic.*

Proof. This follows from Theorem 14 and [17, Theorem 3.5(2)]. ■

3. Applications. Let E be a full Hilbert $\mathscr{A}$-module and let G be a locally compact group. A *continuous action* of G on E is a group homomorphism η from G to $\mathrm{Aut}(E)$, the group of all isomorphisms of Hilbert C^*-modules from E on E, such that the map $t \mapsto \eta_t(x)$ from G to E is continuous for each $x \in E$. Any continuous action η of G on E induces a continuous action α^η of G on $\mathscr{A}$ by $\alpha^\eta_g(\langle x, y\rangle) = \langle \eta_g(x), \eta_g(y)\rangle$ for all $x, y \in E$ and $g \in G$. The linear space $C_c(G, E)$ of all continuous functions from G to E with compact support has a pre-Hilbert $G \times_{\alpha^\eta} \mathscr{A}$-module structure with the action of $G \times_{\alpha^\eta} \mathscr{A}$ on $C_c(G, E)$ given by

$$(\widehat{x}f)(s) = \int_G \widehat{x}(t)\alpha^\eta_t(f(t^{-1}s))\, dt$$

for all $\widehat{x} \in C_c(G, X)$ and all $f \in C_c(G, \mathscr{A})$ and the inner product given by

$$\langle \widehat{x}, \widehat{y}\rangle(s) = \int_G \alpha^\eta_{t^{-1}}(\langle \widehat{x}(t), \widehat{y}(ts)\rangle)\, dt.$$

The *crossed product of E by η*, denoted by $G \times_\eta E$, is the Hilbert $G \times_{\alpha^\eta} \mathscr{A}$-module obtained by the completion of the pre-Hilbert $G \times_{\alpha^\eta} \mathscr{A}$-module $C_c(G, E)$ (see, for example, [10]).

A continuous action η of G on E induces a continuous action β^η of G on $\mathcal{K}_{\mathscr{A}}(E)$ given by $\beta^\eta_g(\theta_{x,y}) = \theta_{\eta_g(x),\eta_g(y)}$ and a continuous action γ^η of G on the linking algebra $\mathcal{L}(E)$ given by $\gamma^\eta_g(\theta_{a\oplus x, b\oplus y}) = \theta_{\alpha^\eta_g(a)\oplus\eta_g(x), \alpha^\eta_g(b)\oplus\eta_g(y)}$.

Recall that two continuous actions α and β of a locally compact group G on the C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, are *Morita equivalent* if there is a full Hilbert C^*-module E over $\mathscr{A}$ and a continuous action η of G on E such that the C^*-algebras $\mathcal{K}_{\mathscr{A}}(E)$ and $\mathscr{B}$ are isomorphic, $\alpha = \alpha^\eta$ and $\varphi \circ \beta = \beta^\eta \circ \varphi$, where φ is an isomorphism from $\mathscr{B}$ onto $\mathcal{K}_{\mathscr{A}}(E)$.

DEFINITION 17. Two continuous actions η and μ of a locally compact group G on full Hilbert C^*-modules E and F over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, are *Morita equivalent* if the actions β^η and β^μ of G on $\mathcal{K}_{\mathscr{A}}(E)$ and $\mathcal{K}_{\mathscr{B}}(F)$, respectively, are Morita equivalent.

The following proposition extends [7, Theorem 1] and [6, Theorem, p. 299] in the context of Hilbert C^*-modules.

PROPOSITION 18. *Let η and μ be continuous actions of a locally compact group G on full Hilbert C^*-modules E and F, respectively. If η and μ are Morita equivalent, then the Hilbert C^*-modules $G\times_\eta E$ and $G\times_\mu F$ are Morita equivalent.*

Proof. If the actions η and μ are Morita equivalent, then the actions γ^η and γ^μ are Morita equivalent, since the actions β^η and γ^η are Morita equivalent, the actions β^μ and γ^μ are Morita equivalent (see [6, p. 297]), and Morita equivalence of actions of groups on C^*-algebras is an equivalence relation. Then, by [6, Theorem, p. 299], the C^*-algebras $G\times_{\gamma^\eta}\mathcal{L}(E)$ and $G\times_{\gamma^\mu}\mathcal{L}(F)$ are Morita equivalent. On the other hand, the C^*-algebras $G\times_{\gamma^\eta}\mathcal{L}(E)$ and $\mathcal{L}(G\times_\eta E)$ are isomorphic, as also are the C^*-algebras $G\times_{\gamma^\mu}\mathcal{L}(F)$ and $\mathcal{L}(G\times_\mu F)$ (see the proof of [8, Theorem 4.1]). Hence the C^*-algebras $\mathcal{L}(G\times_\eta E)$ and $\mathcal{L}(G\times_\mu F)$ are Morita equivalent. From these facts and Proposition 8, we conclude that the Hilbert C^*-modules $G\times_\eta E$ and $G\times_\mu F$ are Morita equivalent. ∎

The vector space $C_0(G,E)$ of all continuous functions from G to E vanishing at infinity has a canonical Hilbert C^*-module structure over the C^*-algebra $C_0(G,\mathscr{A})$, which can be identified with $C_0(G)\otimes\mathscr{A}$. Moreover, $C_0(G,E)$ is full and can be identified with $C_0(G)\otimes E$.

The following theorem extends Green's theorem to the context of Hilbert C^*-modules.

THEOREM 19. *Let G be a locally compact group, G_0 a closed subgroup of G, E a full Hilbert $\mathscr{A}$-module, η a continuous action of G on E, and σ^η a continuous action of G on the Hilbert $C_0(G/G_0,\mathscr{A})$-module $C_0(G/G_0,E)$ defined by*

$$\sigma^\eta_t(f)(sG_0)=\eta_t(f(t^{-1}sG_0)).$$

Then the Hilbert C^*-modules $G\times_{\sigma^\eta}C_0(G/G_0,E)$ and $G_0\times_{\eta|_{G_0}}E$ are Morita equivalent.

Proof. Because the linking algebra of $C_0(G/G_0,E)$ can be identified with $C_0(G/G_0,\mathcal{L}(E))$, the action of G on $\mathcal{L}(C_0(G/G_0,E))$ induced by σ^η can be identified with the action σ^{γ^η} of G on $C_0(G/G_0,\mathcal{L}(E))$ given by $\sigma^{\gamma^\eta}_g(f)(sG_0)=\gamma^\eta_g(f(g^{-1}sG_0))$. Hence the C^*-algebras $\mathcal{L}(G\times_{\sigma^\eta}C_0(G/G_0,E))$ and $G\times_{\sigma^{\gamma^\eta}}C_0(G/G_0,\mathcal{L}(E))$ are isomorphic (see the proof of [8, Theorem 4.1]).

Clearly $\gamma^\eta|_{G_0}=\gamma^{\eta|_{G_0}}$, so by [14, Theorem 4.21], the C^*-algebra $G_0\times_{\gamma^\eta|_{G_0}}\mathcal{L}(E)$ is Morita equivalent to the C^*-algebra $G\times_{\sigma^{\gamma^\eta}}C_0(G/G_0,\mathcal{L}(E))$ and is isomorphic to $\mathcal{L}(G_0 \times_{\eta|_{G_0}} E)$ (see the proof of [8, Theorem 4.1]). Hence the C^*-algebras $G \times_{\sigma^{\gamma^\eta}} C_0(G/G_0, \mathcal{L}(E))$ and $\mathcal{L}(G_0 \times_{\eta|_{G_0}} E)$ are Morita equivalent.

We have therefore showed that the C^*-algebras $\mathcal{L}(G \times_{\sigma^\eta} C_0(G/G_0, E))$ and $\mathcal{L}(G_0 \times_{\eta|_{G_0}} E)$ are Morita equivalent, whence, by Proposition 8, we have $G \times_{\sigma^\eta} C_0(G/G_0, E) \backsim_M G_0 \times_{\eta|_{G_0}} E$. ■

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References

[1] P. Ara, *Morita equivalence and Pedersen ideals*, Proc. Amer. Math. Soc. 129 (2001), 1041–1049.

[2] D. Bakić and B. Guljǎs, *On a class of module maps of Hilbert C^*-modules*, Math. Comm. 7 (2002), 177–192.

[3] W. Beer, *On Morita equivalence of nuclear C^*-algebras*, J. Pure Appl. Algebra 26 (1982), 249–267.

[4] D. P. Blecher, P. S. Muhly and V. I. Paulsen, *Categories of operator modules (Morita equivalence and projective modules)*, Mem. Amer. Math. Soc. 143 (2000), no. 681.

[5] L. G. Brown, P. Green and M. A. Rieffel, *Stable isomorphism and strong Morita equivalence of C^*-algebras*, Pacific J. Math. 71 (1977), 349–363.

[6] F. Combes, *Crossed products and Morita equivalence*, Proc. London Math. Soc. 49 (1984), 289–306.

[7] R. E. Curto, S. P. Muhly and D. P. Williams, *Cross products of strongly Morita equivalent C^*-algebras*, Proc. Amer. Math. Soc. 90 (1984), 528–530.

[8] S. Echterhoff, S. Kaliszewski, J. Quigg and I. Raeburn, *Naturality and induced representations*, Bull. Austral. Math. Soc. 61 (2000), 415–438.

[9] M. Joiţa, *A note on countably generated Hilbert modules*, Results Math. 55 (2009), 101–109.

[10] M. Kusuda, *Duality for crossed products of Hilbert C^*-modules*, J. Operator Theory 60 (2008), 85–112.

[11] E. C. Lance, *Hilbert C^*-Modules*, London Math. Soc. Lecture Note Ser. 210, Cambridge Univ. Press, Cambridge, 1995.

[12] P. S. Muhly and B. Solel, *On the Morita equivalence of tensor algebras*, Proc. London Math. Soc. 81 (2000), 113–168.

[13] I. Raeburn and S. J. Thompson, *Countably generated Hilbert modules, the Kasparov stabilisation theorem, and frames with Hilbert modules*, Proc. Amer. Math. Soc. 131 (2003), 1557–1564.

[14] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*, Math. Surveys Monogr. 60, Amer. Math. Soc., Providence, RI, 1998.

[15] M. A. Rieffel, *Induced representations of C^*-algebras*, Adv. Math. 13 (1974), 176–257.

[16] M. Skeide, *Unit vectors, Morita equivalence and endomorphisms*, Publ. RIMS Kyoto Univ. 45 (2009), 475–518.

[17] —, *Classification of E_0-semigroups by product systems*, arXiv:0901.1798v2.

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