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Factorization of sequences in discrete Hardy spaces

by

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Abstract. The purpose of this paper is to obtain a discrete version for the Hardy spaces $H^p(\mathbb{Z})$ of the weak factorization results obtained for the real Hardy spaces $H^p(\mathbb{R}^n)$ by Coifman, Rochberg and Weiss for $p > n/(n+1)$, and by Miyachi for $p \leq n/(n+1)$. It represents an extension, in the one-dimensional case, of the corresponding result by A. Uchiyama who obtained a factorization theorem in the general context of spaces X of homogeneous type, but with some restrictions on the measure that exclude the case of points of positive measure on X and, hence, $\mathbb{Z}$. In order to obtain the factorization theorem, we first study the boundedness of some bilinear maps defined on discrete Hardy spaces.

1. Introduction. The work of Coifman and Weiss [7] established, by means of an atomic characterization, an extension of the theory of Hardy spaces to the general context of spaces of homogeneous type. Also the work of Macías and Segovia [10] extends the study of Hardy spaces, in this case, via the boundedness of a grand maximal function. These two references included in their respective hypotheses the space $\mathbb{Z}^n$. But this is not the case of some other works dealing with other characterizations of Hardy spaces. In this connection, we mention [16] where a maximal characterization is given for Hardy spaces on spaces of homogeneous type, or [8], where the atomic decomposition of Triebel–Lizorkin spaces is studied. Both references exclude in their assumptions the possibility of points of positive measure, and hence $\mathbb{Z}^n$.

In [2], in the case of dimension one, or in [3] in the case of several variables, the equivalence of some other characterizations of Hardy spaces was obtained in the discrete setting. In particular, the discrete Hardy space $H^p(\mathbb{Z})$, $0 < p < \infty$, is defined as the space of sequences $c = \{c(n)\}_{n\in\mathbb{Z}}$ such that

$$\|c\|_{H^p(\mathbb{Z})} = \|c\|_p + \|\mathcal{H}c\|_p < \infty,$$

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where $\mathcal{H}c$ is the discrete Hilbert transform of the sequence c given by

$$\mathcal{H}c(n) = \sum_{m \neq n} \frac{c(m)}{n-m}, \qquad n \in \mathbb{Z}.$$

The boundedness of $\mathcal{H}$ in $\ell^p(\mathbb{Z})$, $1 < p < \infty$, (see the work of Plancherel and Pólya [14]) leads to the norm equivalence between $H^p(\mathbb{Z})$ and $\ell^p(\mathbb{Z})$ in this range.

In [2], as in the euclidean case, it is proved that this characterization of the discrete Hardy spaces is equivalent to a maximal one in terms of the discrete Poisson kernel or in terms of other maximal discrete operators, and, most importantly, it is equivalent to the original definition of $H^p(\mathbb{Z})$ in terms of atoms that appears in the literature when we consider $\mathbb{Z}$ as a space of homogeneous type (see Definition 3.2). Some other works dealing with discrete Hardy spaces are [9], where a molecular decomposition of the spaces was obtained, and [5], where discrete Hardy spaces appear in the study of synthesis operators defined on $H^p(\mathbb{R}^n)$.

The celebrated work of Coifman, Rochberg and Weiss [6] established a factorization theorem for the real Hardy space $H^1(\mathbb{R}^n)$. Their result also contains a new characterization of $\mathrm{BMO}(\mathbb{R}^n)$ in terms of the boundedness on L^p of the commutator of a singular integral operator with a multiplication operator. Later on, in [15], A. Uchiyama proved a refinement of that result and, in [17], the factorization theorem was extended to $H^p(X)$, in the range $0 \leq \varepsilon_X < p \leq 1$ where X is a space of homogeneous type under certain assumptions that, as previously mentioned, exclude the discrete setting.

The works of A. Miyachi [11] and [12] deal with the extension of the factorization results in real Hardy spaces $H^p(\mathbb{R}^n)$ to the range $p \leq n/(n+1)$. In [13], the boundedness of more general multilinear operators defined on $H^p(\mathbb{R}^n)$ is studied. The factorization result, specified to the one-dimensional case and in terms of the Hilbert transform, can be stated as follows:

Theorem 1.1 ([6, Theorem II], [15, Corollary to Theorem 1], [17], [11], [12]). *Let H be the Hilbert transform. For $h \in L^2 \cap H^q(\mathbb{R})$ and $g \in L^2 \cap H^r(\mathbb{R})$, set*

$$P_N(h,g) = \sum_{j=0}^{N} \binom{N}{j} H^j h\, H^{N-j} g.$$

Then, if $p, q, r > 0$ satisfy $1/p = 1/q + 1/r < N+1$, there is a constant $C > 0$ depending on q, r and N such that, for all $h \in L^2 \cap H^q(\mathbb{R})$ and $g \in L^2 \cap H^r(\mathbb{R})$,

$$\|P_N(h,g)\|_{H^p(\mathbb{R})} \leq C \|h\|_{H^q(\mathbb{R})} \|g\|_{H^r(\mathbb{R})}.$$

Conversely, if $p \le 1$ is as above, every $f \in H^p(\mathbb{R})$ can be decomposed as

$$f = \sum_{j=1}^{\infty} \lambda_j P_N(h_j, g_j),$$

where λ_j is a sequence of real numbers, $h_j \in L^2 \cap H^q(\mathbb{R})$ and $g_j \in L^2 \cap H^r(\mathbb{R})$, and

$$\|h_j\|_{H^q(\mathbb{R})} \|g_j\|_{H^r(\mathbb{R})} \le C, \qquad \Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{1/p} \le C \|f\|_{H^p(\mathbb{R})}$$

with a constant C depending on q, r and N.

In this paper we shall deal with the corresponding result for the discrete Hardy spaces $H^p(\mathbb{Z})$, $0 < p \le 1$. In Section 2, the boundedness of some bilinear maps acting on discrete Hardy spaces is proved. The starting point is the characterization of $H^p(\mathbb{Z})$ in terms of the boundedness of the discrete Hilbert transform. In Section 3, the factorization result is shown in the discrete setting. The proof consists in adapting Miyachi's result to our context and it combines the decomposition of sequences in $H^p(\mathbb{Z})$ in terms of atoms (see Definition 3.2 and Theorem 3.3) with the decomposition in terms of those sequences whose periodic Fourier transform vanishes in a zero neighborhood and which have a controlled ℓ^2-norm (see Lemma 3.6).

In Section 4, we conclude with an application of the main result to a new proof of the boundedness in $\ell^p(\mathbb{Z})$ of the commutator of the discrete Hilbert transform with multiplication by a sequence in $\mathrm{BMO}(\mathbb{Z})$.

We will use C to denote constants that may change from one occurrence to the next. We shall write $\star$ for convolution of sequences. For a given set I of integers, $\#I$ will denote the cardinality of I.

2. Product of sequences in discrete Hardy spaces. Let us start with the following result that states the boundedness of some discrete convolution operators and which is a direct consequence of the characterization of $H^p(\mathbb{Z})$ in terms of the discrete Hilbert transform.

PROPOSITION 2.1. *Let $j \ge 1$ be an integer, and define the discrete convolution operator C_j by*

$$(C_j a)(n) = \sum_{m \ne 0} \frac{a(n-m)}{m^j}.$$

Then C_j is a bounded operator from $H^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$ for any $j \ge 1$ and all $0 < p < \infty$.

Proof. In the range $1 < p < \infty$ the result is well known, since $H^p = \ell^p$ and, for $j \ge 2$, C_j is a convolution operator whose kernel is in $\ell^1(\mathbb{Z})$, and hence bounded in $\ell^p(\mathbb{Z})$ for $p > 1$. For $j = 1$, C_1 is the discrete Hilbert transform which is also bounded in ℓ^p for $p > 1$. Therefore, we restrict our attention to $0 < p \le 1$.

We proceed by induction on j. For $j = 1$, the discrete operator C_1 is the discrete Hilbert transform which acts from $H^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$.

Assume that the conclusion holds for all $1 \le j \le j_0$, and consider C_{j_0+1}.

Let k be a positive integer, and consider the kernel

$$K_k(m) = \frac{1}{m^{j_0}(m-1)\cdots(m-k)} \qquad \text{for } m \in \mathbb{Z},\ m \neq 0, 1, \ldots, k,$$

and $K_k(m) = 0$ otherwise. Let us see that $\{K_k(m)\}_{m\in\mathbb{Z}}$ is the convolution kernel of an operator that sends $H^p(\mathbb{Z})$ into $\ell^p(\mathbb{Z})$. By decomposing the kernel $\{K_k(m)\}_{m\in\mathbb{Z}}$ into partial fractions, we find that for any sequence a,

$$(K_k \star a)(n) = \sum_{i=0}^{j_0-1} \alpha_i \sum_{m\neq 0} \frac{a(n-m)}{m^{j_0-i}} + \sum_{i=1}^{k} \beta_i \sum_{m\neq i} \frac{a(n-m)}{m-i} + \sum_{i=0}^{k} \gamma_i a(n-i)$$

for some constants α, β and γ. Hence the discrete operator with kernel K_k is the sum of the operators C_j, $1 \le j \le j_0$, plus some translates of the discrete Hilbert transform and also translates of the identity operator. Then, by the induction hypothesis, $K_k \star a \in \ell^p(\mathbb{Z})$ for $a \in H^p(\mathbb{Z})$.

Similarly, for any integer $k \ge 1$, define

$$J_k(m) = \frac{1}{m^{j_0+1}(m-1)\cdots(m-k)} \qquad \text{for } m \in \mathbb{Z},\ m \neq 0, 1, \ldots, k,$$

and $J_k(m) = 0$ otherwise.

For $m \neq 0, 1, \ldots, k$, we have

$$K_k(m) - J_{k-1}(m) = kJ_k(m). \tag{2.1}$$

The sequence J_k is a convolution kernel in $\ell^p(\mathbb{Z})$ for $p > 1/(k + j_0 + 1)$. Using (2.1), we obtain

$$(J_{k-1} \star a)(n) = (K_k \star a)(n) - k(J_k \star a)(n) + \frac{a(n-k)}{k!k^{j_0}}.$$

For this reason, if $p > 1/(k + j_0 + 1)$, there exists a constant C depending on k such that

$$\|J_{k-1} \star a\|_p \le \|K_k \star a\|_p + k\|J_k \star a\|_p + \frac{1}{k!k^{j_0}}\|a\|_p \le C\|a\|_{H^p(\mathbb{Z})}.$$

Analogously, for $m \neq 0, 1, \ldots, k-1$, we have

$$K_{k-1}(m) - J_{k-2}(m) = (k-1)J_{k-1}(m),$$

and hence the operator with kernel J_{k-2} is bounded from $H^p(\mathbb{Z})$ into $\ell^p(\mathbb{Z})$ for any $p > 1/(k + j_0 + 1)$. Iterating this process, we deduce that the operator with kernel J_1 also sends $H^p(\mathbb{Z})$ into $\ell^p(\mathbb{Z})$ continuously. Moreover, if we denote also by C_{j_0+1} the kernel of C_{j_0+1}, we see that for $m \neq 0, 1$,

$$K_1(m) - C_{j_0+1}(m) = \frac{1}{m^{j_0}(m-1)} - \frac{1}{m^{j_0+1}} = \frac{1}{m^{j_0+1}(m-1)} = J_1(m).$$

Hence, the convolution operator C_{j_0+1} is also bounded from $H^p(\mathbb{Z})$ to $\ell^p(\mathbb{Z})$ for any $p > 1/(k+j_0+1)$. Taking the integer k large enough shows that the result holds for any $0 < p \leq 1$. ∎

Corollary 2.2. *For $0 < p \leq 1$ the discrete Hilbert operator is bounded from $H^p(\mathbb{Z})$ into itself.*

Proof. Due to the characterization of $H^p(\mathbb{Z})$ in terms of the boundedness of the discrete Hilbert transform, it is enough to prove that

$$\|\mathcal{H}^2 a\|_p \leq C\|a\|_{H^p(\mathbb{Z})}.$$

We observe that, for any $n \in \mathbb{Z}$,

$$\mathcal{H}^2 a(n) = \sum_{m\in\mathbb{Z}} (K \star K)(m)a(n-m),$$

where $\{K(n)\}_{n\in\mathbb{Z}}$ is the sequence that corresponds to the kernel of the discrete Hilbert transform. Easy calculations show that

$$(K \star K)(0) = -\sum_{n\neq 0} \frac{1}{n^2} = -\frac{\pi^2}{3},$$
$$(K \star K)(m) = -\sum_{n\neq 0,m} \frac{1}{n(n-m)} = -\frac{2}{m^2}, \qquad m \neq 0.$$

Therefore,

$$\mathcal{H}^2 a(n) = -2(C_2 a)(n) - \frac{\pi^2}{3} a(n),$$

and applying Proposition 2.1 to the operator C_2, we conclude that

$$\|\mathcal{H}^2 a\|_p \leq C(\|C_2 a\|_p + \|a\|_p) \leq \|a\|_{H^p(\mathbb{Z})}. \ ∎$$

The following proposition establishes the boundedness of some bilinear maps defined on discrete Hardy spaces; it will be fundamental to obtaining Theorem 2.4.

Proposition 2.3. *Let $j_0 \geq 1$ be an integer, and let $q, r > 0$. Let Γ_{j_0} be the bilinear operator defined as*

$$\Gamma_{j_0}(a,b) := C_{j_0}[(\mathcal{H}a)b + a(\mathcal{H}b)], \qquad a \in H^q(\mathbb{Z}),\ b \in H^r(\mathbb{Z}).$$

Then $\Gamma_{j_0} : H^q(\mathbb{Z}) \times H^r(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ for $1/p = 1/q + 1/r < j_0 + 1$, that is, there exists a constant C depending on q, r, j_0 such that

$$\|\Gamma_{j_0}(a,b)\|_p \leq C\|a\|_{H^q(\mathbb{Z})}\|b\|_{H^r(\mathbb{Z})}.$$

Proof. We can restrict ourselves to sequences $a \in H^q(\mathbb{Z})$ and $b \in H^r(\mathbb{Z})$ of finite support. We observe that, for any $n \in \mathbb{Z}$,

$$
\begin{aligned}
(2.2)\quad \Gamma_{j_0}(a,b)(n) &= \sum_{m\neq n} \frac{b(m)}{(n-m)^{j_0}} \sum_{k\neq m} \frac{a(k)}{m-k} + \sum_{k\neq n} \frac{a(k)}{(n-k)^{j_0}} \sum_{m\neq k} \frac{b(m)}{k-m} \\
&= \sum_{m\neq n} \sum_{k\neq m,n} a(k)b(m) \frac{1}{m-k} \left(\frac{1}{(n-m)^{j_0}} - \frac{1}{(n-k)^{j_0}} \right) \\
&\quad - a(n)(C_{j_0+1}b)(n) - b(n)(C_{j_0+1}a)(n) \\
&= \sum_{m\neq n} \sum_{k\neq m,n} a(k)b(m) \sum_{j=1}^{j_0} \frac{1}{(n-m)^{j_0+1-j}(n-k)^j} \\
&\quad - a(n)(C_{j_0+1}b)(n) - b(n)(C_{j_0+1}a)(n) \\
&= \sum_{j=1}^{j_0} (C_j a)(n)(C_{j_0+1-j}b)(n) - j_0(C_{j_0+1}(ab))(n) \\
&\quad - a(n)(C_{j_0+1}b)(n) - b(n)(C_{j_0+1}a)(n).
\end{aligned}
$$

As the product of a and b is in $\ell^p(\mathbb{Z})$ and $p(j_0+1) > 1$, we have

$$\|C_{j_0+1}(ab)\|_p \le C\|a\|_q\|b\|_r.$$

We use this last estimate in expression (2.2), and the result follows as a consequence of Hölder's inequality and Proposition 2.1. ∎

The main result of this section is the following:

THEOREM 2.4. *Let $N \ge 1$ be an integer, and let $q, r > 0$. For $a \in H^q(\mathbb{Z})$ and $b \in H^r(\mathbb{Z})$, define the bilinear operator Λ_N by*

$$\Lambda_N(a,b) = \sum_{j=0}^{N} \binom{N}{j} (\mathcal{H}^j a)(\mathcal{H}^{N-j} b).$$

Then $\Lambda_N : H^q(\mathbb{Z}) \times H^r(\mathbb{Z}) \to H^p(\mathbb{Z})$ for $1/p = 1/q + 1/r < N+1$, that is, there exists a constant C depending on q, r, N such that

$$\|\Lambda_N(a,b)\|_{H^p(\mathbb{Z})} \le C\|a\|_{H^q(\mathbb{Z})}\|b\|_{H^r(\mathbb{Z})}.$$

Proof. Let a and b be sequences in $H^q(\mathbb{Z})$ and $H^r(\mathbb{Z})$, respectively. Then Hölder's inequality and Corollary 2.2 imply

$$\|\Lambda_N(a,b)\|_p \le \sum_{j=0}^{N} \binom{N}{j} \|\mathcal{H}^j a\|_q \|\mathcal{H}^{N-j} b\|_r \le C\|a\|_{H^q(\mathbb{Z})}\|b\|_{H^r(\mathbb{Z})}.$$

To estimate the p-norm of $\mathcal{H}(\Lambda_N(a,b))$ we will prove that

$$(2.3)\qquad \mathcal{H}[\Lambda_N(a,b)] = O_N(a,b) + (-1)^{N-1}(N-1)!\, \Gamma_N(a,b),$$

where O_N is a bilinear operator defined recursively such that, for any $p = (1/q + 1/r)^{-1}$,

$$\|O_N(a,b)\|_p \le C\|a\|_{H^q(\mathbb{Z})}\|b\|_{H^r(\mathbb{Z})}, \tag{2.4}$$

and, as a consequence of Proposition 2.3, since $(N+1)p > 1$,

$$\|\Gamma_N(a,b)\|_p \le C\|a\|_{H^q(\mathbb{Z})}\|b\|_{H^r(\mathbb{Z})}.$$

Thus, we reduce the proof to (2.3). We proceed by induction on N. For $N = 1$, we observe that

$$\mathcal{H}[\Lambda_1(a,b)] = \mathcal{H}[(\mathcal{H}a)b + a(\mathcal{H}b)] = \Gamma_1(a,b),$$

and from Proposition 2.3 the result follows.

Assume (2.3) holds for an integer N. By Corollary 2.2, $\mathcal{H}a \in H^q(\mathbb{Z})$ and $\mathcal{H}b \in H^r(\mathbb{Z})$, and using the recursive formula

$$\Lambda_{N+1}(a,b) = \Lambda_N(\mathcal{H}a,b) + \Lambda_N(a,\mathcal{H}b),$$

which can be easily checked, we see that

$$\begin{aligned}\mathcal{H}[\Lambda_{N+1}(a,b)] = \mathcal{H}[\Lambda_N(\mathcal{H}a,b) + \Lambda_N(a,\mathcal{H}b)] &= O_N(\mathcal{H}a,b) + O_N(a,\mathcal{H}b)\\ &\quad + (-1)^{N-1}(N-1)!\,(\Gamma_N(\mathcal{H}a,b) + \Gamma_N(a,\mathcal{H}b)).\end{aligned}$$

From this last equation, substituting the expression for Γ_N given by (2.2), we obtain

$$\mathcal{H}[\Lambda_{N+1}(a,b)] = O_{N+1}(a,b) + (-1)^N N!\,\Gamma_{N+1}(a,b),$$

where the bilinear operator O_{N+1} is defined in terms of O_N as follows:

$$\begin{aligned}&O_{N+1}(a,b)\\ &\quad= (-1)^{N-1}(N-1)!\Big[\sum_{j=1}^{N}(C_j(\mathcal{H}a))(C_{N+1-j}b) + (C_ja)(C_{N+1-j}(\mathcal{H}b))\\ &\qquad - (\mathcal{H}a)(C_{N+1}b) - (\mathcal{H}b)(C_{N+1}a) - a(C_{N+1}(\mathcal{H}b)) - b(C_{N+1}(\mathcal{H}a))\Big]\\ &\qquad + O_N(\mathcal{H}a,b) + O_N(a,\mathcal{H}b).\end{aligned}$$

O_{N+1} also satisfies the estimate (2.4) by the induction hypothesis applied to O_N, Hölder's inequality, Proposition 2.1 and Corollary 2.2. ∎

3. Factorization in $H^p(\mathbb{Z})$. We denote by m the periodic multiplier corresponding to $\mathcal{H}$. By computing the corresponding Fourier series we observe that

$$m(\xi) = -\pi i\,\mathrm{sign}(\xi)(1-2|\xi|), \qquad \xi \in [-1/2,1/2].$$

By using the Fourier transform, the discrete bilinear operator of Theorem 2.4 can be expressed as follows:

$$\widehat{\Lambda_N}(a,b)(\xi) = \int_{-1/2}^{1/2} \hat{a}(\theta)\hat{b}(\xi-\theta)(m(\xi-\theta)+m(\theta))^N \, d\theta,$$

where for the sequence $a \in \ell^2(\mathbb{Z})$, $\hat{a}$ denotes the periodic function whose Fourier coefficients are $\{a(n)\}_n$, that is

$$\hat{a}(\xi) = \sum_{n\in\mathbb{Z}} a(n)e^{-2\pi i n\xi}.$$

We observe that for $a \in H^p(\mathbb{Z})$, $0 < p \le 1$, since a is also in $\ell^1(\mathbb{Z})$, its Fourier transform $\hat{a}$ is a continuous 1-periodic function. Note that, in the definitions above, we are identifying the one-dimensional torus $\mathbb{T}$ with the interval $[-1/2, 1/2]$.

The converse of Theorem 2.4 can be formulated as follows.

Theorem 3.1. *Let $N \ge 1$ be an integer and let $p, q, r > 0$ satisfy $1 \le 1/p = 1/q + 1/r < N + 1$. Then every $c \in H^p(\mathbb{Z})$ can be decomposed as*

$$c = \sum_{j=1}^{\infty} \lambda_j \Lambda_N(a_j, b_j),$$

where $\{\lambda_j\}_j \in \ell^p(\mathbb{Z})$, $a_j \in H^q(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$, $b_j \in H^r(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$ and

$$\|a_j\|_{H^q}\|b_j\|_{H^r} \le C, \qquad \Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{1/p} \le C\|c\|_{H^p(\mathbb{Z})},$$

with a constant C depending only on p, q, r.

As already mentioned in the introduction, the proof of Theorem 3.1 consists in adapting the proof given by Miyachi in [12] to the discrete situation. As in [12], the theorem is obtained by using the expression in terms of the Fourier transform of the bilinear operators involved and will be based on the decomposition of sequences in discrete Hardy spaces into atoms.

The atomic decomposition which yields the original definition of $H^p(\mathbb{Z})$ (see [7]) consists in the following:

Definition 3.2. Let $0 < p \le 1$. We say that the finite sequence a is a *p-atom* in $\mathbb{Z}$ if:

(a) The support of a is contained in a ball B in $\mathbb{Z}$ centered at an integer m_0; denote its cardinality by $\#B$.
(b) $\|a\|_\infty \le 1/(\#B)^{1/p}$.
(c) $\sum n^\alpha a(n) = 0$ for all integers α satisfying $0 \le \alpha \le p^{-1} - 1$.

We define the atomic space $H^p_{\mathrm{at}}(\mathbb{Z})$ as the set of sequences a that admit the decomposition

$$a = \sum_{j=0}^{\infty} \lambda_j a_j, \tag{3.1}$$

where a_j is a p-atom and $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. For $a \in H^p_{\mathrm{at}}(\mathbb{Z})$, define the p-norm

$$\|a\|_{H^p_{\mathrm{at}}(\mathbb{Z})} = \inf \Big\{ \Big(\sum_{j\geq 0} |\lambda_j|^p \Big)^{1/p} \Big\},$$

where the infimum is taken over all the representations of a in the form (3.1).

Theorem 3.3 (see [2, Theorems 3.10 and 3.14]). *Let $0 < p \leq 1$. Then*

$$\|a\|_{H^p(\mathbb{Z})} \simeq \|a\|_{H^p_{\mathrm{at}}(\mathbb{Z})}.$$

Also, let us now introduce the following class of sequences whose continuous counterpart was already introduced by Miyachi [12] and which represents the main ingredient to obtain the factorization result (Theorem 3.1).

Definition 3.4 (see [12]). For $p > 0$, $t > 2$ and a nonnegative integer M, we denote by $\mathcal{A}_{p,M}(t)$ the set of sequences $a \in \ell^2(\mathbb{Z})$ such that

$$\hat{a}(\xi) = 0 \quad \text{for } |\xi| \leq 1/t$$

and

$$\|D^\alpha \hat{a}\|_{L^2(\mathbb{T})} \leq t^{\alpha - 1/p + 1/2} \quad \text{for any integer } 0 \leq \alpha \leq M.$$

Let us prove that $\mathcal{A}_{p,M}(t)$ is a subset of $H^p(\mathbb{Z})$ with uniformly bounded H^p-norm.

Lemma 3.5. *Let $0 < p \leq 2$ and $M > 1/p - 1/2$. Then $\mathcal{A}_{p,M}(t) \subset H^p(\mathbb{Z})$ and there is a constant $C > 0$ depending on p such that*

$$\|a\|_{H^p(\mathbb{Z})} \leq C \quad \textit{for all } a \in \mathcal{A}_{p,M}(t),\ t > 2.$$

Proof. We assume $M = [1/p - 1/2] + 1$. We shall prove

$$\|\mathcal{H}a\|_p \leq C \quad \text{for all } a \in \mathcal{A}_{p,M}(t),\ t > 2.$$

Since the multiplier m trivially satisfies the estimate, valid for $\xi \in [-1/2, 1/2]$, $|D^\alpha m(\xi)| \leq C|\xi|^{-\alpha}$, $0 \leq \alpha \leq M$, as a consequence of the Leibnitz formula, for any $a \in \mathcal{A}_{p,M}(t)$, we obtain

$$\|D^\alpha \widehat{\mathcal{H}a}\|_{L^2(\mathbb{T})} = \|D^\alpha (m\hat{a})\|_{L^2(\mathbb{T})} \leq C t^{\alpha - 1/p + 1/2} \quad \text{for } \alpha \leq M,$$

and, then, by Parseval's identity,

$$\big\| |n|^k \mathcal{H}a \big\|_2 \leq C t^{k - 1/p + 1/2}, \quad k = 0, 1, \ldots, M.$$

From this estimate, if $0 < p \le 2$ and $1/p = 1/2 + 1/q$ we conclude by Hölder's inequality that

$$\sum_{|n|<t} |\mathcal{H}a(n)|^p \le \|\mathcal{H}a\|_2^p t^{p/q} \le Ct^{-1+p/2+p/q} = C,$$

and

$$\sum_{|n|\ge t} |\mathcal{H}a(n)|^p \le \||n|^M \mathcal{H}a\|_2^p \Big(\sum_{|n|\ge t} |n|^{-Mq}\Big)^{p/q} \le Ct^{Mp-1+p/2} t^{(1-Mq)p/q} = C,$$

where the last inequality holds since $Mq > 1$. ■

LEMMA 3.6. *Let $0 < p \le 2$ and $M > 1/p - 1/2$. Then any sequence $a \in H^p(\mathbb{Z})$ can be decomposed as*

$$a = \sum_{j=1}^{\infty} \lambda_j a_j(\cdot - n_j),$$

where $\lambda_j \in \mathbb{R}$, $a_j \in \mathcal{A}_{p,M}(t_j)$ for some $t_j > 2$, and $n_j \in \mathbb{Z}$, and

$$\Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{1/p} \le A'\|a\|_{H^p(\mathbb{Z})},$$

with the constant A' depending on M and p.

Proof. Let $a \in H^p(\mathbb{Z})$. By Theorem 3.3,

$$a(n) = \sum_{j=1}^{\infty} \lambda_j a_j(n)$$

where every a_j is a p-atom centered at an integer n_j, and there exists a positive constant A such that

$$\Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{1/p} \le A\|a\|_{H^p(\mathbb{Z})}. \tag{3.2}$$

Let us prove the lemma for any p-atom a centered at n_0. We will prove that we can take A'' depending on p and M, and $c \in \mathcal{A}_{p,M}(t)$, $t > 2$, such that

$$\|a - A''c(\cdot - n_0)\|_{H^p(\mathbb{Z})} \le A/2, \tag{3.3}$$

where A corresponds to the constant appearing in (3.2).

For the moment, let us assume (3.3). Applying it to each atom a_j gives

$$a(n) = \sum_{j=1}^{\infty} \lambda_j A'' c_j(n - n_j) + a_{(1)}(n)$$

where $c_j \in \mathcal{A}_{p,M}(t_j)$, for some $t_j > 2$ and

$$\|a_{(1)}\|_{H^p(\mathbb{Z})} \le \tfrac{1}{2}\|a\|_{H^p(\mathbb{Z})}.$$

Next apply the same process to $a_{(1)}$ to obtain a smaller error and so on. Eventually we obtain, for each N,

$$a(n) = \sum_{k=0}^{N}\sum_{j=1}^{\infty} \lambda_j^k A'' c_j^k(n - n_j^k) + a_{(N+1)}(n),$$

where $c_j^k \in \mathcal{A}_{p,M}(t_j^k)$ for some $t_j^k > 2$, and

$$\Big(\sum_{j=1}^{\infty} |\lambda_j^k|^p\Big)^{1/p} \le 2^{-k} A\|a\|_{H^p(\mathbb{Z})}, \qquad \|a_{(N+1)}\|_{H^p(\mathbb{Z})} \le \frac{1}{2^{N+1}}\|a\|_{H^p(\mathbb{Z})}.$$

The decomposition of the lemma is obtained by letting $N \to \infty$ since

$$\Big(\sum_{k=0}^{\infty}\sum_{j=1}^{\infty} |\lambda_j^k A''|^p\Big)^{1/p} \le \Big(\sum_{k=0}^{\infty} 2^{-kp}\Big)^{1/p} A'' A\|a\|_{H^p(\mathbb{Z})} = A'\|a\|_{H^p(\mathbb{Z})}.$$

Let us see, then, the approximation (3.3). We can assume that a is a p-atom with cardinality ρ and centered at 0. The Fourier transform of a satisfies

$$\|D^\alpha \hat{a}\|_{L^2(\mathbb{T})} \le C_\alpha \rho^{\alpha - 1/p + 1/2}, \tag{3.4}$$

as a consequence of Parseval's identity and the size condition (b) in Definition 3.2. Also,

$$|D^\alpha \hat{a}(\xi)| \le C_\alpha \rho^{[1/p]+1-1/p}|\xi|^{[1/p]-\alpha} \qquad \text{for } |\xi| \le \rho^{-1}. \tag{3.5}$$

If $\alpha \le [1/p - 1]$, this last inequality holds by Taylor's formula and the following facts that are consequences of the cancelation and the size properties of a, respectively:

$$D^\beta D^\alpha \hat{a}(0) = 0 \qquad \text{for } \beta \le [1/p - 1] - \alpha,$$
$$\|D^\beta D^\alpha \hat{a}\|_\infty \le C\rho^{[1/p]+1-1/p} \qquad \text{for } \beta = [1/p - 1] - \alpha + 1.$$

For $\alpha > [1/p - 1]$, we use the fact that $\|D^\alpha \hat{a}\|_\infty \le C_\alpha \rho^{\alpha+1-1/p}$; then, for $\rho \le |\xi|^{-1}$, (3.5) follows.

For T large enough, let us consider the 1-periodic function defined in $[-1/2, 1/2]$ as $\Phi(T\rho\,\cdot)\hat{a}(\cdot)$, where Φ is a C^∞ function such that $\Phi \equiv 1$ for $|\xi| \ge 2$ and $\Phi \equiv 0$ for $|\xi| \le 1$.

Let b_T be the sequence defined in terms of its Fourier transform as

$$\widehat{b_T}(\xi) = \Phi(T\rho\,\xi)\hat{a}(\xi), \qquad \xi \in [-1/2, 1/2].$$

From estimates (3.4) and (3.5) we will deduce

$$\|D^\alpha \widehat{b_T}\|_{L^2(\mathbb{T})} \le C_\alpha T^\alpha \rho^{\alpha-1/p+1/2}, \tag{3.6}$$

$$\|a - b_T\|_{H^p(\mathbb{Z})} \le CT^{-[1/p]-1+1/p}, \tag{3.7}$$

with constants independent on T, ρ and a. Once we have proved (3.6) and (3.7), the approximation (3.3) follows by considering

$$c = \frac{1}{A''} b_T \in \mathcal{A}_{p,M}(T\rho),$$

with A'' and T large enough depending on M and p. The inequality (3.6) follows from (3.4), and (3.7) is obtained by decomposing

$$a - b_T = \sum_{j=0}^{\infty} a_j,$$

where for each $j \ge 0$, the sequence a_j is defined in terms of its Fourier transform as

$$\widehat{a_j}(\xi) = (\Phi(2^{j+1}T\rho\,\xi) - \Phi(2^j T\rho\,\xi))\hat{a}(\xi), \qquad \xi \in [-1/2, 1/2].$$

We observe that the Fourier transform of each a_j is a 1-periodic function with support in $[2^{-1-j}(T\rho)^{-1}, 2^{1-j}(T\rho)^{-1}]$, and hence from (3.5) we have

$$\|D^\alpha \widehat{a_j}\|_{L^2(\mathbb{T})} \le C_\alpha (2^j T)^{-[1/p]-1+1/p}(2^j T\rho)^{\alpha-1/p+1/2}.$$

Thus, using Lemma 3.5,

$$\|a_j\|_{H^p(\mathbb{Z})} \le C(2^j T)^{-[1/p]-1+1/p}.$$

Finally,

$$\|a - b_T\|_{H^p(\mathbb{Z})} \le \Big(\sum_{j=0}^{\infty} \|a_j\|^p_{H^p(\mathbb{Z})}\Big)^{1/p} \le CT^{-[1/p]-1+1/p},$$

which is inequality (3.7). ∎

Proof of Theorem 3.1. Since $1 \le 1/p = 1/q + 1/r$, one of the exponents q or r is less than or equal to 2. Let us assume $r \le 2$. We will prove that for all $c \in \mathcal{A}_{p,M}(t)$, $t > 2$, and $M = [1/p - 1/2] + 2$ there exist sequences $a_j \in H^q(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$, $b_j \in H^r(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$ and numbers λ_j such that

$$\Big\| c - \sum_{j=1}^{\infty} \lambda_j \Lambda_N(a_j, b_j)\Big\|_{H^p(\mathbb{Z})} \le \frac{1}{2A'},$$

$$\|a_j\|_{H^q}\|b_j\|_{H^r} \le C, \qquad \Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{1/p} \le C,$$

where A' is the constant appearing in Lemma 3.6 corresponding to $M = [1/p - 1/2] + 2$ and the constant C depends only on N, p, q, r. Once this estimate is proved, the result follows from Lemma 3.6 using an approximation argument in a similar way as applied there, where the corresponding result was obtained from the atomic decomposition.

Take then $c \in \mathcal{A}_{p,M}(t)$, $t > 2$, $M = [1/p - 1/2] + 2$, and consider, for $\delta > 0$ to be fixed, the points $\nu_1 = \delta t^{-1}$ and $\nu_2 = -\delta t^{-1}$ belonging to the intervals I_1 and I_2 respectively in such a way that $I_1 \cup I_2$ covers the fundamental interval $[-1/2, 1/2]$. We observe that there exists a constant $C > 0$ such that

$$\inf_{\xi \in I_k} |m(\xi) + m(\nu_k)| > C, \qquad k = 1, 2. \tag{3.8}$$

Decompose $c = c_1 + c_2$, where c_1 and c_2 are defined in terms of their periodic Fourier transforms by

$$\widehat{c_k}(\xi) = \varphi_k(\xi)\hat{c}(\xi), \qquad \xi \in [-1/2, 1/2],\ k = 1, 2,$$

and $\{\varphi_1, \varphi_2\}$ is a smooth partition of unity on $[-1/2, 1/2]$ associated to the covering I_1, I_2.

It is enough to prove that for $k = 1, 2$ there exist sequences $a_k \in H^q(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$, $b_k \in H^r(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$ such that

$$\|c_k - \Lambda_N(a_k, b_k)\|_{H^p(\mathbb{Z})} \le (2A')^{-1}, \qquad \|a_k\|_{H^q}\|b_k\|_{H^r} \le C. \tag{3.9}$$

To prove (3.9), let us define b_k and a_k via

$$\widehat{b_k}(\xi) = (m(\xi) + m(\nu_k))^{-N}\widehat{c_k}(\xi),$$
$$\widehat{a_k}(\xi) = (t/\epsilon)\theta((t/\epsilon)(\xi - \nu_k)),$$

where θ is a C^∞ function with support in $(-1, 1)$ and $\int \theta(x)\, dx = 1$, and ϵ is a small positive number with $\epsilon < \delta/2$ and $\delta + \epsilon < 1/2$.

We shall prove that

$$\|b_k\|_{H^r(\mathbb{Z})} \le Ct^{-1/p+1/r}, \tag{3.10}$$

$$\|a_k\|_{H^q(\mathbb{Z})} \le C(t/\epsilon)^{1/q}, \tag{3.11}$$

$$\|c_k - \Lambda_N(a_k, b_k)\|_{H^p(\mathbb{Z})} \le C(\delta + \delta^{-1}\epsilon), \tag{3.12}$$

where C depends only on the multiplier m and the exponents p, q and r. The estimates in (3.9) are obtained by taking δ and ϵ small enough depending also on m, p, q and r.

Let us prove (3.10). By (3.8), the function

$$B(\xi) = (m(\xi) + m(\nu_k))^{-N}$$

satisfies $|D^\alpha B(\xi)| \le C_\alpha |\xi|^{-\alpha}$ on the support of $\widehat{c_k}$. This estimate guarantees, due to the multiplier theorem (see [9, Theorem 3]), that B is an $H^r(\mathbb{Z})$-multiplier, and hence

$$\|b_k\|_{H^r(\mathbb{Z})} \le C\|c_k\|_{H^r(\mathbb{Z})} \le C\|c\|_{H^r(\mathbb{Z})} \le Ct^{1/r-1/p},$$

where the last inequality is a consequence of Lemma 3.5.

To see (3.11), we observe that if $q > 2$,

$$\|a_k\|_{H^q(\mathbb{Z})} = \|a_k\|_q \le C(t/\epsilon)^{1/q},$$

whereas for $q \le 2$, (3.11) is again a consequence of Lemma 3.5, since

$$\|D^\alpha \widehat{a_k}\|_{L^2(\mathbb{T})} \le C_\alpha (t/\epsilon)^{\alpha+1/2},$$

and $\widehat{a_k}(\xi) = 0$ for $|\xi| \le \epsilon/t$.

Finally, to see (3.12) we write, using $\int \theta(x)\,dx = 1$,

$$\begin{aligned}
&(c_k - \Lambda_N(a_k, b_k))\widehat{\ }(\xi) \\
&= \int_{-1/2}^{1/2} \widehat{a_k}(\eta)\big(\widehat{c_k}(\xi) - \widehat{b_k}(\xi-\eta)(m(\xi-\eta)+m(\eta))^N\big)\,d\eta \\
&= \int_{-1/2}^{1/2} \widehat{a_k}(\eta)\bigg(\widehat{c_k}(\xi) - \frac{\widehat{c_k}(\xi-\eta)}{(m(\xi-\eta)+m(\nu_k))^N}(m(\xi-\eta)+m(\eta))^N\bigg)\,d\eta \\
&= \int_{-1/2}^{1/2} \widehat{a_k}(\eta)(\widehat{c_k}(\xi) - \widehat{c_k}(\xi-\eta))\,d\eta \\
&\quad + \int_{-1/2}^{1/2} \widehat{a_k}(\eta)\widehat{c_k}(\xi-\eta)\bigg(1 - \frac{(m(\xi-\eta)+m(\eta))^N}{(m(\xi-\eta)+m(\nu_k))^N}\bigg)\,d\eta = \widehat{I}(\xi) + \widehat{II}(\xi).
\end{aligned}$$

The supports, relative to the fundamental interval $[-1/2, 1/2]$, of the periodic functions $\widehat{I}$ and $\widehat{II}$ are contained in the set

$$\{\xi \in [-1/2,1/2] : \operatorname{dist}(\xi, \operatorname{supp}(\widehat{c_k})) \le (\delta+\epsilon)t^{-1}\} \subset \{1/(2t) < |\xi| \le 1/2\}.$$

For $\widehat{I}$, the mean value theorem implies, for $\alpha \le M - 1 = [1/p - 1/2] + 1$,

$$\begin{aligned}
\|D^\alpha \widehat{I}\|_{L^2(\mathbb{T})} &\le \|D^{\alpha+1}\widehat{c_k}\|_{L^2(\mathbb{T})} \int_{-1/2}^{1/2} |\widehat{a_k}(\eta)|\,|\eta|\,d\eta \\
&\le C_\alpha t^{\alpha+1-1/p+1/2}\bigg(\frac{\epsilon}{t}\int|\theta(r)|\,|r|\,dr + \frac{\delta}{t}\int|\theta(r)|\,dr\bigg) \\
&\le C_\alpha \delta t^{\alpha-1/p+1/2}.
\end{aligned}$$

For $\widehat{II}$, we observe that for $\xi - \eta \in \operatorname{supp}(\widehat{c_k})$ and $z \in (\nu_k - \epsilon/t, \nu_k + \epsilon/t)$,

$$\left|\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \xi}\right)^\alpha \frac{(m(\xi-\eta)+m(z))^N}{(m(\xi-\eta)+m(\nu_k))^N}\right| \le C_\alpha|\xi-\eta|^{-\alpha}|z|^{-1} \le C_\alpha \delta^{-1} t|\xi-\eta|^{-\alpha}. \tag{3.13}$$

As a consequence, if $\xi - \eta \in \operatorname{supp}(\widehat{c_k})$ and $\eta \in \operatorname{supp}(\widehat{a_k})$, the mean value theorem implies that

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(1-\frac{(m(\xi-\eta)+m(\eta))^N}{(m(\xi-\eta)+m(\nu_k))^N}\right)\right| \le C_\alpha\delta^{-1}t|\xi-\eta|^{-\alpha}|\nu_k-\eta| \\ \le C_\alpha\epsilon\delta^{-1}|\xi-\eta|^{-\alpha} \le C_\alpha\epsilon\delta^{-1}t^{\alpha}.$$

Again taking into account that $c \in \mathcal{A}_{p,M}(t)$ and this last inequality, we find that for all $\alpha \le M$,

$$\|D^\alpha \widehat{II}\|_{L^2(\mathbb{T})} \le C_\alpha\epsilon\delta^{-1}t^{\alpha-1/p+1/2}.$$

Finally, the use of Lemma 3.5 leads to

$$\|c_k - \Lambda_N(a_k,b_k)\|_{H^p(\mathbb{Z})} \le C(\|I\|_{H^p(\mathbb{Z})} + \|II\|_{H^p(\mathbb{Z})}) \le C(\delta+\delta^{-1}\epsilon),$$

as we wanted to prove. ∎

4. Application: the commutator on sequence spaces. Let $b = \{b(n)\}_{n\in\mathbb{Z}}$ and consider the commutator of the discrete Hilbert transform with multiplication by the sequence b given by

$$[b,\mathcal{H}]a(n) := b(n)\mathcal{H}a(n) - \mathcal{H}(ba)(n) = \sum_{k\neq 0}\frac{b(n)-b(n-k)}{k}a(n-k).$$

In [1] (see also [4] for a proof in the context of spaces of homogeneous type) it is proved that the set of sequences b for which $[b,\mathcal{H}]$ is a bounded operator on $\ell^p(\mathbb{Z})$, $1<p<\infty$, coincides with $\mathrm{BMO}(\mathbb{Z})$, defined as

$$\mathrm{BMO}(\mathbb{Z}) = \left\{ b = \{b(n)\}_{n\in\mathbb{Z}} : \sup_I \frac{1}{\#I}\sum_{k\in I}|b(k)-b_I| = \|b\|_{\mathrm{BMO}(\mathbb{Z})} < \infty \right\},$$

where the supremum above is taken over all finite intervals in $\mathbb{Z}$ and $b_I = (\#I)^{-1}\sum_{k\in I} b(k)$.

The use of the $H^1(\mathbb{Z})$-$\mathrm{BMO}(\mathbb{Z})$ duality (see [7]) and the results of Theorems 2.4 and 3.1 allow us to obtain another proof of this fact.

Corollary 4.1.

(a) *Let* $b \in \mathrm{BMO}(\mathbb{Z})$ *and* $1<p<\infty$. *Then there exists a constant* $C>0$ *such that, for all* $a \in \ell^p(\mathbb{Z})$,

$$\|[b,\mathcal{H}]a\|_p \le C\|b\|_{\mathrm{BMO}(\mathbb{Z})}\|a\|_p.$$

(b) *Conversely, if* $[b,\mathcal{H}]$ *is bounded on* $\ell^p(\mathbb{Z})$ *for some* p *such that* $1 < p < \infty$, *then* b *is in* $\mathrm{BMO}(\mathbb{Z})$ *and we have, for some* $C>0$,

$$\|b\|_{\mathrm{BMO}(\mathbb{Z})} \le C\|[b,\mathcal{H}]\|_{\ell^p(\mathbb{Z})\to\ell^p(\mathbb{Z})}.$$

Proof. To see (a), just observe that for $a_1 \in \ell^p(\mathbb{Z})$ and $a_2 \in \ell^{p'}(\mathbb{Z})$ we can write, making use of Theorem 2.4 for $N=1$,

$$
\begin{aligned}
|\langle [b,\mathcal{H}]a_1, a_2\rangle| &= |\langle b, a_1\mathcal{H}a_2 + a_2\mathcal{H}a_1\rangle| = |\langle b, \Lambda_1(a_1,a_2)\rangle| \\
&\le \|b\|_{\mathrm{BMO}(\mathbb{Z})}\|\Lambda_1(a_1,a_2)\|_{H^1(\mathbb{Z})} \le C\|b\|_{\mathrm{BMO}(\mathbb{Z})}\|a_1\|_p\|a_2\|_{p'}.
\end{aligned}
$$

For the proof of (b), let $c \in H^1(\mathbb{Z})$. Then by the factorization result of Theorem 3.1,

$$
\begin{aligned}
|\langle b, c\rangle| &\le \sum_k |\lambda_k|\,|\langle b, a_k\mathcal{H}b_k + b_k\mathcal{H}a_k\rangle| \\
&= \sum_k |\lambda_k|\,|\langle b_k, b\mathcal{H}a_k - \mathcal{H}(ba_k)\rangle| \\
&\le \sum_k |\lambda_k|\,\|b_k\|_{p'}\|[b,\mathcal{H}]a_k\|_p \le C\sum_k |\lambda_k|\,\|b_k\|_{p'}\|a_k\|_p \\
&\le C\|c\|_{H^1(\mathbb{Z})}.
\end{aligned}
$$

By the duality theorem between $H^1(\mathbb{Z})$ and $\mathrm{BMO}(\mathbb{Z})$, the sequence b is in $\mathrm{BMO}(\mathbb{Z})$ and $\|b\|_{\mathrm{BMO}(\mathbb{Z})}$ is bounded by the norm of the commutator as a bounded operator in $\ell^p(\mathbb{Z})$. ∎

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