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The spectral topology in rings

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Abstract. The spectral topology of a ring is easily defined, has familiar applications in elementary Banach algebra theory, and appears relevant to abstract Fredholm and stable range theory.

0. Introduction. The spectral topology of a ring is motivated by the simple Banach algebra observation that, for $x \in A$,

$$\|x\| < 1 \ \Rightarrow \ 1 - x \in A^{-1}. \tag{0.1}$$

This leads to simple observations about the (norm) closure of the invertible group and its intersection with the one-sided invertibles, relatively regular and Fredholm elements. In a topological algebra the relationship between the spectral and the original topology is able to decide whether or not the invertible group is open, and the spectral topology offers another version of the idea of "stable rank". Precisely what is the spectral closure of the invertibles in the familiar disc algebra seems to be a delicate question.

In this writing each definition and theorem ushers in a new section. We define the spectral closure in §1, prove it gives a topology in §2, with jointly continuous addition and multiplication; in §3 we check some "spectral permanence" and work out some obvious examples. In §4 we make the observation that the spectral topology is the weakest ring topology giving an open group of invertibles, and in §5 we suggest that it offers a possible extension of the "quasinilpotent" concept to general rings. In §6 we chart the interaction between the spectral topology and the "regular" elements of the ring. In §7 we find that onto homomorphisms $T : A \to B$ are automatically continuous, that Gelfand homomorphisms are relatively open, and in §8 that if the homomorphism has inverse closed range then the nearly invertible "Fredholm" elements are "Weyl". In §9 we make an extension of the spectral topology to the space A^n of n-tuples, and use it to show, in §10,

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that the "Bass stable rank" of a ring is never greater than its "spectral topological stable rank". We conclude by comparing our spectral topology with the rather different concept of Ara, Pedersen and Perera.

Suppose A is a ring, with identity 1 and invertible group A^{-1}; then we make the following

1. Definition. The *spectral closure* of a subset $K \subseteq A$ is the set

$$(1.1)\quad \mathrm{Cl}(K) \equiv \mathrm{Cl}_A(K) = \{a \in A : \forall \text{ finite } J \subseteq A\ \exists a' \in K,\ 1 - J(a-a') \subseteq A^{-1}\}.$$

Equivalently

$$(1.2)\quad a \in \mathrm{Cl}(K) \Leftrightarrow \forall \text{ finite } J, L \subseteq A\ \exists a' \in K,\ 1 - J(a-a')L \subseteq A^{-1}.$$

Indeed, this is because for arbitrary $a, b, c, a' \in A$, Jacobson's Lemma ([11, Theorem 3.1.3]) says

$$(1.3)\quad 1 - cb(a-a') \in A^{-1} \Leftrightarrow 1 - b(a-a')c \in A^{-1}.$$

We remark that this is a simple refinement of the "algebraic closure" we [6] offered previously, in which only singleton sets $J = \{y\}$ were allowed:

$$(1.4)\quad \mathrm{cl}_{\mathrm{alg}}(K) \equiv \mathrm{cl}^A_{\mathrm{alg}}(K) = \{a \in A : \forall y \in A\ \exists a' \in K,\ 1 - y(a-a') \in A^{-1}\}.$$

The algebraic closure operation (1.4) satisfies all but one of the Kuratowski conditions, and qualifies as a "closure" but not quite as a topology; in contrast the spectral closure really does give a topology:

2. Theorem. *If A is a ring then the spectral closure defines a topology, giving jointly continuous addition and multiplication. The topology is separated if and only if the ring is semisimple.*

Proof. Ninety percent of the argument ([6, Theorem 2]) closely follows that for the singleton version (1.4); we only check the final condition

$$(2.1)\quad \mathrm{Cl}(K \cup H) \subseteq \mathrm{Cl}(K) \cup \mathrm{Cl}(H).$$

Thus suppose that

$$x \in \mathrm{Cl}(K \cup H) \setminus \mathrm{Cl}(H),$$

so that there is a finite subset $L \subseteq A$ for which

$$\forall h \in H, \quad 1 - L(x-h) \not\subseteq A^{-1}.$$

Now for arbitrary finite $J \subseteq A$ we have $x' \in K \cup H$ for which

$$1 - L(x-x') \subseteq 1 - (J \cup L)(x-x') \subseteq A^{-1},$$

and necessarily $x' \in K$. Finally it is easily checked that

$$(2.2)\quad \mathrm{Cl}(\{a\}) = a + \mathrm{Rad}(A),$$

where

$$(2.3)\quad \mathrm{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\}. \ \blacksquare$$

For example if A is a Banach algebra then the norm closure is still a subset of the spectral closure:

$$\mathrm{cl}(\cdot) \subseteq \mathrm{Cl}_A(\cdot) \subseteq \mathrm{cl}^A_{\mathrm{alg}}(\cdot); \tag{2.4}$$

indeed, using the geometric series,

$$\max_{y \in J} \|y\| \, \|x - x'\| < 1 \ \Rightarrow\ 1 - J(x - x') \subseteq A^{-1}. \tag{2.5}$$

The reverse inclusions are not clear: for example it is clear from (2.1), together with the failure ([6, Example 3]) of the analogue for (1.4), that the spectral closure is liable to be genuinely smaller than the singleton version for two-dimensional algebras. Also the spectral and the norm closure differ for singletons unless the Banach algebra is "semisimple"; we shall see below that even for the simplest semisimple Banach algebras the spectral topology is much weaker than the norm.

The spectral topology respects cartesian products:

3. Theorem. *If A and B are rings, and $A \oplus B$ is given its direct sum addition and multiplication, then the spectral topology for $A \oplus B$ is the cartesian product of the spectral topology for the factors. The same is true one way round for arbitrary products; in particular if $A = D^\Omega$ then*

$$H \subseteq D \ \Rightarrow\ (\mathrm{Cl}_D(H))^\Omega \subseteq \mathrm{Cl}_A(H^\Omega). \tag{3.1}$$

Proof. If $K \subseteq A$ and $H \subseteq B$ then necessary and sufficient for $(x, y) \in \mathrm{Cl}(K \oplus H)$ is that for arbitrary finite $J \subseteq A$ and $L \subseteq B$ there is $(x', y') \in K \oplus H$ for which

$$(1, 1) - (J \oplus L)(x - x', y - y') \subseteq (A \oplus B)^{-1} = A^{-1} \oplus B^{-1}, \tag{3.2}$$

which holds if and only if $1 - J(x - x') \subseteq A^{-1}$ and $1 - L(y - y') \subseteq B^{-1}$. More generally

$$J \subseteq A = \prod_{t \in \Omega} A_t \ \Rightarrow\ J \subseteq \prod_{t \in \Omega} J_t,$$

with finite $J_t \subseteq A_t$, and if $x_t \in K_t$ for each $t \in \Omega$ then there is $x' \in A$ with $x'_t \in K_t$ for which

$$1 - J(x - x') \subseteq \prod_{t \in \Omega} (1_t - J_t(x_t - x'_t)) \subseteq \prod_{t \in \Omega} A_t^{-1} = A^{-1}$$

and hence

$$K = \prod_{t \in \Omega} K_t \subseteq A \ \Rightarrow\ \prod_{t \in \Omega} \mathrm{Cl}_{A_t}(K_t) \subseteq \mathrm{Cl}_A(K). \tag{3.3}$$

Specialising to the case in which $A_t = D$ and $K_t = H$ for each $t \in \Omega$ gives (3.1). ∎

We shall see below (Theorem 7) that the spectral topology of the quotient by a two-sided ideal is no stronger than the quotient of the spectral topology, and similarly the spectral topology of a subring which is invertibly closed is no stronger than the relative topology. We note also that Theorem 3 does not say, for a linear algebra A, that in its spectral topology it is a topological vector space.

For the ring of integers $\mathbb{Z}$, whose invertible group $\mathbb{Z}^{-1} = \{-1, 1\}$ is rather small, we get back the familiar discrete topology:

$$K \subseteq \mathbb{Z} \;\Rightarrow\; \mathrm{Cl}(K) = K; \tag{3.4}$$

indeed, if $J \subseteq \mathbb{Z}$ has more than two nonzero elements then the only way for $1 - Jy \subseteq \mathbb{Z}^{-1}$ is $y = 0$. The same is true if A is Boolean, with

$$\forall x \in A, \quad x + x = x = x^2, \tag{3.5}$$

since now $A^{-1} = \{1\}$. At the other extreme, if $A = A^{-1} \cup \{0\}$ is a division ring then the spectral closure gives the "cofinite topology", for which

$$\#K < \infty \;\Rightarrow\; \mathrm{Cl}(K) = K; \qquad \#K \geq \infty \;\Rightarrow\; \mathrm{Cl}(K) = A, \tag{3.6}$$

since there cannot be more $y \in A$ for which $1 \in Jy$ than there are elements in $J \subseteq A$. This of course applies to $\mathbb{R}$ and $\mathbb{C}$: the spectral topology of a Banach algebra is therefore in general a much weaker topology than the norm topology. More generally if $A = A^{-1} \cup \mathrm{Rad}(A)$ is a *local ring* then $K \subseteq A$ is closed iff it is finite modulo the radical.

For finite powers A^n of division algebras, or more generally finite products, the spectral topology is a cartesian product of cofinite topologies.

For the local ring $A = \mathbb{C}[[z]]$ of all formal power series a subset $K \subseteq A$ is closed iff the set of its leading coefficients $K_0 \subseteq \mathbb{C}$ is finite. If on the other hand $A \subseteq \mathbb{C}[[z]]$ consists of the functions whose coefficient sequences belong to one of the ℓ_p spaces, so that the invertible group A^{-1} is contained in $\mathbb{C}$, then the spectral topology is again discrete: for example, looking at neighbourhoods of $0 \in \mathbb{C} \subseteq A$, there is implication

$$J \not\subseteq \mathbb{C} \;\Rightarrow\; \{x \in A : 1 - Jx \subseteq A^{-1} \subseteq \mathbb{C}\} = \{0\}. \tag{3.7}$$

In the spectral topology, the invertible group is an open set and inversion is continuous:

4. Theorem. *There is inclusion*

$$\mathrm{Cl}(A \setminus A^{-1}) \subseteq A \setminus A^{-1}, \tag{4.1}$$

and if $K \subseteq A^{-1}$ *is arbitrary, inclusion*

$$(A^{-1} \cap \mathrm{Cl}(K))^{-1} \subseteq \mathrm{Cl}(K^{-1}). \tag{4.2}$$

If A is a topological ring with separately continuous multiplication, then necessary and sufficient for the spectral topology to be weaker than the original is that the invertible group be open:

$$\mathrm{cl}(\cdot) \subseteq \mathrm{Cl}(\cdot) \;\Leftrightarrow\; A^{-1} \subseteq \mathrm{int}(A^{-1}). \tag{4.3}$$

The spectral topology is therefore the weakest ring topology for which the invertible group is open.

Proof. If $a \in A^{-1}$ is arbitrary then

$$J = \{a^{-1}\} \Rightarrow 1 - J(a - A^{-1}) \subseteq A^{-1}, \tag{4.4}$$

giving (4.1). If $K \subseteq A^{-1}$ and $a \in \mathrm{Cl}(K) \cap A^{-1}$ then for arbitrary finite $J \subseteq A$ there is $c \in K$ for which $1 - J(a - c) \subseteq A^{-1}$ and hence

$$1 - L(c^{-1} - a^{-1}) \subseteq A^{-1} \quad \text{with} \quad L = cJa. \tag{4.5}$$

Towards (4.3), if A^{-1} is open then by separate continuity of multiplication there is for each $y \in A$ a neighbourhood $U_y \in \mathrm{Nbd}(0)$ for which

$$1 + yU_y \subseteq A^{-1}$$

and then, with

$$U_J = \bigcap_{y \in J} U_y,$$

implication that if $x \in \mathrm{cl}(K)$ and $J \in \mathrm{Finite}(A)$ there is $x' \in K$ for which $x - x' \in U_J$, giving

$$1 - J(x - x') \subseteq 1 - JU_J \subseteq A^{-1}.$$

The other way round is (4.1): A^{-1} is open relative to the spectral closure, and therefore also for any stronger topology. ∎

The spectral closure of the invertibles participates in the familiar curious minuet with the left and the right invertibles:

$$A^{-1}_{\mathrm{left}} \cap \mathrm{Cl}(A^{-1}_{\mathrm{right}}) = A^{-1} = A^{-1}_{\mathrm{right}} \cap \mathrm{Cl}(A^{-1}_{\mathrm{left}}). \tag{4.6}$$

We simply observe ([6, Theorem 4]) that if $a'a = 1$ and $1 - a'(a - a'') \in A^{-1}$ with $a'' \in A^{-1}_{\mathrm{right}}$ then a'', and hence a', and hence a, must have two-sided inverses. If we refer to what is in the spectral closure of the invertibles as nearly invertible then (4.6) says that a nearly invertible with a one-sided inverse has to be invertible. The analogue of (4.6) for the topological closure is familiar in Banach algebras [11], and follows at once from (2.4) and (4.6): indeed, the Banach algebra arguments are really about the spectral, or the algebraic, closure in disguise. We might remark that the passage from (1.4) to (1.1) in no way simplifies the argument; however, having a genuine topology enables us freely to pass between concepts of closure, neighbourhoods and open sets. For the record, basic neighbourhoods of the origin $U_J \in \mathrm{Nbd}(0)$ come from finite subsets $J \subset A$:

$$U_J = \{x \in A : 1 - Jx \subseteq A^{-1}\}. \tag{4.7}$$

For a sequence (x_n) to converge to zero there has to be a family (N_J) of natural numbers, indexed by finite $J \subseteq A$, for which

(4.8) $$n \geq N_J \ \Rightarrow \ 1 + Jx_n \subseteq A^{-1}.$$

The spectral closure appears to offer [7] an extension to general rings of the Banach algebra concept of "quasinilpotent":

5. Definition. $a \in A$ will be said to be *quasinilpotent*, written $a \in \mathrm{QN}(A)$, provided

(5.1) for arbitrary $m \in \mathbb{N}$ and $\{c, d\} \subseteq A^m$, $c_{(n)}a^n d^{(n)} \to 0 \ (n \to \infty)$,

where

(5.2) $$b \in A^m \ \Rightarrow \ b^{(n)} = b_1^n b_2^n \dots b_m^n, \ b_{(n)} = b_m^n \dots b_2^n b_1^n.$$

The analogous condition, with norm convergence, gives back ([3, (7.4.4.5)]) the usual concept when A is a Banach algebra. Our spectral closure version of quasinilpotents behaves ([7, Theorem 3.1]) mostly as we would wish:

(5.3) $$0 \in \{a^k : k \in \mathbb{N}\} \ \Rightarrow \ a \in \mathrm{QN}(A),$$

there is implication

(5.4) $$a \in \mathrm{QN}(A), \, ba = ab \ \Rightarrow \ ab \in \mathrm{QN}(A),$$

and inclusion

(5.5) $$1 - \mathrm{QN}(A) \subseteq A^{-1} \ ;$$

thus certainly

(5.6) $$\mathrm{Rad}(A) \subseteq \mathrm{QN}(A) \subseteq \mathrm{Cl}(A^{-1}).$$

We have not however been able to settle whether or not there is implication

(5.7) $$a, b \in \mathrm{QN}(A), \ ba = ab \ \Rightarrow \ a + b \in \mathrm{QN}(A).$$

The spectral closure intervenes in generalized inverse theory:

6. Proposition. *With*

(6.1) $$A^{\cap} = \{a \in A : a \in aAa\} \quad \textit{and} \quad A^{\cup} = \{a \in A : a \in aA^{-1}a\}$$

there is inclusion

(6.2) $$A^{\cap} \cap \mathrm{Cl}(A^{-1}) \subseteq A^{\cup}.$$

Necessary and sufficient for equality in (6.2) *is that*

(6.3) $$A^{\bullet} \equiv \{p \in A : p = p^2\} \subseteq \mathrm{Cl}(A^{-1}).$$

Proof. The argument is the same as for the singleton version: if $a = aa'a \in A^{\cap}$, so that $a'a = p = p^2$ is idempotent, and if

$$b \in A^{-1} \quad \text{with} \quad 1 + (b - a)a' = c^{-1} \subseteq A^{-1},$$

then

$$a = (cb)p \in A^{\cup}.$$

For equality in (6.2), observe [10], [11]

$$A^{\cup} = A^{-1}A^{\bullet} = A^{\bullet}A^{-1}. \blacksquare \tag{6.4}$$

In words, nearly invertibles with generalized inverses have invertible generalized inverses. In Banach algebras, (6.3) is clear [10], [11]: generally

$$0 \notin \operatorname{int}(\sigma(a)) \ \Rightarrow\ a \in \operatorname{cl}(A^{-1}) \subseteq \operatorname{Cl}(A^{-1}), \tag{6.5}$$

where $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}\}$ is the usual spectrum.

The spectral closure also intervenes in abstract Fredholm theory. Classically Fredholm operators on Banach spaces always have generalized inverses, and have invertible generalized inverses iff they are of index zero; thus Theorem 6 says that nearly invertible Fredholm operators have index zero. This survives in an abstract context, where homomorphisms $T : A \to B$ between rings bring with them Weyl and Fredholm [10], [11], [18] elements:

$$A^{-1} \subseteq A^{-1} + T^{-1}(0) \subseteq T^{-1}(B^{-1}). \tag{6.6}$$

We note first that a homomorphism which is onto is automatically continuous:

7. THEOREM. *If the homomorphism $T : A \to B$ is onto, in the sense that*

$$B \subseteq T(A), \tag{7.1}$$

then it is continuous, in the sense that

$$K \subseteq A \ \Rightarrow\ T(\operatorname{Cl}_A(K)) \subseteq \operatorname{Cl}_B(T(K)); \tag{7.2}$$

conversely, T if has the Gelfand property, in the sense ([11, (9.6.0.1)], [12]) *that*

$$T^{-1}(B^{-1}) \subseteq A^{-1}, \tag{7.3}$$

then it is relatively open, in the sense that

$$K \subseteq A \ \Rightarrow\ T(A) \cap \operatorname{Cl}_B(T(K)) \subseteq T(\operatorname{Cl}_A(K)). \tag{7.4}$$

If more generally T has inverse closed range,

$$T(A) \cap B^{-1} \subseteq T(A)^{-1}, \tag{7.5}$$

then

$$K \subseteq A \ \Rightarrow\ \operatorname{Cl}_A(K) \cap T^{-1}(B^{-1}) \subseteq T^{-1}(0) + A^{-1}K. \tag{7.6}$$

Proof. If $T : A \to B$ is onto and $x \in \operatorname{Cl}_A(K) \subseteq A$ then for arbitrary finite $L = T(J) \subseteq B$ there is $y' = Tx' \in T(K)$ for which $T1 - L(Tx - y) \subseteq T(A^{-1}) \subseteq B^{-1}$, giving (7.2). Conversely, if $y = Tx \in \operatorname{Cl}_B(T(K))$ then for arbitrary finite $J \subseteq A$ there is $y' = Tx' \in T(K)$ for which $T(1 - J(x - x')) \subseteq B^{-1}$, giving by (7.3) inclusion $1 - J(x - x') \subseteq A^{-1}$. Towards (7.6), if $a \in T^{-1}(B^{-1})$ then by (7.5) there is $d \in A$ for which

$$\{1 - ad, 1 - da\} \subseteq T^{-1}(0), \tag{7.7}$$

and if also $a \in \mathrm{Cl}_A(K)$ there is $c \in K$ for which

$$1 - (a - c)d = e^{-1} \in A^{-1}. \tag{7.8}$$

But now

$$\begin{aligned} a = e(1 - (a - c)d)a &= e(1 - ad)a + ec(da - 1) + ec \\ &\in T^{-1}(0) + T^{-1}(0) + A^{-1}K. \ \blacksquare \end{aligned}$$

(7.1) applies in particular when $B = A/J$ is the quotient of A by a two-sided ideal $J \subseteq A$, and (7.2) tells us that the spectral topology for A/J is weaker than or equal to the quotient of the spectral topology of A; (7.3) applies when $A \subseteq B$ is an inverse closed subring, and (7.4) tells us that its spectral topology is weaker than or equal to the restriction of the spectral topology of B. Each of the conditions (7.1) and (7.3) implies inverse closure (7.5), which is enough to ensure that nearly invertible Fredholm elements are Weyl:

8. Theorem. *If $T : A \to B$ is a homomorphism then sufficient for inclusion*

$$\mathrm{Cl}(A^{-1}) \cap T^{-1}(B^{-1}) \subseteq A^{-1} + T^{-1}(0) \tag{8.1}$$

is that T has inverse closed range, in the sense of (7.5). *Also necessary and sufficient for inclusion*

$$A^{-1} + T^{-1}(0) \subseteq \mathrm{Cl}(A^{-1}) \cap T^{-1}(B^{-1}) \tag{8.2}$$

is that the kernel $T^{-1}(0)$ is weakly Riesz, in the sense (*cf.* [9]) *that*

$$1 + T^{-1}(0) \subseteq \mathrm{Cl}(A^{-1}). \tag{8.3}$$

Proof. To see why (7.5) implies (8.1) apply (7.6) with $K = A^{-1}$. Towards (8.2), if the ideal $J = T^{-1}(0)$ satisfies the weakly Riesz condition (8.3) then

$$A^{-1} + T^{-1}(0) = A^{-1}(1 + T^{-1}(0)) \subseteq A^{-1}\,\mathrm{Cl}(A^{-1}) = \mathrm{Cl}(A^{-1});$$

the reverse implication is clear. $\blacksquare$

In words, nearly invertible Fredholms are Weyl. Theorem 8 is derived from a result of Xue [18], which deals with Banach algebras and the topological closure. We have however separated his result into two autonomous components, analogous [12] to an "Arens" and a "Royden" property of the homomorphism T. When $T : A \to B$ is a bounded homomorphism of Banach algebras then the condition (8.1) implies the corresponding condition with norm closure, which in turn follows from "Property (F)" of Xue [18], which is equivalent to the analogue of (7.5) in which invertible groups are replaced by the *generalized exponentials* ([11, Definition 7.11.3]), the connected components of their identities:

$$\mathrm{Exp}(B) \cap T(A) \subseteq T(\mathrm{Exp}(A)). \tag{8.4}$$

We might notice that the analogue of the invertibility closure condition (7.5), and more, always holds for the subsemigroup $1 + T^{-1}(0)$:

$$(1 + T^{-1}0) \cap A^{\cap} \subseteq (1 + T^{-1}(0))^{\cap}, \tag{8.5}$$

since if $a = aa'a$ with $Ta = 1$ then necessarily $Ta' = 1$. The condition (8.3) says that the ideal $J = T^{-1}(0)$ is in a sense of "stable rank one",

$$1 + J \subseteq \mathrm{Cl}_{1+J}(1 + J)^{-1}, \tag{8.6}$$

since there is implication

$$\begin{aligned} 1 + J \subseteq \mathrm{Cl}_A(A^{-1}) \;&\Rightarrow\; 1 + J \subseteq \mathrm{Cl}_A(1 + J)^{-1} \\ &\Rightarrow\; 1 + J \subseteq \mathrm{Cl}_{1+J}(1 + J)^{-1}. \end{aligned} \tag{8.7}$$

Indeed, it is clear at once that the middle condition implies both the first and the last; if $J = T^{-1}(0)$ is weakly Riesz then for $a \in 1 + J$ and arbitrary finite $L \subseteq A$ there is $c \in A^{-1}$ for which, changing sign, $1 + (L \cup LaL)(a - c) = U^{-1} \subseteq A^{-1}$. Now, with $U' = cU$,

$$U' \subseteq 1 + J$$

and

$$\begin{aligned} 1 - L(a - U') \subseteq 1 - L(a - cU) &\subseteq ((1 - La)(1 + La - Lc) + Lc)U \\ &= 1 + LaL(a - c) \subseteq A^{-1}. \end{aligned}$$

We have not settled whether we can reverse the second implication of (8.7), which is of course a triviality for the norm closure.

We can extend the spectral closure to tuples:

9. Definition. For arbitrary $K \subseteq A^n$,

$$\begin{aligned} &\mathrm{Cl}^{(n)}_{\mathrm{left}}\, K \\ &\quad = \{x \in A^n : \forall \text{ finite } J \subseteq A^n\ \exists x' \in K,\ 1 - J \cdot (x - x') \subseteq A^{-1}\}, \end{aligned} \tag{9.1}$$

and similarly for $\mathrm{Cl}^{(n)}_{\mathrm{right}}(K)$, where we write

$$a' \cdot a = \sum_{j=1}^{n} a'_j a_j \in A \qquad (a, a' \in A^n). \tag{9.2}$$

We remark that these are weaker topologies on A^n than the cartesian product of Theorem 3. We can also define left and right invertible tuples: in the notation (9.2) we set ([11, Definition 11.1.1])

$$\begin{aligned} A^{-n}_{\mathrm{left}} &= \Big\{a \in A^n : 1 \in A^n \cdot a = \sum_{j=1}^{n} A a_j\Big\}, \\ A^{-n}_{\mathrm{right}} &= \Big\{a \in A^n : 1 \in a \cdot A^n = \sum_{j=1}^{n} a_j A\Big\}. \end{aligned} \tag{9.3}$$

As with the singleton version, the Kuratowski conditions continue to hold among n tuples, in particular now the union condition (2.1) holds. Note that we do not have a multivariable analogue of Jacobson's Lemma (1.3): if $vu = 1 \neq uv$ then

$$(9.4) \qquad \begin{aligned} 1-(1/2)(1-uv)-(1/2)vu &= (1/2)uv, \\ 1-(1-uv)(1/2)-u(1/2)v &= 1/2. \end{aligned}$$

Thus (9.1) is not equivalent to its "right" analogue.

Theorem 3 and the first two implications of Theorem 7 also extend: if $T : A \to B$ is onto (7.1) then its extension to A^n is still continuous,

$$(9.5) \qquad K \subseteq A^n \ \Rightarrow\ T(\mathrm{Cl}_A^{(n)}(K)) \subseteq \mathrm{Cl}_B^{(n)}(T(K));$$

conversely, if T has the Gelfand property (7.3) then its extension is still relatively open,

$$(9.6) \qquad K \subseteq A^n \ \Rightarrow\ T(A^n) \cap \mathrm{Cl}_B^{(n)}(T(K)) \subseteq T(\mathrm{Cl}_A^{(n)}(K)).$$

When $n \in \mathbb{N}$ we declare [3] that a ring A *has left (Bass) stable rank* $\leq n$ provided

$$(9.7) \qquad \forall (a,b) \in A^n \times A, \quad (a,b) \in A_{\mathrm{left}}^{-n-1} \ \Rightarrow\ \exists c \in A^n,\ a - cb \in A_{\mathrm{left}}^{-n}.$$

Corach and Suarez [5], Blackadar [4] and Herman and Vaserstein [13] have considered this kind of situation when A is commutative, or a C^*-algebra. We have the following curious hybrid result:

10. THEOREM. *Sufficient for the ring A to have left Bass stable rank $\leq n$ is that*

$$(10.1) \qquad A^n \subseteq \mathrm{Cl}_{\mathrm{right}}^{(n)}(A_{\mathrm{right}}^{-n}).$$

Proof. Suppose

$$a' \cdot a + b'b = 1 \text{ with } a' \in \mathrm{Cl}_{\mathrm{right}}^{(n)}(A_{\mathrm{right}}^{-n}),$$

so that there are $a'', a''' \in A^n$ with

$$b'b = 1 - a' \cdot a = d - a'' \cdot a \quad \text{with} \quad d \in A^{-1},\ a'' \cdot a''' = 1;$$

then $d^{-1}a'' \cdot (a + a'''b'b) = d^{-1}(a'' \cdot a + b'b)$, giving

$$a - cb = a'''d \in A_{\mathrm{left}}^{-n} \quad \text{with} \quad c = -a'''b'. \ ■$$

This is essentially Theorem 2.3 of Rieffel [17]. Notice the difference when $b = 0$ between this and (4.6). When $A = D^\Omega$ then the extended version of (3.1), with $H = D_{\mathrm{right}}^{-n}$, says that condition (10.1) is transmitted from D to A. If Ω is a topological space and D is a topological ring then the inclusion $T : A = C(\Omega, D) \to D^\Omega = B$ satisfies the condition (7.5) but not in general the Gelfand condition (7.3), so that (10.1) need not be transmitted from D^Ω to $C(\Omega, D)$. For a commutative Banach algebra A of course the Gelfand homomorphism $T : A \to B = C(\sigma(A))$ does satisfy (7.3); this does not however say either that the condition (10.1) is transmitted from $B = C(\sigma(A))$ to A, or conversely.

For a C^*-algebra A the condition (10.1), its norm analogue, and (9.7) are [13] equivalent; in contrast when $A = A(\mathbb{D})$ is the disc algebra and $n = 1$ then (10.2) holds ([14, Theorem 1]) while the norm analogue of (10.1) fails. Indeed (cf. [16, Example 2.3]) the homomorphism $T : A \to B = C(\partial\mathbb{D})$ of restriction and embedding is isometric and, with $a = z \in A$, whose spectrum is $\mathbb{D}$,

$$\{\lambda \in \mathbb{C} : a - \lambda \in \partial A^{-1}\} \subseteq \{\lambda \in \mathbb{C} : T(a - \lambda) \in \partial B^{-1}\} = \partial\mathbb{D}, \tag{10.2}$$

excluding $a = z \in A$ from the (norm) closure of A^{-1}. When $n = 1$ condition (10.1) holds for $A = C(\partial\mathbb{D})$ and fails for $A = C(\mathbb{D})$, and is undecided for $A = A(\mathbb{D})$. In one tiny constructive step towards (10.1) for the disc algebra Walter Hayman, Finbarr Holland, Anthony O'Farrell and Rupert Levene have each noticed that, for example,

$$b = \frac{1}{2}a + i\sqrt{2} \;\Rightarrow\; \{b, 1 - a(a - b)\} \subseteq A^{-1}. \tag{10.3}$$

Ara, Pedersen and Perera [1], [2], using (9.7), have a different kind of "algebraic closure": let us write $a \in \mathrm{cl}^{\sim}_{\mathrm{left}}(K)$ to mean that for arbitrary $b \in A$ there is implication

$$(a, b) \in A^{-2}_{\mathrm{left}} \subseteq A^2 \;\Rightarrow\; (a - Ab) \cap K \neq \emptyset \subseteq A. \tag{10.4}$$

This satisfies all but the last of the Kuratowski conditions. We believe that it can be converted to a topology with a modification similar to ours if we instead define, for $K \subseteq A^n$,

$$a \in \mathrm{Cl}^{\sim}_{\mathrm{left}}(K) \subseteq A^n \;\Leftrightarrow\; \forall J \in \mathrm{Finite}(A), \quad (a, J) \subseteq A^{-n-1}_{\mathrm{left}} \;\Rightarrow\; \bigcap_{b \in J} (a - A^n b) \cap K \neq \emptyset. \tag{10.5}$$

Such a modification would make a cosmetic change in the concept (9.7) of "Bass stable rank", which might or might not actually alter it. With this notation the implication (10.1)⇒(9.7) says

$$A^n \subseteq \mathrm{Cl}^{(n)}_{\mathrm{right}}(A^{-n}_{\mathrm{right}}) \;\Rightarrow\; A^n \subseteq \mathrm{Cl}^{\sim}_{\mathrm{left}}(A^{-n}_{\mathrm{left}}). \tag{10.6}$$

We are unable to decide whether in general there is inclusion, for suitably related pairs $K, H \subseteq A^n$,

$$\mathrm{Cl}^{(n)}_{\mathrm{right}}(K) \subseteq \mathrm{Cl}^{\sim}_{\mathrm{left}}(H) \subseteq A^n. \tag{10.7}$$

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