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Continuation of holomorphic functions with growth conditions and some of its applications

by

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Abstract. We prove a generalization of the well-known Hörmander theorem on continuation of holomorphic functions with growth conditions from complex planes in $\mathbb{C}^p$ into the whole $\mathbb{C}^p$. We apply this result to construct special families of entire functions playing an important role in convolution equations, interpolation and extension of infinitely differentiable functions from closed sets. These families, in their turn, are used to study optimal or canonical, in a certain sense, weight sequences defining inductive and projective type spaces of entire functions with O-growth conditions. Finally, we give a natural and complete description of multipliers for spaces given by canonical weight sequences.

1. Introduction. Let φ be a plurisubharmonic (psh) function in $\mathbb{C}^p$ such that, for some $C_0 > 0$,

$$|\varphi(z) - \varphi(\zeta)| \le C_0 \quad \text{for all } z, \zeta \in \mathbb{C}^p \text{ with } |z - \zeta| \le 1. \tag{1.1}$$

In this case, it is natural to say that φ is *stable* (or *slowly varying*) with respect to the distance function $\rho(t) \equiv 1$. Next, let f be a holomorphic function on a complex subspace Σ of dimension k in $\mathbb{C}^p$ and

$$A_f := \int_{\Sigma} |f(z)|^2 e^{-2\varphi(z)}\, d\lambda_z < \infty, \quad \text{where } \lambda \text{ is the Lebesgue measure.}$$

Then, according to Hörmander's well-known result [7, Theorem 4.4.3], f can be continued as an entire function F on $\mathbb{C}^p$ (i.e., $F|_{\Sigma} = f$) satisfying the condition

$$\int_{\mathbb{C}^p} |F(z)|^2 e^{-2\varphi(z)} (1 + |z|^2)^{-3(p-k)}\, d\lambda_z \le C A_f,$$

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where C is an absolute constant depending on C_0, p and k and independent of φ, Σ and f. This easily implies a similar statement with uniform estimates for f and F instead of integral ones.

The Hörmander theorem is very useful when we have some fact for $p = 1$ and need to extend it to the multivariate case. It has been successfully applied in the theory of entire functions, approximation and interpolation, duality of function spaces, and convolution equations. On the other hand, for some problems the condition (1.1) is too restrictive. To study the problem of approximation of psh functions by $\log|f|$ with an entire function f, Youlmukhametov (see [13, Lemma 1]) obtained an analog of the Hörmander theorem under the weaker condition

$$|\varphi(z) - \varphi(\zeta)| \le C_0 \quad \text{for all } z, \zeta \in \mathbb{C}^p \text{ with } |z - \zeta| \le \frac{1}{(1 + |z|)^s}, \tag{1.2}$$

where s is a positive constant. Now φ is slowly varying with respect to $\rho(t) = (1 + t)^{-s}$. Later, Musin [12] used this analog to describe, via the Fourier–Borel transformation of functionals, the dual space to a certain weighted space of infinitely differentiable functions in $\mathbb{R}^n$ with prescribed growth of all derivatives at infinity.

In Section 2 of the present paper we prove a general Hörmander type theorem (see Theorem 2.1), where the stability of a psh function φ is controlled by a distance function ρ satisfying some natural properties.

In Section 3 we use Theorem 2.1 to construct special families of entire functions having uniform upper growth bounds and prescribed values, close to those bounds, at each individual point of $\mathbb{C}^p$. Families of such type are useful in many problems concerning convolution equations, interpolation, sufficient sets, and extension of infinitely differentiable functions from closed sets (see, e.g., [1], [3], [6], [8]–[11]). They are closely connected with optimal, in a certain sense, weight sequences used for spaces of entire functions satisfying O-growth conditions.

In Section 4 we discuss such sequences (called *canonical*) and give sufficient conditions for a weight sequence to be canonical in two different cases, inductive and projective. It should be noted that the projective case is more complicated and has some particularities. Our definitions of canonical weight sequences is based on the notion of associate weight function introduced in Bierstedt–Bonet–Taskinen [5] (see also [4]).

Finally, in Section 5 we give, in terms of canonical weight sequences, a complete description of multipliers for weighted inductive and projective spaces of entire functions.

2. Continuation of holomorphic functions with growth conditions. In this section we obtain a Hörmander type result concerning the continuation of holomorphic functions with growth conditions. Without further reference we shall use the standard notation from complex analysis (see [7]).

A decreasing C^1-function $\rho : [0,\infty) \to (0,1]$ is called a *regular distance function* if

$$\rho'(t) \to 0 \quad \text{as } t \to \infty, \tag{2.1}$$

$$\log \rho(e^x) \text{ is concave on } \mathbb{R}. \tag{2.2}$$

The following functions are regular distances:

(1) $\rho(t) = 1$;
(2) $\rho(t) = 1/(1+t)^s,\ s > 0$;
(3) $\rho(t) = e^{-at^s},\ a > 0, s > 0$;
(4) $\rho(t) = e^{-\exp(\exp(\ldots(\exp t))\ldots)}$.

Notice some common properties of such functions. Using (2.1), we have

$$A_0 := \max_{t\in[0,\infty)} |\rho'(t)| < \infty$$

and there exists $t_0 \ge 0$ such that $\rho'(t) \ge -1/2$ for all $t \ge t_0$. Finding $\xi = \xi(t)$ in $(t, t+\rho(t))$ with $\rho(t+\rho(t)) = (1+\rho'(\xi))\rho(t)$, we then have $\rho(t) \le 2\rho(t+\rho(t))$ for all $t \ge t_0$. Hence, there always exists $B_0 > 1$ such that

$$\rho(t) \le B_0\rho(t+\rho(t)) \quad \text{for all } t \ge 0.$$

This implies

$$\rho(t) \le B_0\rho(t') \text{ and } \rho(t') \le B_0\rho(t) \text{for all } t, t' \ge 0 \text{ with } |t-t'| \le \rho(t). \tag{2.3}$$

Also, put $\rho(z) := \rho(|z|)$ for $z \in \mathbb{C}^p$ ($p \in \mathbb{N}$). Since ρ decreases and satisfies (2.2), the function $-\log \rho(z)$ is psh in $\mathbb{C}^p$.

We say that a function φ is *ρ-stable* in $\mathbb{C}^p$ if there exists $C_0 \in [0,\infty)$ such that

$$|\varphi(z) - \varphi(\zeta)| \le C_0 \quad \text{for all } z, \zeta \in \mathbb{C}^p \text{ with } |z-\zeta| \le \rho(z). \tag{2.4}$$

The main result of this section is the following theorem.

THEOREM 2.1. *Let φ be a ρ-stable psh function in $\mathbb{C}^p$ and C_0 the constant from (2.4). Then for any complex plane Σ in $\mathbb{C}^p$ of dimension k and a holomorphic function f on Σ with $\log|f(z)| \le \varphi(z)$ $(z \in \Sigma)$ there exists an entire function F in $\mathbb{C}^p$ such that $F|_\Sigma = f$ and, for all $z \in \mathbb{C}^p$,*

$$\log|F(z)| \le \varphi(z) + (2p-k)\log\frac{1}{\rho(z)} + \frac{3p-2k+1}{2}\log(1+|z|^2) + \frac{p-k}{2}\log(1+d_\Sigma^2) + M,$$

where d_Σ is the distance from the origin to Σ and M an absolute constant depending on A_0, B_0, C_0, p and k and independent of φ, Σ, f and ρ.

To prove this theorem, we need some preparations.

Given $1 \le k < p$ and $c := (c_{k+1}, \dots, c_p) \in \mathbb{C}^{p-k}$, fix $n \in \mathbb{N}$ with $k \le n < p$ and denote

$$z := (z_1, \dots, z_n, z_{n+1}) \in \mathbb{C}^{n+1}, \quad z' := (z_1, \dots, z_n) \in \mathbb{C}^n,$$
$$(z, c) := (z_1, \dots, z_n, z_{n+1}, c_{n+2}, \dots, c_p) \in \mathbb{C}^p,$$
$$(z', c') := (z_1, \dots, z_n, c_{n+1}, \dots, c_p) \in \mathbb{C}^p.$$

We put $h(z,c) := h((z,c))$ and $h(z',c') := h((z',c'))$ for a function h defined on $\mathbb{C}^p$. Denote by $\lambda_{z'}$ and λ_z the Lebesgue measures in $\mathbb{C}^n_{z'} \cong \mathbb{R}^{2n}$ and $\mathbb{C}^{n+1}_z \cong \mathbb{R}^{2n+2}$, respectively.

Lemma 2.2. *Let g be an entire function on $\mathbb{C}^n_{z'}$ such that*

$$I(g) := \int\limits_{\mathbb{C}^n} \frac{\rho^{2l}(z',c')|g(z')|^2}{(1+|z'|^2)^m} e^{-2\varphi(z',c')}\, d\lambda_{z'} < \infty$$

for some $l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then there exists an entire function G in $\mathbb{C}^{n+1}_z$ with $G(z', c_{n+1}) = g(z')$ on $\mathbb{C}^n_{z'}$ and

$$\int\limits_{\mathbb{C}^{n+1}} \frac{\rho^{2l+2}(z,c)|G(z)|^2}{(1+|z|^2)^{m+3}} e^{-2\varphi(z,c)} d\lambda_z \le N_0^2 B_0^{2l+2}(1+|c_{n+1}|^2) I(g),$$

where $N_0 > 0$ is an absolute constant depending only on A_0 and C_0.

Proof. Take an infinitely differentiable function η on $[0,\infty)$ with $\eta(t) = 1$ on $[0,1/2]$, $0 \le \eta(t) \le 1$ on $[1/2,1]$, and $\eta(t) = 0$ on $[1,\infty)$. Denote $\alpha_0 := \max_{t\ge 0} |\eta'(t)|$ and $\tau(z) := |z_{n+1} - c_{n+1}|/\rho(z',c')$, $z \in \mathbb{C}^{n+1}$. It is clear that the $(0,1)$-form

$$V(z) := \frac{g(z')}{z_{n+1} - c_{n+1}} \overline{\partial}(\eta(\tau(z)))$$

is well-defined on $\mathbb{C}^{n+1}$ and $\overline{\partial}$-closed (i.e., $\overline{\partial} V = 0$). Let

$$B(z') := \{w \in \mathbb{C} : \rho(z',c')/2 \le |w - c_{n+1}| \le \rho(z',c')\}, \quad z' \in \mathbb{C}^n,$$
$$E := \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : z_{n+1} \in B(z'),\ z' \in \mathbb{C}^n\}.$$

Then $V(z) = 0$ outside E and, for $z \in E$,

$$V(z) = \frac{g(z')}{z_{n+1} - c_{n+1}} \eta'(\tau(z)) \overline{\partial}(\tau(z)) = \frac{g(z')}{2(z_{n+1} - c_{n+1})} \frac{\eta'(\tau(z))}{\rho(z',c')}$$
$$\times \left(\frac{z_{n+1} - c_{n+1}}{|z_{n+1} - c_{n+1}|} d\overline{z}_{n+1} - \frac{|z_{n+1} - c_{n+1}|\rho'(z',c')}{\rho(z',c')} \frac{1}{|(z',c')|} \sum_{j=1}^n z_j d\overline{z}_j \right).$$

It follows that for any $z \in E$,

$$\|V(z)\| \le \alpha_0 \frac{|g(z')|}{\rho^2(z',c')} \sqrt{1+(\rho'(z',c'))^2} \le \alpha_0 \sqrt{1+A_0^2}\, \frac{|g(z')|}{\rho^2(z',c')}. \tag{2.5}$$

Here and below, $\|V(z)\|$ is the Euclidean norm of the $(0,1)$-form $V(z)$ in the corresponding space (now that is $\mathbb{C}^{n+1}$).

Notice that $|(z,c)-(z',c')| = |z_{n+1}-c_{n+1}| \le \rho(z',c')$ on E. Then, using (2.3)–(2.5), we obtain

$$\begin{aligned}
&\int\limits_{\mathbb{C}^{n+1}} \frac{\rho^{2l+2}(z,c)\|V(z)\|^2}{(1+|z|^2)^m} e^{-2\varphi(z,c)}\, d\lambda_z = \int\limits_{E} \frac{\rho^{2l+2}(z,c)\|V(z)\|^2}{(1+|z|^2)^m} e^{-2\varphi(z,c)}\, d\lambda_z \\
&\le \alpha_0^2(1+A_0^2) B_0^{2l+2} e^{2C_0} \int\limits_{E} \frac{\rho^{2l}(z',c')|g(z')|^2}{(1+|z'|^2)^m} e^{-2\varphi(z',c')}\, \frac{d\lambda_z}{\rho^2(z',c')} \\
&\le \alpha_0^2(1+A_0^2) B_0^{2l+2} e^{2C_0} \int\limits_{\mathbb{C}^n} \frac{\rho^{2l}(z',c')|g(z')|^2}{(1+|z'|^2)^m} e^{-2\varphi(z',c')}\, d\lambda_{z'} \int\limits_{z_{n+1}\in B(z')} \frac{d\lambda_{z_{n+1}}}{\rho^2(z',c')} \\
&= M_0^2 B_0^{2l+2} e^{2C_0} I(g), \qquad \text{where } M_0 := (\alpha_0/2)\sqrt{3\pi(1+A_0^2)}.
\end{aligned}$$

Since $\varphi(z,c) - (l+1)\log\rho(z,c) + (m/2)\log(1+|z|^2)$ is psh in $\mathbb{C}^{n+1}$, by Hörmander's well-known result [7, Theorem 4.4.2] we can find a function u with $\overline{\partial} u = V$ and

$$I(u) := \int\limits_{\mathbb{C}^{n+1}} \frac{\rho^{2l+2}(z,c)|u(z)|^2}{(1+|z|^2)^{m+2}} e^{-2\varphi(z,c)}\, d\lambda_z \le \frac{1}{2} M_0^2 B_0^{2l+2} e^{2C_0} I(g). \tag{2.6}$$

Consider the function

$$G(z) := g(z')\eta(\tau(z)) - (z_{n+1}-c_{n+1})u(z), \qquad z \in \mathbb{C}^{n+1}.$$

Since $\overline{\partial} G = 0$, G is entire in $\mathbb{C}^{n+1}$. Obviously, $G(z', c_{n+1}) = g(z')$ on $\mathbb{C}^n$. Denote

$$\begin{aligned}
B_0(z') &:= \{w \in \mathbb{C} : |w - c_{n+1}| \le \rho(z',c')\}, \qquad z' \in \mathbb{C}^n, \\
E_0 &:= \{z = (z', z_{n+1}) \in \mathbb{C}^{n+1} : z_{n+1} \in B_0(z'),\ z' \in \mathbb{C}^n\}
\end{aligned}$$

and notice that $|(z,c)-(z',c')| = |z_{n+1}-c_{n+1}| \le \rho(z',c')$ on E_0. Using (2.3), (2.4), and (2.6), we then have

$$\begin{aligned}
J(g) &:= \int\limits_{\mathbb{C}^{n+1}} \frac{\rho^{2l+2}(z,c)|g(z')|^2\eta^2(\tau(z))}{(1+|z|^2)^{m+3}} e^{-2\varphi(z,c)}\, d\lambda_z \\
&= \int\limits_{E_0} \frac{\rho^{2l+2}(z,c)|g(z')|^2\eta^2(\tau(z))}{(1+|z|^2)^{m+3}} e^{-2\varphi(z,c)}\, d\lambda_z
\end{aligned}$$

$$\le B_0^{2l+2} e^{2C_0} \int\limits_{E_0} \frac{\rho^{2l+2}(z',c')|g(z')|^2}{(1+|z'|^2)^m} e^{-2\varphi(z',c')}\, d\lambda_z$$
$$\le B_0^{2l+2} e^{2C_0} \int\limits_{\mathbb{C}^n} \frac{\rho^{2l}(z',c')|g(z')|^2}{(1+|z'|^2)^m} e^{-2\varphi(z',c')}\, d\lambda'_z \int\limits_{z_{n+1}\in B_0(z')} d\lambda_{z_{n+1}}$$
$$\le \pi B_0^{2l+2} e^{2C_0} I(g).$$

Next,

$$J(u) := \int\limits_{\mathbb{C}^{n+1}} \frac{\rho^{2l+2}(z,c)|z_{n+1}-c_{n+1}|^2|u(z)|^2}{(1+|z|^2)^{m+3}} e^{-2\varphi(z,c)}\, d\lambda_z$$
$$\le (1+|c_{n+1}|^2) \int\limits_{\mathbb{C}^{n+1}} \frac{\rho^{2l+2}(z,c)|u(z)|^2}{(1+|z|^2)^{m+2}} e^{-2\varphi(z,c)}\, d\lambda_z = (1+|c_{n+1}|^2) I(u).$$

Thus, using the last two estimates and (2.6), we get

$$\int\limits_{\mathbb{C}^{n+1}} \frac{\rho^{2l+2}(z,c)|G(z)|^2}{(1+|z|^2)^{m+3}} e^{-2\varphi(z,c)}\, d\lambda_z \le 2(J(g)+J(u))$$
$$\le 2\pi B_0^{2l+2} e^{2C_0} I(g) + (1+|c_{n+1}|^2) M_0^2 B_0^{2l+2} e^{2C_0} I(g)$$
$$\le N_0^2 B_0^{2l+2}(1+|c_{n+1}|^2) I(g), \quad \text{where } N_0 := \sqrt{2\pi(1+\alpha_0^2)(1+A_0^2)}\, e^{C_0}. \ \blacksquare$$

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Each complex plane of dimension k can be defined by a system

$$\begin{cases} \langle a^{k+1}, z\rangle = c_{k+1}, \\ \dots\dots\dots\dots\dots\dots \\ \langle a^p, z\rangle = c_p, \end{cases} \tag{2.7}$$

where $(a^j)_{j=k+1}^p$ is part of an orthonormal basis $(a^j)_{j=1}^p$ in $\mathbb{C}^p$ and $c_j \in \mathbb{C}$ $(k+1 \le j \le p)$. Let $a^j = (a_{j1}, \dots, a_{jp})$. Then $A := (a_{jn})_{j,n=1}^p$ is a unitary matrix and, using the transformation $z \mapsto \widetilde{z} := Az$, we can rewrite (2.7) as

$$\begin{cases} \widetilde{z}_{k+1} = c_{k+1}, \\ \dots\dots\dots\dots \\ \widetilde{z}_p = c_p. \end{cases}$$

Since the transformation $z \mapsto Az$ and its inverse $z \mapsto A^{-1}z$ preserve holomorphy and plurisubharmonicity and $|Az| = |A^{-1}z| = |z|$ for all $z \in \mathbb{C}^p$, in what follows we can assume that Σ is defined by the system

$$\begin{cases} z_{k+1} = c_{k+1}, \\ \dots\dots\dots\dots \\ z_p = c_p. \end{cases}$$

Note that $d_\Sigma = \sqrt{|c_{k+1}|^2 + \cdots + |c_p|^2}$.

Put $z^k := (z_1, \ldots, z_k) \in \mathbb{C}^k$, $c^k := (c_{k+1}, \ldots, c_p)$, and $(z^k, c^k) := (z_1, \ldots, z_k, c_{k+1}, \ldots, c_p)$. As before, write $h(z^k, c^k) := h((z^k, c^k))$ for a function h defined on $\mathbb{C}^p$. The entire function $g(z^k) := f(z^k, c^k)$ satisfies the condition

$$\int\limits_{\mathbb{C}^k} \frac{|g(z^k)|^2}{(1+|z^k|^2)^{k+1}} e^{-2\varphi(z^k, c^k)}\, d\lambda_{z^k} \le \int\limits_{\mathbb{C}^k} \frac{d\lambda_{z^k}}{(1+|z^k|^2)^{k+1}} = \frac{\pi^k}{k!} < e^2.$$

Applying Lemma 2.2 $(p-k)$ times, we find an entire function F in $\mathbb{C}^p$ such that $F(z^k, c^k) = g(z^k) = f(z^k, c^k)$ and

$$\begin{aligned} I^2(F) &:= \int\limits_{\mathbb{C}^p} \frac{\rho^{2(p-k)}(z)|F(z)|^2}{(1+|z|^2)^{3p-2k+1}} e^{-2\varphi(z)}\, d\lambda_z \\ &\le e^2 N_0^{2(p-k)} B_0^{2(p-k)} (1+|c_{k+1}|^2)\ldots(1+|c_p|^2) \\ &\le M_1^2 (1+d_\Sigma^2)^{p-k}, \qquad \text{where } M_1 := e N_0^{p-k} B_0^{p-k}. \end{aligned}$$

Denote by v_p the volume of the unit ball in $\mathbb{C}^p \cong \mathbb{R}^{2p}$ and apply standard arguments for integral and uniform estimates of entire functions. Then, using (2.3) and (2.4), we get, for every $z \in \mathbb{C}^p$,

$$\begin{aligned} |F(z)|^2 &\le \frac{1}{v_p \rho^{2p}(z)} \int\limits_{|\zeta - z| \le \rho(z)} |F(\zeta)|^2\, d\lambda_\zeta \\ &\le \frac{1}{v_p \rho^{2p}(z)} I^2(F) \max_{|\zeta - z| \le \rho(z)} \frac{(1+|\zeta|^2)^{3p-2k+1}}{\rho^{2(p-k)}(\zeta)} e^{2\varphi(\zeta)} \\ &\le \frac{1}{v_p} I^2(F) 2^{3p-2k+1} B_0^{2(p-k)} e^{2C_0} \frac{(1+|z|^2)^{3p-2k+1}}{\rho^{2(2p-k)}(z)} e^{2\varphi(z)} \\ &= e^{2M} \frac{(1+d_\Sigma^2)^{p-k}(1+|z|^2)^{3p-2k+1}}{\rho^{2(2p-k)}(z)} e^{2\varphi(z)}, \end{aligned}$$

where $M := \log(v_p^{-1/2} 2^{(3p-2k+1)/2} B_0^{p-k} e^{C_0} M_1)$. Consequently,

$$|F(z)| \le e^M \frac{(1+d_\Sigma^2)^{(p-k)/2}(1+|z|^2)^{(3p-2k+1)/2}}{\rho^{2p-k}(z)} e^{\varphi(z)} \qquad \text{for all } z \in \mathbb{C}^p,$$

which is equivalent to the desired estimate for $\log|F(z)|$. ∎

COROLLARY 2.3. *Let φ be as in Theorem* 2.1. *Then for any subspace Σ in $\mathbb{C}^p$ of dimension k and a holomorphic function f on Σ with $\log|f(z)| \le \varphi(z)$ $(z \in \Sigma)$ there exists an entire function F on $\mathbb{C}^p$ such that $F|_\Sigma = f$ and, for all $z \in \mathbb{C}^p$,*

$$(2.8)\quad \log|F(z)| \le \varphi(z) + (2p-k)\log\frac{1}{\rho(z)} + \frac{3p-2k+1}{2}\log(1+|z|^2) + M,$$

where M is an absolute constant depending on A_0, B_0, C_0, p and k and independent of φ, Σ, f and ρ.

3. Families of entire functions with lower and upper bounds. In this section we prove the existence of special families of entire functions having uniform upper and local lower bounds of the same type. We start with functions of one variable and then consider the multivariable case by using Theorem 2.1.

PROPOSITION 3.1. *Let δ, φ, ψ be real-valued functions on $\mathbb{C}$ such that $0 < \delta(z) \leq 1$ for all $z \in \mathbb{C}$, ψ is subharmonic in $\mathbb{C}$, and*

$$\exists C_0 > 0: \quad \varphi(z) \leq \inf_{|t|\leq\delta(z)} \psi(z+t) + C_0, \quad \forall z \in \mathbb{C}. \tag{3.1}$$

Then there exists a family $\mathcal{G} := \{g_\xi : \xi \in \mathbb{C}\}$ of entire functions in $\mathbb{C}$ satisfying the following conditions:

$$g_\xi(\xi) = \delta(\xi)e^{\varphi(\xi)}, \qquad \forall \xi \in \mathbb{C}, \tag{3.2}$$

$$|g_\xi(z)| \leq \frac{A}{\delta(z)}(1+|z|^2)^2 e^{\psi^*_\delta(z)}, \quad \forall z, \xi \in \mathbb{C}, \tag{3.3}$$

where $\psi^*_\delta(z) := \sup_{|t|\leq\delta(z)} \psi(z+t)$ and A is an absolute constant depending only on C_0.

Proof. The method and steps of the proof are the same as for Lemma 2.2. For this reason we omit the details and calculations.

For fixed $\xi \in \mathbb{C}$, let

$$V_\xi(z) := \eta'\left(\frac{|z-\xi|}{\delta(\xi)}\right)\frac{1+|\xi|}{2|z-\xi|}e^{\varphi(\xi)}, \qquad z \in \mathbb{C},$$

where the function η is as in the proof of Lemma 2.2. Then from (3.1) it follows that

$$\int_{\mathbb{C}} \frac{|V_\xi(z)|^2}{1+|z|^2} e^{-2\psi(z)}\, d\lambda_z \leq A_1,$$

where A_1 depends only on C_0 and $\alpha_0 = \max_{t\in\mathbb{R}} |\eta'(t)|$ (for example, $A_1 = 6\pi\alpha_0^2 e^{2C_0}$ works).

By [7, Theorem 4.4.2], there exists a function u_ξ with $\partial u_\xi/\partial\overline{z} = V_\xi$ and

$$\int_{\mathbb{C}} \frac{|u_\xi(z)|^2}{(1+|z|^2)^3} e^{-2\psi(z)}\, d\lambda_z \leq \frac{A_1}{2}.$$

Using this and (3.1), for the function

$$g_\xi(z) := \delta(\xi)\eta\left(\frac{|z-\xi|}{\delta(\xi)}\right)e^{\varphi(\xi)} - \frac{z-\xi}{1+|\xi|}u_\xi(z), \qquad z \in \mathbb{C},$$

we have

$$\int_{\mathbb{C}} \frac{|g_\xi(z)|^2}{(1+|z|^2)^4} e^{-2\psi(z)}\, d\lambda_z \le 2\int_{\mathbb{C}} \frac{\delta^2(\xi)}{(1+|z|^2)^4}\, \eta^2\Big(\frac{|z-\xi|}{\delta(\xi)}\Big) e^{2\varphi(\xi)-2\psi(z)}\, d\lambda_z$$
$$+\, 2\int_{\mathbb{C}} \frac{|z-\xi|^2}{(1+|\xi|)^2}\, \frac{|u_\xi(z)|^2}{(1+|z|^2)^4} e^{-2\psi(z)} d\lambda_z \le A_2,$$

where A_2 is some constant depending only C_0 (for example, we can take $A_2 = (2\pi/3)e^{2C_0} + A_1$). Since $\partial g_\xi/\partial\overline{z} = 0$ in $\mathbb{C}$, g_ξ is entire. Obviously, $g_\xi(\xi) = \delta(\xi)e^{\varphi(\xi)}$, that is, g satisfies (3.2). In addition, applying the last inequality and a standard argument for uniform and integral estimates of the subharmonic function $|g_\xi|^2$, we get

$$|g_\xi(z)|^2 \le \frac{1}{\pi\delta^2(z)} \int_{|\zeta-z|\le\delta(z)} |g_\xi(\zeta)|^2 d\lambda_\zeta$$
$$\le \frac{A_2}{\pi\delta^2(z)} \max_{|\zeta-z|\le\delta(z)} (1+|\zeta|^2)^4 e^{2\psi(\zeta)} \le \frac{16A_2}{\pi\delta^2(z)}(1+|z|^2)^4 e^{2\psi^*_\delta(z)}.$$

This yields (3.3) with $A = 4\sqrt{A_2/\pi}$. ∎

Corollary 3.2. *Let δ be as in Proposition* 3.1 *and φ a subharmonic function in $\mathbb{C}$ such that*

$$\exists C_0 > 0 : \quad |\varphi(z) - \varphi(\zeta)| \le C_0 \quad \textit{for all } z,\zeta \in \mathbb{C} \textit{ with } |z-\zeta| \le \delta(z).$$

Then there exists a family $\mathcal{G} = \{g_\xi : \xi \in \mathbb{C}\}$ of entire functions in $\mathbb{C}$ satisfying (3.2) *and*

$$|g_\xi(z)| \le \frac{A}{\delta(z)}(1+|z|^2)^2 e^{\varphi(z)}, \qquad \forall z,\xi \in \mathbb{C}, \tag{3.4}$$

where A is an absolute constant depending only on C_0.

Proposition 3.3. *Let ρ be a regular distance function and φ a ρ-stable psh function in $\mathbb{C}^p$ satisfying* (2.4). *Then there exists a family $\mathcal{G} = \{g_\xi : \xi \in \mathbb{C}^p\}$ of entire functions in $\mathbb{C}^p$ such that the following conditions hold:*

$$g_\xi(\xi) = \rho(\xi)e^{\varphi(\xi)}, \qquad \forall \xi \in \mathbb{C}^p, \tag{3.5}$$

$$|g_\xi(z)| \le \frac{M}{\rho^{2p}(z)}(1+|z|^2)^{3p+1} e^{\varphi(z)} \qquad \textit{for all } z \in \mathbb{C}^p, \tag{3.6}$$

where M is an absolute constant depending only on A_0, B_0 and C_0.

Proof. For any $\xi \ne 0$ in $\mathbb{C}^p$ the linear transformation

$$w \mapsto z_\xi(w) := w\frac{\xi}{|\xi|}, \qquad w \in \mathbb{C},$$

maps the complex plane onto a one-dimensional complex plane in $\mathbb{C}^p$. Denote the latter plane by Σ. Obviously, $\varphi_\xi(w) := \varphi(z_\xi(w))$ is subharmonic in $\mathbb{C}$.

Since $|z_\xi(w)| = |w|$ and $z_\xi(\cdot)$ is linear, from (2.4) it follows that

$$|\varphi_\xi(w) - \varphi_\xi(u)| \le C_0 \qquad \text{for all } w, u \in \mathbb{C} \text{ with } |w-u| \le \rho(w).$$

Then, by Corollary 3.2 with ρ in place of δ, there exist an absolute constant $A > 0$, depending only on C_0, and an entire function f_ξ in $\mathbb{C}$ such that

$$f_\xi(|\xi|) = \rho(|\xi|)e^{\varphi_\xi(|\xi|)} \tag{3.7}$$

and

$$|f_\xi(w)| \le \frac{A}{\rho(w)}(1+|w|^2)^2 e^{\varphi_\xi(w)}, \qquad \forall w \in \mathbb{C}. \tag{3.8}$$

Let $w_\xi : \Sigma \to \mathbb{C}_w$ be the inverse transformation to z_ξ. Clearly, $|w_\xi(z)| = |z|$ for every $z \in \Sigma$. Then the function $F_\xi(z) := f_\xi(w_\xi(z))$ is holomorphic on Σ and, by (3.8), for all $z \in \Sigma$,

$$\begin{aligned}|F_\xi(z)| = |f_\xi(w_\xi(z))| &\le \frac{A}{\rho(w_\xi(z))}(1+|w_\xi(z)|^2)^2 e^{\varphi_\xi(w_\xi(z))} \\ &= \frac{A}{\rho(z)}(1+|z|^2)^2 e^{\varphi(z)}.\end{aligned}$$

From (2.3) and (2.4) it follows that the psh function

$$\psi(z) := \varphi(z) + \log\frac{1}{\rho(z)} + 2\log(1+|z|^2), \qquad z \in \mathbb{C}^p,$$

satisfies the condition

$$\begin{aligned}|\psi(z) - \psi(\zeta)| &\le |\varphi(z) - \varphi(\zeta)| + \left|\log\frac{\rho(z)}{\rho(\zeta)}\right| + 2\left|\log\frac{1+|z|^2}{1+|\zeta|^2}\right| \\ &\le C_0 + \log B_0 + 6 \qquad \text{for all } z, \zeta \in \mathbb{C}^p \text{ with } |z-\zeta| \le \rho(z).\end{aligned}$$

Therefore, applying Corollary 2.3 to ψ and F_ξ in place of φ and f, respectively, we can find an entire function g_ξ in $\mathbb{C}^p$ such that $g_\xi|_\Sigma = F_\xi$ and

$$|g_\xi(z)| \le \frac{M}{\rho^{2p}(z)}(1+|z|^2)^{3p+1}e^{\varphi(z)} \qquad \text{for all } z \in \mathbb{C}^p,$$

where M is an absolute constant depending only on A_0, B_0 and C_0, that is, (3.6) holds. In addition, (3.7) and the equality $w_\xi(\xi) = |\xi|$ yield

$$g_\xi(\xi) = F_\xi(\xi) = f_\xi(w_\xi(\xi)) = f_\xi(|\xi|) = \rho(|\xi|)e^{\varphi_\xi(|\xi|)} = \rho(\xi)e^{\varphi(\xi)}.$$

Thus, g_ξ satisfies (3.5).

In the case $\xi = 0$ the same arguments can be applied to $\varphi_0(w) := \varphi(w, 0, \ldots, 0)$ and $\Sigma = \{z = (w, 0, \ldots, 0) \in \mathbb{C}^p : w \in \mathbb{C}\}$. ■

4. Canonical weights and weight sequences. Given φ, a continuous real-valued function on $\mathbb{C}^p$, let

$$\|f\|_\varphi := \sup_{z\in\mathbb{C}^p}\frac{|f(z)|}{e^{\varphi(z)}}, \qquad f \in H(\mathbb{C}^p),$$

and

$$E(\varphi) := \{f \in H(\mathbb{C}^p) : \|f\|_\varphi < \infty\},$$

which is a Banach space with respect to the norm $\|\cdot\|_\varphi$. Denote by V the set of all φ such that the corresponding space $E(\varphi)$ is nonvanishing on $\mathbb{C}^p$. Recall that a class $E \subset H(\mathbb{C}^p)$ is called *nonvanishing at* $z_0 \in \mathbb{C}^p$ if there is $f \in E$ with $f(z_0) \neq 0$; and E is said to be *nonvanishing on* $G \subset \mathbb{C}^p$ if it is nonvanishing at each $z \in G$. Elements of V are called *weights*.

We say that a weight φ is *dominated* by a weight ψ ($\varphi \prec \psi$) if there is $C \geq 0$ such that $\varphi(z) \leq \psi(z) + C$ for all $z \in \mathbb{C}^p$. If $\varphi \prec \psi$ and $\psi \prec \varphi$, then φ and ψ are called *equivalent* ($\varphi \sim \psi$). Obviously, $E(\varphi) \hookrightarrow E(\psi)$ whenever $\varphi \prec \psi$, and $E(\varphi) = E(\psi)$ whenever $\varphi \sim \psi$.

For $\varphi \in V$ define the *holomorphically regularized* (or simply *regularized*) weight $\overline{\varphi}$ by the rule

$$\overline{\varphi}(z) := \sup\{\log|f(z)| : f \in B(\varphi)\}, \qquad z \in \mathbb{C}^p,$$

where $B(\varphi)$ is the unit ball in $E(\varphi)$. Note that $\overline{\varphi} = \log{(e^\varphi)}^\sim$, where $(e^\varphi)^\sim$ is the associated function of e^φ in the sense of [5]. As is well known, for each $\varphi \in V$ the following conditions hold:

(a) $\overline{\varphi}$ is a continuous psh function in $\mathbb{C}^p$;
(b) $\overline{\varphi} \leq \varphi$ on $\mathbb{C}^p$;
(c) $\|f\|_{\overline{\varphi}} = \|f\|_\varphi$ for all $f \in H(\mathbb{C}^p)$;
(d) $E(\overline{\varphi}) = E(\varphi)$.

A weight φ is called *canonical* if $\varphi \sim \overline{\varphi}$, or equivalently, $\varphi \prec \overline{\varphi}$. Denote by W the set of all canonical weights. Note that for every $\varphi \in V$, $B(\varphi)$ is a closed bounded subset in $H(\mathbb{C}^p)$ and $B(\varphi)$ is nonvanishing on $\mathbb{C}^p$. On the other hand, if B is an arbitrary subset in $H(\mathbb{C}^p)$ with the same properties, then the function

$$\varphi_B(z) := \sup\{\log|f(z)| : f \in B\}, \qquad z \in \mathbb{C}^p,$$

is a weight and $\overline{\varphi}_B = \varphi_B$ (see, e.g., [5, Example 1.4]). In this notation, a subset $B \subset B(\varphi)$ closed in $H(\mathbb{C}^p)$ and nonvanishing on $\mathbb{C}^p$ is called *defining* for φ if $\varphi_B = \overline{\varphi}$.

Denote by $V^\uparrow$ the family of all sequences $\Phi = (\varphi_n)_{n=1}^\infty$ with $\varphi_n \in V$ ($n \in \mathbb{N}$) and $\varphi_1 \prec \varphi_2 \prec \cdots$. For each $\Phi \in V^\uparrow$, define the locally convex space $I(\Phi) := \bigcup_{n=1}^\infty E(\varphi_n)$ endowed with the natural inductive limit topology. The elements of $V^\uparrow$ are called *inductive weight sequences*.

We say that $\Phi \in V^\uparrow$ is *dominated* by $\Psi = (\psi_n)_{n=1}^\infty \in V^\uparrow$ ($\Phi \prec \Psi$) if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\varphi_n \prec \psi_m$. If $\Phi \prec \Psi$ and $\Psi \prec \Phi$, then Φ and Ψ are called *equivalent*, $\Phi \sim \Psi$. It is clear that $I(\Phi) \hookrightarrow I(\Psi)$ for $\Phi \prec \Psi$. Consequently, if $\Phi \sim \Psi$, then $I(\Phi)$ and $I(\Psi)$ coincide as sets and topologically. Since a weight sequence is equivalent to each of its subsequences, we may assume that the condition of domination or other conditions of such type hold for $m = n + 1$.

By the above, to define the space $I(\Phi)$, we can replace the weight sequence Φ with $\overline{\Phi} := (\overline{\varphi}_n)_{n=1}^{\infty}$. As is known, always $\overline{\Phi} \prec \Phi$. Moreover, $\overline{\varphi}_n \le \varphi_n$ for every $n \in \mathbb{N}$. A sequence $\Phi \in V^{\uparrow}$ is called *canonical* if $\Phi \sim \overline{\Phi}$. Denote by $W^{\uparrow}$ the family of all canonical inductive weight sequences. It is easy to see that a weight sequence $\Phi \in V^{\uparrow}$ is canonical if and only if it is equivalent to some $\Psi \in W^{\uparrow}$ consisting of canonical weights.

From Proposition 3.3 we have the following conditions for an inductive weight sequence to be canonical.

THEOREM 4.1. *Let ρ be a regular distance function and $\Phi = (\varphi_n)_{n=1}^{\infty}$ an inductive weight sequence consisting of psh ρ-stable functions. If for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $D_n > 0$ such that*

$$\varphi_n(z) + \log \frac{1 + |z|^2}{\rho(z)} \le \varphi_m(z) + D_n \quad \textit{for all } z \in \mathbb{C}^p, \tag{4.1}$$

then Φ is canonical.

Proof. Fix any $n \in \mathbb{N}$ and find $m \in \mathbb{N}$ and D_n so that (4.1) holds. By Proposition 3.3, there exists a family $\{g_\xi : \xi \in \mathbb{C}^p\}$ of entire functions in $\mathbb{C}^p$ such that

$$g_\xi(\xi) = \rho(\xi) e^{\varphi_m(\xi)} \quad \text{for every } \xi \in \mathbb{C}^p, \tag{4.2}$$

$$|g_\xi(z)| \le \frac{M}{\rho^{2p}(z)} (1 + |z|^2)^{3p+1} e^{\varphi_m(z)} \quad \text{for all } z, \xi \in \mathbb{C}^p, \tag{4.3}$$

where M does not depend on $z, \xi \in \mathbb{C}^p$. Applying (4.1) $(3p + 1)$ times, we find $k \in \mathbb{N}$ and $L_m > 0$ such that

$$\varphi_m(z) + (3p + 1) \log \frac{1 + |z|^2}{\rho(z)} \le \varphi_k(z) + L_m \quad \text{for all } z \in \mathbb{C}^p.$$

From this and (4.3) it follows that the functions $f_\xi := (e^{-L_m}/M) g_\xi$ satisfy the estimate

$$|f_\xi(z)| \le e^{-L_m} \left(\frac{1 + |z|^2}{\rho(z)} \right)^{3p+1} e^{\varphi_m(z)} \le e^{\varphi_k(z)} \quad \text{for all } z, \xi \in \mathbb{C}^p.$$

Hence, $f_\xi \in B(\varphi_k)$ for every $\xi \in \mathbb{C}^p$. On the other hand, (4.1) and (4.2) imply that, for any $\xi \in \mathbb{C}^p$,

$$\log |f_\xi(\xi)| = \log |g_\xi(\xi)| - L_m - \log M \ge \varphi_n(\xi) - M_m,$$

where $M_m := L_m + D_n + \log M$. Consequently, $\varphi_n(z) \le \overline{\varphi}_k(z) + M_m$ for all $z \in \mathbb{C}^p$. Thus, $\Phi \prec \overline{\Phi}$ or $\Phi \sim \overline{\Phi}$. ∎

In the "dual" projective case, when $\varphi_1 \succ \varphi_2 \succ \cdots$, we consider the Fréchet space $P(\Phi) := \bigcap_{n=1}^{\infty} E(\varphi_n)$; the topology in $P(\Phi)$ is given by the norm system $(\|\cdot\|_{\varphi_n})_{n=1}^{\infty}$. Clearly, this space may be vanishing at some points (moreover, it may be trivial), although each $E(\varphi_n)$ is nonvanishing on $\mathbb{C}^p$. In this connection, note that $I(\Phi)$ is nonvanishing on $\mathbb{C}^p$ for any $\Phi \in V^{\uparrow}$. Taking this into account, denote by $V^{\downarrow}$ the set of weight sequences Φ such that $\varphi_1 \succ \varphi_2 \succ \cdots$ and $P(\Phi)$ is nonvanishing on $\mathbb{C}^p$. The elements of $V^{\downarrow}$ are called *projective weight sequences*. We say that $\Phi \in V^{\downarrow}$ is *dominated* by $\Psi \in V^{\downarrow}$ ($\Phi \prec \Psi$) if for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\varphi_m \prec \psi_n$. As in the inductive case, Φ and Ψ from $V^{\downarrow}$ are called *equivalent* ($\Phi \sim \Psi$) if $\Phi \prec \Psi$ and $\Psi \prec \Phi$. Clearly, $P(\Phi) \hookrightarrow P(\Psi)$ for $\Phi \prec \Psi$ and $P(\Phi) = P(\Psi)$ for $\Phi \sim \Psi$. Although $P(\Phi)$, just as $I(\Phi)$, can be given by the corresponding sequence $\overline{\Phi}$, it is more natural to define it by another sequence of regularized weights.

Indeed, let

$$\underline{\varphi}_n(z) := \sup\{\log|f(z)| : f \in P(\Phi) \cap B(\varphi_n)\} \quad (n \in \mathbb{N}).$$

Note that $\underline{\varphi}_n(z) \le \overline{\varphi}_n(z) \le \varphi_n(z)$ for all $z \in \mathbb{C}^p$ ($n \in \mathbb{N}$) and $P(\Phi) = P(\overline{\Phi}) = P(\underline{\Phi})$, where $\underline{\Phi} := (\underline{\varphi}_n)_{n=1}^{\infty}$. From the open mapping theorem it also follows that the topologies of $P(\Phi)$ and $P(\underline{\Phi})$ coincide. A projective weight sequence Φ is called *canonical* if $\Phi \sim \underline{\Phi}$. Evidently, $\Phi \sim \overline{\Phi}$ for a canonical weight sequence $\Phi \in V^{\downarrow}$. It is useful to know that a converse statement holds. To obtain some sufficient conditions for this, we need the following notions. A projective weight sequence Φ is called *reduced* if the corresponding projective sequence $(E(\varphi_n))_{n=1}^{\infty}$ of Banach spaces is reduced, that is, $P(\Phi)$ is dense in each $E(\varphi_n)$ ($n \in \mathbb{N}$). We say that Φ is *weakly reduced* if for each $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $P(\Phi)$ is dense in $E(\varphi_n)$ with respect to the norm $\|\cdot\|_{\varphi_k}$. Obviously, every reduced sequence is weakly reduced. The following result is only a slight revision of [2, Proposition 2].

PROPOSITION 4.2. *Suppose that $\Phi \in V^{\downarrow}$ is weakly reduced and $\Phi \sim \overline{\Phi}$. Then Φ is canonical.*

Proof. Fix any $k \in \mathbb{N}$ and find $n \in \mathbb{N}$ so that $P(\Phi)$ is dense in $E(\varphi_n)$ with respect to the norm $\|\cdot\|_{\varphi_k}$. Since $\Phi \sim \overline{\Phi}$, there exist $m \in \mathbb{N}$ and $C > 0$ such that

$$\varphi_m(z) \le \overline{\varphi}_n(z) + C \quad \text{for all } z \in \mathbb{C}^p. \tag{4.4}$$

It can be assumed that $k \le n \le m$. From (4.4) it follows that for each $z_0 \in \mathbb{C}^p$ there is $g \in B(\varphi_n)$ with

$$\log|g(z_0)| \ge \varphi_m(z_0) - 2C. \tag{4.5}$$

Take $\varepsilon_0 := \min\{1, \frac{1}{2}\exp(-2C+\varphi_m(z_0)-\varphi_k(z_0))\}$ and $D > 0$ so that $\varphi_n(z) \le \varphi_k(z) + \log D$ for all $z \in \mathbb{C}^p$. The density of $P(\Phi)$ in $E(\varphi_n)$ with respect to the norm $\|\cdot\|_{\varphi_k}$ implies that there exists $h \in P(\Phi)$ with $\|g-h\|_{\varphi_k} \le \varepsilon_0$. Therefore,

$$|h(z)| \le \varepsilon_0 e^{\varphi_k(z)} + |g(z)| \le (D+1)e^{\varphi_k(z)} \quad \text{for all } z \in \mathbb{C}^p,$$

and consequently $h_0 := \frac{1}{D+1} h \in P(\Phi) \cap B(\varphi_k)$. On the other hand, using (4.5), we have

$$|h(z_0)| \geq |g(z_0)| - \varepsilon_0 e^{\varphi_k(z_0)} \geq e^{-2C+\varphi_m(z_0)} - \tfrac{1}{2} e^{-2C+\varphi_m(z_0)} = \tfrac{1}{2} e^{-2C+\varphi_m(z_0)}.$$

Hence,

$$\underline{\varphi}_k(z_0) \geq \log|h_0(z_0)| = \log|h(z_0)| - \log(D+1) \geq \varphi_m(z_0) - 2C - \log 2(D+1),$$

where the constant $-2C - \log 2(D+1)$ does not depend on $z_0 \in \mathbb{C}^p$. ∎

COROLLARY 4.3. *A reduced $\Phi \in V^{\downarrow}$ is canonical if and only if $\Phi \sim \overline{\Phi}$.*

Now we are ready to get an analog of Theorem 4.1 in the projective case.

THEOREM 4.4. *Let ρ be a regular distance function and $\Phi = (\varphi_n)_{n=1}^{\infty}$ a weakly reduced weight sequence consisting of psh ρ-stable functions. If for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $D_n > 0$ such that*

$$\varphi_m(z) + \log \frac{1+|z|^2}{\rho(z)} \leq \varphi_n(z) + D_n \quad \text{for all } z \in \mathbb{C}^p, \tag{4.6}$$

then Φ is canonical.

Proof. As in the inductive case (see the proof of Theorem 4.1), we find that $\Phi \sim \overline{\Phi}$. Since Φ is weakly reduced, it remains to apply Proposition 4.2. ∎

5. Multipliers of weighted spaces. In this section we apply the above results to the description of multipliers acting in weighted spaces of inductive or projective type. To do this, we recall some definitions and results.

Let E, F be locally convex spaces of entire functions in $\mathbb{C}^p$ such that $E, F \hookrightarrow H(\mathbb{C}^p)$; $H(\mathbb{C}^p)$ is endowed with its natural topology of uniform convergence on compact subsets of $\mathbb{C}^p$. An entire function μ is called a *multiplier* from E into F if $\mu f \in F$ for every $f \in E$. Denote by $\mathcal{M}(E, F)$ the set of all multipliers from E into F. Obviously, $\mathcal{M}(E, F)$ is a linear subspace of $H(\mathbb{C}^p)$ and each $\mu \in \mathcal{M}(E, F)$ generates the corresponding multiplication operator $\Lambda_\mu : f \in E \mapsto \mu f \in F$. Clearly, Λ_μ is linear and from the uniqueness theorem for holomorphic functions it follows that Λ_μ is injective whenever μ is nontrivial. Since $E, F \hookrightarrow H(\mathbb{C}^p)$, this operator always has a closed graph.

Returning to weighted spaces of inductive and projective type, we see that $I(\Phi)$ is an (LB)-space, while $P(\Phi)$ is a Fréchet space. Consequently, spaces of these two types are ultrabornological and for them the Banach (for $P(\Phi)$) and Grothendieck (for $(I(\Phi))$) closed graph theorems are valid. Thus, the operator $\Lambda_\mu : I(\Phi) \to I(\Psi)$ (or $\Lambda_\mu : P(\Phi) \to P(\Psi)$) is continuous for any $\mu \in \mathcal{M}(I(\Phi), I(\Psi))$ (resp., $\mu \in \mathcal{M}(P(\Phi), P(\Psi))$).

Theorem 5.1. *Let Φ and Ψ be inductive or projective weight sequences. If Φ is canonical, then*

$$\mathcal{M}(I(\Phi), I(\Psi)) = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E(\psi_m - \varphi_n) \tag{5.1}$$

or, respectively,

$$\mathcal{M}(P(\Phi), P(\Psi)) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E(\psi_m - \varphi_n). \tag{5.2}$$

Proof. We prove (5.2); the proof for (5.1) is similar. It is easy to see that

$$\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E(\psi_m - \varphi_n) \subset \mathcal{M}(P(\Phi), P(\Psi))$$

without any restrictions on Φ.

On the other hand, let $\mu \in \mathcal{M}(P(\Phi), P(\Psi))$. Since the operator $\Lambda_\mu : P(\Phi) \to P(\Psi)$ is continuous, for each m there exist $k \in \mathbb{N}$ and $A > 0$ such that

$$\|\mu f\|_{\psi_m} \le A\|f\|_{\varphi_k}, \quad \forall f \in P(\Phi).$$

Then, for each $z \in \mathbb{C}^p$,

$$|\mu(z)|\,|f(z)| \le A \exp \psi_m(z), \qquad \forall f \in P(\Phi) \cap B(\varphi_k).$$

This yields

$$|\mu(z)| \exp \underline{\varphi}_k(z) \le A \exp \psi_m(z), \qquad \forall z \in \mathbb{C}^p.$$

Using the fact that Φ is canonical, we can find $n \in \mathbb{N}$ and $B > 0$ so that

$$\varphi_n(z) \le \underline{\varphi}_k(z) + B, \qquad \forall z \in \mathbb{C}^p.$$

Therefore,

$$|\mu(z)| \le Ae^B \exp(\psi_m(z) - \varphi_n(z)), \qquad \forall z \in \mathbb{C}^p.$$

Hence, $\mu \in \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E(\psi_m - \varphi_n)$, and consequently

$$\mathcal{M}(P(\Phi), P(\Psi)) \subset \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E(\psi_m - \varphi_n). \ \blacksquare$$

As an immediate consequence of Theorems 4.1, 4.4, and 5.1, we have the following result.

Corollary 5.2. *Let ρ be a regular distance function and Φ an inductive (weakly reduced projective) weight sequence consisting of psh ρ-stable functions satisfying condition* (4.1) (*resp.*, (4.6)). *Then for any $\Psi \in V^{\uparrow}$* (*resp.*, *$\Psi \in V^{\downarrow}$*) *the description* (5.1) (*resp.*, (5.2)) *holds.*

In its turn, this corollary implies the next, more concrete, result.

COROLLARY 5.3. *Let Φ be an inductive* (*weakly reduced projective*) *weight sequence consisting of psh functions and satisfying the following conditions:*

(i) *for some $s > 0$ and each $n \in \mathbb{N}$,*

$$\sup\{|\varphi_n(z+\zeta) - \varphi_n(z)| : |\zeta| \le 1/(1+|z|)^s,\ z \in \mathbb{C}^p\} < \infty;$$

(ii) *for every $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $D_n > 0$ such that, for all $z \in \mathbb{C}^p$,*

$$\varphi_n(z) + \log(1+|z|) \le \varphi_m(z) + D_n (\textit{resp.},\ \varphi_m(z) + \log(1+|z|) \le \varphi_n(z) + D_n).$$

Then the description (5.1) (*resp.,* (5.2)) *holds for any weight sequence Ψ of inductive* (*resp., projective*) *type.*

REMARK 5.4. In [8], for the inductive case, and in [2], for the projective case, results similar to Corollary 5.3 were established under the condition

$$\sup\{|\varphi_n(z+\zeta) - \varphi_n(z)| : |\zeta| \le 1,\ z \in \mathbb{C}^p\} < \infty \quad \text{for every } n \in \mathbb{N},$$

which is more restrictive than (i).

References

[1] A. V. Abanin, *Certain criteria for weak sufficiency*, Math. Notes 40 (1986), 757–764.

[2] —, *Dense spaces and analytic multipliers*, Izv. Vyssh. Uchebn. Zaved. Sev.-Kavk. Reg. Estestv. Nauki 4 (1994), 3–10 (in Russian).

[3] —, *On Whitney's extension theorem for spaces of ultradifferentiable functions*, Math. Ann. 320 (2001), 115–126.

[4] J. M. Anderson and J. Duncan, *Duals of Banach spaces of entire functions*, Glasgow Math. J. 32 (1990), 215–220.

[5] K. D. Bierstedt, J. Bonet and J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. 127 (1998), 137–168.

[6] J. Bonet, A. Galbis, and S. Momm, *Nonradial Hörmander algebras of several variables and convolution operators*, Trans. Amer. Math. Soc. 353 (2001), 2275–2291.

[7] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North-Holland, 1973.

[8] Yu. F. Korobeĭnik, *On multipliers of weighted functional spaces*, Anal. Math. 15 (1989), 105–114 (in Russian).

[9] —, *Interpolation problems and dense sets*, Siberian Math. J. 31 (1990), 940–949.

[10] R. Meise and B. A. Taylor, *Whitney's extension theorem for ultradifferentiable functions of Beurling type*, Ark. Mat. 26 (1988), 265–287.

[11] S. Momm, *Closed principal ideals in nonradial Hörmander algebras*, Arch. Math. (Basel) 58 (1992), 47–55.

[12] I. Kh. Musin, *Fourier–Laplace transformation of functionals on a weighted space of infinitely smooth functions on $\mathbb{R}^n$*, Sb. Math. 195 (2004), 1477–1501.

[13] R. S. Yulmukhametov, *Entire functions of several variables with given behaviour at infinity*, Izv. Math. 60 (1996), 857–876.

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