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James boundaries and σ-fragmented selectors

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Abstract. We study the boundary structure for w^*-compact subsets of dual Banach spaces. To be more precise, for a Banach space X, $0 < \varepsilon < 1$ and a subset T of the dual space X^* such that $\bigcup\{B(t,\varepsilon) : t \in T\}$ contains a James boundary for B_{X^*} we study different kinds of conditions on T, besides T being countable, which ensure that

$$X^* = \overline{\operatorname{span} T}^{\|\cdot\|}. \tag{SP}$$

We analyze two different non-separable cases where the equality (SP) holds: (a) if $J : X \to 2^{B_{X^*}}$ is the duality mapping and there exists a σ-fragmented map $f : X \to X^*$ such that $B(f(x),\varepsilon) \cap J(x) \neq \emptyset$ for every $x \in X$, then (SP) holds for $T = f(X)$ and in this case X is Asplund; (b) if T is weakly countably K-determined then (SP) holds, X^* is weakly countably K-determined and moreover for every James boundary B of B_{X^*} we have $B_{X^*} = \overline{\operatorname{co}(B)}^{\|\cdot\|}$. Both approaches use Simons' inequality and ideas exploited by Godefroy in the separable case (i.e., when T is countable). While proving (a) we show that X is Asplund if, and only if, the duality mapping has an ε-selector, $0 < \varepsilon < 1$, that sends separable sets into separable ones. A consequence is that the dual unit ball B_{X^*} is norm fragmented if, and only if, it is norm ε-fragmented for some fixed $0 < \varepsilon < 1$. Our analysis is completed by a characterization of those Banach spaces (not necessarily separable) without copies of ℓ^1 via the structure of the boundaries of w^*-compact sets of their duals. Several applications and complementary results are proved. Our results extend to the non-separable case results by Godefroy, Contreras–Payá and Rodé.

1. Introduction. Given a Banach space X and a w^*-compact subset $K \subset X^*$, a *James boundary* for K is a subset B of K such that for every $x \in X$ there exists some $b \in B$ such that $b(x) = \sup\{k(x) : k \in K\}$. If K is moreover convex the classical James boundary $\operatorname{Ext} K$ of the set of extreme points of K allows us to recover K through the equality $K = \overline{\operatorname{co}(\operatorname{Ext} K)}^{w^*}$. In general, James boundaries can even be disjoint from the set of extreme

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points. Therefore the idea of studying how properties of a given boundary are reflected on K has been of continuous interest, with applications to the theory of general Banach spaces, optimization, Fourier analysis, etc. Here is a non-exhaustive list of papers and books dealing with this kind of problems: [8, 9, 13, 16, 19, 20, 26, 27, 30, 32, 42, 44, 51]; the reference [17] offers an excellent survey about infinite-dimensional convexity and in particular about integral representation theorems and boundaries. Along these lines, the so-called "boundary problem" is also worth mentioning; we comment on it in the section "Some open problems" at the end of this paper.

Our starting point is the following result:

Theorem 1.1 ([8, 17, 20]). *Let X be a Banach space, $0<\varepsilon<1$ and T a countable subset of X^* such that $\bigcup\{B(t,\varepsilon) : t\in T\}$ contains a James boundary for B_{X^*}. Then*

$$X^*=\overline{\operatorname{span} T}^{\|\cdot\|}$$

and therefore X^* is separable.

Note that under the conditions in Theorem 1.1, once we know that X^* is separable, a result by Rodé [44] can be used to deduce that for every James boundary B of B_{X^*} we have $B_{X^*}=\overline{\operatorname{co}(B)}^{\|\cdot\|_{X^*}}$. Using Fonf and Lindenstrauss' [16] terminology, when the last equality holds we will say that B has *property* (S); here (S) stands for *strong.*

In this paper we aim to extend the results above to the non-separable case (T will then be uncountable) and thus answer a question raised in [42]. We envisage two different ways of extending the previous results to the non-separable case:

• *Using ε-selectors for the duality mapping.* If $(X,\|\cdot\|)$ is a Banach space, the *duality mapping* $J : X\to 2^{B_{X^*}}$ is defined at each $x\in X$ by

$$J(x):=\{x^*\in B_{X^*} : x^*(x)=\|x\|\}.$$

Our main result here, Theorem 4.1, states that if $f : X\to X^*$ is a σ-fragmented map such that $B(f(x),\varepsilon)\cap J(x)\neq\emptyset$ for every $x\in X$ then $X^*=\overline{\operatorname{span} f(X)}^{\|\cdot\|}$, and in this case X is Asplund. The notion of σ-fragmented map (see Definition 1) is truly wide: in particular, if we can split $X=\bigcup_{n=1}^{\infty}X_n$ in such a way that for every $n\in\mathbb{N}$ and every closed set $F\subset X_n$ the restriction $f_{|F}$ has at least one point of norm continuity then f is σ-fragmented. Therefore each function with countable range is σ-fragmented and consequently Theorem 4.1 extends Theorem 1.1.

• *Using descriptive properties of T.* Our result here, Theorem 5.1, says that if $T\subset X^*$ is weakly countably K-determined and there exists $0<\varepsilon<1$ such that $\bigcup\{B(t,\varepsilon) : t\in T\}$ contains a James boundary for B_{X^*}, then $X^* = \overline{\operatorname{span} T}^{\|\cdot\|}$ is weakly countably K-determined and every James boundary B of B_{X^*} has property (S). Since every separable metric space is countably K-determined this second approach clearly extends Theorem 1.1 and its consequence above.

A brief description of the contents of the paper follows. Section 2 is devoted to the study of the notion of σ-fragmentability for single-valued and set-valued maps; in the single-valued case, this is a large class that contains all Borel measurable maps. We use ideas from [30] to obtain a characterization of set-valued σ-fragmented maps via ε-selectors that are either piecewise barely constant or piecewise barely continuous (Theorem 2.1). The σ-fragmented maps are precisely the uniform limits of piecewise barely constants maps (Corollary 2.2); pointwise cluster points of σ-fragmented maps are σ-fragmented (Proposition 2.3). The relationship between σ-fragmentability and networks is stated in one of the key results in this paper, Theorem 2.5. Theorem 2.8, whose proof appears in [37, Theorem 2.15], states how σ-fragmented maps send separable sets into separable ones.

In Section 3 we specialize the results of the previous section to the identity map from a Banach space equipped with its weak topology into itself with the norm metric. By doing so we exhibit several properties of σ-fragmented Banach spaces following the scheme presented in [36] for the renorming case.

In Section 4 we prove one of our main results already commented on, Theorem 4.1. We also characterize Asplund spaces as those Banach spaces X for which the duality mapping J has an ε-selector, $0 < \varepsilon < 1$, $f : X \to X^*$ that sends separable subsets of X into separable subsets of X^* (Theorem 4.2 and Corollary 4.3). A consequence is that the dual unit ball B_{X^*} is norm fragmented if, and only if, it is norm ε-fragmented for some fixed $0 < \varepsilon < 1$ (Corollary 4.4).

Section 5 starts with the proof of Theorem 5.1 already presented above. Theorem 5.4 offers a characterization of those Banach spaces (not necessarily separable) without copies of ℓ^1 via the structure of the boundaries of w^*-compact sets of their duals and the topology γ on X^* of uniform convergence on bounded and countable subsets of X. The paper is finished by giving our proof in Corollary 5.6 of the fact that for a dual Banach space X^* with property $\mathcal{C}$ all boundaries for B_{X^*} have property (S).

A bit of terminology. Most of our notation and terminology is standard, otherwise it is either explained here or whenever it is needed; unexplained concepts and terminology can be found in our standard references for Banach spaces [9, 13] and topology [11, 34]. By letters $T, E, X, \dots$ we denote topological spaces. Sometimes the topological spaces we use are assumed to be metric and then the letters $d, \varrho, \dots$ denote metrics on them. If (E, ϱ) is a metric space, $x \in E$ and $\delta > 0$ we denote by $B_\varrho(x,\delta)$ (or $B(x,\delta)$ if no confusion arises) the open ϱ-ball centered at x of radius δ; if $A \subset E$ we write

$$\varrho\text{-diam}(A) := \sup\{\varrho(x,y) : x, y \in A\}.$$

All vector spaces $E, X, \ldots$ are assumed to be real. Sometimes E is assumed to be a normed space with the norm $\|\cdot\|$; the letter X is reserved to denote a Banach space. Given a subset S of a vector space, we write $\mathrm{co}(S)$, $\mathrm{aco}(S)$ and $\mathrm{span}(S)$ to denote, respectively, the convex, absolutely convex and linear hull of S. In the normed space $(E, \|\cdot\|)$ the unit ball $\{x \in E : \|x\| \le 1\}$ is denoted by B_E. Thus the unit ball of E^* is B_{E^*}. If S is a subset of E^*, then $\sigma(E,S)$ denotes the weakest topology for E that makes each member of S continuous, or equivalently, the topology of pointwise convergence on S. Dually, if S is a subset of E, then $\sigma(E^*,S)$ is the topology for E^* of pointwise convergence on S. In particular $\sigma(E,E^*)$ and $\sigma(E^*,E)$ are the weak (w) and weak* (w^*) topologies on E and E^* respectively. Of course, $\sigma(E,S)$ is always a locally convex topology and it is Hausdorff if and only if $E^* = \overline{\mathrm{span}\, S}^{w^*}$, and similarly for $\sigma(E^*,S)$. Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle$ and $x^*(x)$ for the evaluation of x^* at x.

2. σ-fragmented maps. Our main tool is the notion of σ-fragmented map that was introduced in [30] in order to deal with selection problems. Since its introduction this notion has been used in different settings by different authors as for instance in [40]. In this section we will present a detailed study of σ-fragmented maps which is close in spirit to the properties studied for σ-continuous maps in connection with renorming properties of Banach spaces in [37].

Definition 1 ([30]). Let f be a map from a topological space (T,τ) into a metric space (E,ϱ). Let S be a subset of T. We say that $f|_S$ is *ϱ-fragmented down to ε* or *ε-fragmented* for some $\varepsilon > 0$ if whenever C is a non-empty subset of S, there exists a τ-open subset V in T such that $C \cap V \neq \emptyset$ and $\varrho\text{-diam}(f(C \cap V)) < \varepsilon$; we simply use *fragmented* instead of *ϱ-fragmented* when ϱ is understood. Given $\varepsilon > 0$ we say that f is *ε-σ-fragmented* if there exists a countable family of subsets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ that covers T such that $f|_{T_n^\varepsilon}$ is ε-fragmented for every $n \in \mathbb{N}$.

The map f is said to be *σ-fragmented* if it is ε-σ-fragmented for each $\varepsilon > 0$.

For set-valued maps the corresponding notion of σ-fragmentability is recalled below:

Definition 2 ([30]). Let F be a set-valued map from a topological space (T,τ) into the subsets of a metric space (E,ϱ). Let S be a subset of T. We say that $F|_S$ is *fragmented down to ε* for some $\varepsilon > 0$ if whenever C is a non-empty subset of S, there exists a τ-open subset V of S with $V \cap C \neq \emptyset$ and a subset D of E with ϱ-diam$(D) < \varepsilon$ such that $F(t) \cap D \neq \emptyset$ for every $t \in V \cap C$. Given $\varepsilon > 0$ we say that F is *ε-σ-fragmented* if there exists a countable family of subsets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ that covers T such that $F|_{T_n^\varepsilon}$ is ε-fragmented for every $n \in \mathbb{N}$.

The set-valued map F is said to be *σ-fragmented* if it is ε-σ-fragmented for each $\varepsilon > 0$.

It is easily proved that in the above definitions of ε-fragmentability for f and F the sets C can be taken to be closed without loss of generality. The easiest but non-trivial examples of σ-fragmented maps are provided by the class of maps introduced in the following definition; we refer to [35] for the concept of barely continuous function.

DEFINITION 3. A map f from a topological space (T, τ) into a metric space (E, ϱ) is said to be *barely continuous* (resp. *barely constant*) if for every non-empty closed set $A \subset T$ the restriction $f|_A$ has at least one point of continuity (resp. there exists an τ-open set $W \subset T$ such that $W \cap A \neq \emptyset$ and $f|_{A\cap W}$ is constant).

We say that f is *piecewise barely continuous* (resp. *piecewise barely constant*) if there exists a countable family of subsets $\{T_n : n \in \mathbb{N}\}$ that covers T such that $f|_{T_n}$ is barely continuous (resp. barely constant) for each $n \in \mathbb{N}$.

Baire's Great Theorem establishes that a map f from a complete metric space T into a Banach space E is barely continuous if, and only if, f is the pointwise limit of a sequence of continuous functions, i.e., f is a Baire one map (see [9, Theorem I.4.1]). It was proved in [30, Corollary 7] that a map f from a perfectly paracompact space T into a Banach space X is σ-fragmented with closed sets T_n^ε in Definition 1 if, and only if, it is a Baire one map. Corollary 7 in [30] is based upon the approximation result [30, Theorem 5] that is established there for σ-fragmented maps by closed sets T_n^ε; if we drop off the closedness of the T_n^ε's and only care about σ-fragmentability we can prove the following:

THEOREM 2.1. *Let F be a set-valued map from a topological space (T, τ) into the subsets of a metric space (E, ϱ). The following statements are equivalent*:

(i) *F is σ-fragmented*;
(ii) *for every $\varepsilon > 0$ there exists a piecewise barely constant map $f_\varepsilon : T \to E$ such that ϱ-dist$(f_\varepsilon(t), F(t)) < \varepsilon$ for every $t \in T$*;
(iii) *for every $\varepsilon > 0$ there exists a piecewise barely continuous map $f_\varepsilon : T \to E$ such that ϱ-dist$(f_\varepsilon(t), F(t)) < \varepsilon$ for every $t \in T$.*

Proof. (i)⇒(ii). Fix $\varepsilon > 0$. According to Definition 1 let us decompose T as $T = \bigcup_{n=1}^\infty T_n^\varepsilon$ in such a way that for each non-empty subset C of T_n^ε

there exists an open subset V of T and a subset D of E with ϱ-diam$(D) < \varepsilon$ such that $V \cap C \neq \emptyset$ and

$$F(t) \cap D \neq \emptyset \quad \text{for every } t \in V \cap C.$$

Without any loss of generality we can assume that the sets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ are pairwise disjoint. Now we will construct for every n a barely constant function $f_n^\varepsilon : T_n^\varepsilon \to E$ with

$$\varrho\text{-dist}(f_n^\varepsilon(t), F(t)) < \varepsilon \quad \text{for every } t \in T_n^\varepsilon.$$

Once the above has been proved, statement (ii) is satisfied if we define

$$f_\varepsilon(t) := f_n^\varepsilon(t) \quad \text{for } t \in T_n^\varepsilon,\ n \in \mathbb{N}.$$

Let us fix $n \in \mathbb{N}$ and construct f_n^ε. Since $F|_{T_n^\varepsilon}$ is ε-fragmented, a transfinite induction argument provides us with well ordered families $\{G_\gamma : \gamma < \Gamma_\varepsilon^n\}$ of open subsets of T_n^ε together with subsets $\{D_\gamma : \gamma < \Gamma_\varepsilon^n\}$ of E with ϱ-diam$(D_\gamma) < \varepsilon$, $\gamma < \Gamma_\varepsilon^n$, such that for each $\mu < \Gamma_\varepsilon^n$ we have

$$M_\mu := G_\mu \setminus \bigcup \{G_\gamma : \gamma < \mu\} \neq \emptyset \text{ and } F(t) \cap D_\mu \neq \emptyset \text{ for every } t \in M_\mu$$

and

$$T_n^\varepsilon = \bigcup \{G_\gamma : \gamma < \Gamma_\varepsilon^n\}.$$

For each $\gamma < \Gamma_\varepsilon^n$ we pick a point y_γ in D_γ and define $f_n^\varepsilon(t) := y_\gamma$ whenever $t \in M_\gamma$. The map $f_n^\varepsilon : T_n^\varepsilon \to E$ is barely constant. Indeed, if for a non-empty subset A of T_n^ε we define γ_0 to be the first ordinal with $A \cap G_{\gamma_0} \neq \emptyset$ then f_n^ε is constant on $A \cap G_{\gamma_0}$ because

$$A \cap G_{\gamma_0} \subset M_{\gamma_0} = G_{\gamma_0} \setminus \bigcup \{G_\beta : \beta < \gamma_0\}$$

and $f_n^\varepsilon(t) = y_{\gamma_0}$ for every $t \in M_{\gamma_0}$. On the other hand, since $\gamma < \Gamma_\varepsilon^n$ and $t \in M_\gamma$ imply that $f_n^\varepsilon(t) = y_\gamma$ with

$$y_\gamma \in D_\gamma, \quad \varrho\text{-diam}(D_\gamma) < \varepsilon \quad \text{and} \quad F(t) \cap D_\gamma \neq \emptyset,$$

we conclude that

$$\varrho\text{-dist}(f_n^\varepsilon(t), F(t)) < \varepsilon \quad \text{for every } t \in T_n^\varepsilon,$$

and the proof for (i)⇒(ii) is complete.

The implication (ii)⇒(iii) being obvious, we prove (iii)⇒(i). Fix $\varepsilon > 0$ and take $f_\varepsilon : T \to E$ such that we can decompose $T = \bigcup \{T_n^\varepsilon : n \in \mathbb{N}\}$ and $f_\varepsilon|_{T_n^\varepsilon}$ is barely continuous for each $n \in \mathbb{N}$ with

$$\varrho\text{-dist}(f_\varepsilon(t), F(t)) < \varepsilon/3 \quad \text{for every } t \in T.$$

If C is a non-empty closed subset of T_n^ε, then there exists an open subset V of T such that $V \cap C \neq \emptyset$ and ϱ-diam$(f_\varepsilon(V \cap C)) < \varepsilon/3$. If we define

$$D := \{y \in E : \varrho\text{-dist}(y, f_\varepsilon(V \cap C)) < \varepsilon/3\}$$

then ϱ-diam$(D) < \varepsilon$ and $F(t) \cap D \neq \emptyset$ for every $t \in V \cap C$. ∎

When we deal with a single-valued map the above result is the key to proving that the barely constant maps together with countable splitting and limit points produce all σ-fragmented maps.

COROLLARY 2.2. *A map f from a topological space (T,τ) into a metric space (E,ϱ) is σ-fragmented if, and only if, there exists a sequence $\{f_n : T \to E : n \in \mathbb{N}\}$ of piecewise barely constant maps that uniformly converges to f.*

Proof. By Theorem 2.1 the map $f : T \to E$ is σ-fragmented if, and only if, for $\varepsilon = 1/n$, $n \in \mathbb{N}$, there exists a piecewise barely constant map $f_n : T \to E$ with $\varrho(f_n(t), f(t)) < 1/n$ for all $t \in T$. ∎

Next we show that σ-fragmentability is in fact preserved when taking pointwise cluster points of sequences of σ-fragmented maps:

PROPOSITION 2.3. *Let f be a map from a topological space (T,τ) into a metric space (E,ϱ). If there exists a sequence $\{f_n : T \to E : n = 1,2,\ldots\}$ of σ-fragmented maps with*

$$f(t) \in \overline{\{f_n(t) : n = 1,2,\ldots\}}^{\varrho} \quad \text{for every } t \in T,$$

then f is σ-fragmented.

Proof. Let us fix $\varepsilon > 0$ and define

$$S_n^\varepsilon := \{t \in T : \varrho(f(t), f_n(t)) < \varepsilon/3\}, \quad n = 1,2,\ldots.$$

Clearly $T = \bigcup_{n=1}^\infty S_n^\varepsilon$ and for every n we can also decompose $T = \bigcup_{m=1}^\infty T_m^{n,\varepsilon}$ in such a way that $f_n|_{T_m^{n,\varepsilon}}$ is $\varepsilon/3$-fragmented for every $m = 1,2,\ldots.$ Observe that

$$S_n^\varepsilon = \bigcup_{m=1}^\infty S_n^\varepsilon \cap T_m^{n,\varepsilon} \quad \text{and} \quad T = \bigcup_{n,m=1}^\infty S_n^\varepsilon \cap T_m^{n,\varepsilon}.$$

Now, for every pair of $n,m \in \mathbb{N}$ if we take a non-empty subset C of $S_n^\varepsilon \cap T_m^{n,\varepsilon}$, then there exists a non-empty open set V of T with $V \cap C \neq \emptyset$ and $\varrho\text{-diam}(f_n(V \cap C)) < \varepsilon/3$; the last inequality and the fact that $C \subset S_n^\varepsilon$ lead to

$$\varrho\text{-diam}(f(V \cap C)) < \varepsilon,$$

if we bear in mind the definition of S_n^ε. ∎

For maps with values in a normed space $(E,\|\cdot\|)$ the term *σ-fragmented* will always refer to σ-fragmentability with respect to $\|\cdot\|$.

COROLLARY 2.4. *Let f be a map from a topological space (T,τ) into a normed space $(E,\|\cdot\|)$. If there exists a sequence $\{f_n : T \to E : n = 1,2,\ldots\}$ of σ-fragmented maps with*

$$f(t) \in \overline{\{f_n(t) : n = 1,2,\ldots\}}^{w} \quad \text{for every } t \in T,$$

then f is σ-fragmented.

Proof. It is easily checked that linear combinations of σ-fragmented maps are σ-fragmented. Hence the $\mathbb{Q}$-linear combinations of $\{f_n : n = 1, 2, \ldots\}$ form a countable family $\{g_n : T \to E : n = 1, 2, \ldots\}$ of σ-fragmented maps for which the Hahn–Banach theorem [13, Theorem 3.19] gives

$$f(t) \in \overline{\{g_n(t) : n = 1, 2, \ldots\}}^{\|\cdot\|} \quad \text{for every } t \in T.$$

We now apply Proposition 2.3 to finish the proof. ∎

The following definition can be found in [23].

DEFINITION 4 (Hansell, [23]). A family $\mathcal{E}$ of subsets in a topological space T is said to be *scattered* if $\mathcal{E}$ is disjoint and there exists a well ordering $\le$ of $\mathcal{E}$ such that for every $E \in \mathcal{E}$ the set $\bigcup\{M \in \mathcal{E} : M \le E\}$ is open relative to $\bigcup \mathcal{E}$. The family $\mathcal{E}$ is said to be *σ-scattered* if it can be decomposed into a countable union $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ such that every family $\mathcal{E}_n$ is scattered.

Given a map $f : T \to E$ between topological spaces, we say that a family $\mathcal{B}$ of subsets of T is a *function base* for f if, whenever V is open in E, then $f^{-1}(V)$ is a union of sets of $\mathcal{B}$, i.e., $\mathcal{B}$ is a function base for f if it is a *network* in T for the topology given by $\{f^{-1}(V) : V \text{ is open in } E\}$ (see [11, p. 170]) for the notion of network. We recall that a family $\{F_i : i \in I\}$ of subsets of a topological space (T, τ) is said to be *discrete* if for every point $x \in T$ there exists a neighborhood U of x such that U meets at most one F_i [11, p. 33]. Recall also that a family $\{D_j : j \in J\}$ of subsets of T is a *refinement* of a family $\{C_l : l \in L\}$ if $\bigcup_{j \in J} D_j = \bigcup_{l \in L} C_l$ and each D_j is contained in some C_l [11, p. 165].

We note that the next result already appeared in [23, Theorem 1.10] for the very particular case of f being the identity map $\mathrm{id} : (T, \tau) \to (T, \varrho)$ where ϱ is a metric on T whose associated topology is finer than τ. The next theorem exhibits the relationship between the notion of σ-fragmented map and the earlier concept of map with σ-scattered function base introduced by Hansell (see [22] and the references therein).

THEOREM 2.5. *Let f be a map from a topological space (T, τ) into a metric space (E, ϱ). The following statements are equivalent*:

(i) *f is σ-fragmented*;
(ii) *if $\{D_i : i \in I\}$ is a discrete family of subsets in (E, ϱ) then the family $\{f^{-1}(D_i) : i \in I\}$ has a σ-scattered refinement*;
(iii) *f has a σ-scattered function base.*

Proof. (i)⇒(ii). Let $\{D_i : i \in I\}$ be a discrete family in (E, ϱ) and define

$$D_{i,p} := \{x \in D_i : B_\varrho(x, 1/p) \cap D_j = \emptyset \text{ for each } j \in I,\ j \ne i\}$$

for every positive integer p. We clearly have $D_i = \bigcup_{p=1}^{\infty} D_{i,p}$ for every $i \in I$ and the family $\{D_{i,p} : i \in I\}$ is $1/p$-discrete (meaning that the distance between two different elements of the family is at least $1/p$) for every $p = 1, 2, \ldots$. Fix the positive integer p and use the fact that f is $1/p$-σ-fragmented to produce a decomposition $T = \bigcup_{n=1}^{\infty} T_n^{1/p}$ such that for every n we have a well ordered family $\{G_\gamma^{n,p} : \gamma < \Gamma_n^{1/p}\}$ of relatively open subsets of $T_n^{1/p}$ which covers $T_n^{1/p}$ and provides us with the scattered family

$$\Big\{M_\mu^{n,p} := G_\mu^{n,p} \setminus \bigcup\{G_\beta^{n,p} : \beta < \mu\} : \mu < \Gamma_n^{1/p}\Big\}$$

such that

$$\varrho\text{-diam}(f(M_\mu^{n,p})) < 1/p \quad \text{for every } \mu < \Gamma_n^{1/p}.$$

The $1/p$-discreteness of the family $\{D_{i,p} : i \in I\}$ implies that $M_\mu^{n,p}$ meets at most one member of the family $\{f^{-1}(D_{i,p}) : i \in I\}$. Thus the family formed by all the non-void sets of the form

$$\{M_\mu^{n,p} \cap f^{-1}(D_{i,p})\}$$

for $i \in I$ and $\mu < \Gamma_{n,p}^{1/p}$ is scattered for fixed integers n and p because any subset of ordinals is a well ordered set.

All things considered, we conclude that

$$\bigcup_{n,p=1}^{\infty} \{M_\mu^{n,p} \cap f^{-1}(D_{i,p}) : i \in I,\ \mu < \Gamma_n^{1/p}\}$$

is a σ-scattered refinement of $\{f^{-1}(D_i) : i \in I\}$ and (ii) is satisfied.

(ii)⇒(iii). Stone's theorem [11, Theorem 4.4.3] provides us with a σ-discrete base $\mathcal{B}$ for the metric topology in (E, ϱ), that is, $\mathcal{B}$ is a base for the topology and it can be split as $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ with each $\mathcal{B}_n$ discrete. According to (ii) each $f^{-1}(\mathcal{B}_n)$ has a σ-scattered refinement, that is, there exist scattered families $\mathcal{E}_m^n$ in T with $\bigcup_{m=1}^{\infty} \mathcal{E}_m^n$ being a refinement of $f^{-1}(\mathcal{B}_n)$. Observe that $\bigcup_{n,m=1}^{\infty} \mathcal{E}_m^n$ is a σ-scattered refinement of $f^{-1}(\mathcal{B})$. Furthermore, we claim that $\bigcup_{n,m=1}^{\infty} \mathcal{E}_m^n$ is a function base of f. Indeed, given an open set $V \subset E$ and $x \in f^{-1}(V)$ there exists B in some $\mathcal{B}_n$ such that $x \in f^{-1}(B) \subset f^{-1}(V)$; from the equality

$$\bigcup_{B \in \mathcal{B}_n} f^{-1}(B) = \bigcup_m \bigcup_{C \in \mathcal{E}_m^n} C$$

and the facts that $\{f^{-1}(B) : B \in \mathcal{B}_n\}$ are disjoint together with $\bigcup_{m=1}^{\infty} \mathcal{E}_m^n$ being a refinement of $f^{-1}(\mathcal{B}_n)$ we infer that there exists C in some $\mathcal{E}_m^n$ such that

$$x \in C \subset f^{-1}(B) \subset f^{-1}(V),$$

and the proof for this implication is complete.

(iii)⇒(i). Let $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ be a function base for f with $\mathcal{E}_n$ a scattered family for every $n \in \mathbb{N}$, i.e., $\mathcal{E}_n = \{E_\alpha^n : \alpha < \Gamma_n\}$ and $E_\alpha^n \subset U_\alpha^n \setminus \bigcup\{U_\beta^n :$

$\beta < \alpha\}$ for some well ordered family $\{U_\alpha^n : \alpha < \Gamma_n\}$ of open sets in T. If we fix $T_n := \bigcup\{E_\alpha^n : \alpha < \Gamma_n\}$ and for every $\alpha < \Gamma_n$ we choose $t_\alpha^n \in E_\alpha^n$, then we can define $f_n(t) := f(t_\alpha^n)$ for every $t \in T_n$ when $t \in E_\alpha^n$. The function f_n is barely constant on T_n: if $A \subset T_n$ is non-empty and α_0 is the first ordinal with $A \cap U_{\alpha_0}^n \neq \emptyset$ then

$$A \cap U_{\alpha_0}^n \cap T_n \subset E_{\alpha_0}^n \cap T_n,$$

and therefore $f_n(t) = f_n(t_{\alpha_0}^n)$ for every $t \in A \cap U_{\alpha_0}^n \cap T_n$. Now we extend f_n to the whole T by defining it as an (arbitrary) constant function on $T \setminus T_n$. Since $\mathcal{E}$ is a function base for f, we easily see that

$$f(t) \in \overline{\{f_n(t) : n = 1, 2, \ldots\}}$$

for every $t \in T$: indeed, given $\varepsilon > 0$ there exist a positive integer m and $\beta < \Gamma_m$ such that $t \in E_\beta^m \subset f^{-1}(B_\varrho(f(t), \varepsilon))$, thus $f_m(t) = f(t_\beta^m) \in B_\varrho(f(t), \varepsilon)$. Now Proposition 2.3 applies to conclude that f is σ-fragmented. ∎

Remark 2.6. If a scattered function base for a map f can be constructed with sets which are differences of closed sets, then f enjoys properties which are close to measurability; in fact, it is Borel measurable when the domain space is, for instance, a complete metric space or a Gulko compact (see [24, 23, 22]).

We recall that a set $S \subset T$ is a *Suslin-$\mathcal{F}$-set* in the space (T, τ) if S is the result of the Suslin operation applied to closed sets of T, i.e., for some collection

$$\{F_{n_1,\ldots,n_k} : (n_1, \ldots, n_k) \in \mathbb{N}^{(\mathbb{N})}\}$$

of closed sets indexed by the set $\mathbb{N}^{(\mathbb{N})}$ of finite sequences of positive integers, we have

$$S = \bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \bigcap_{k=1}^{\infty} \{F_{\alpha|k}\},$$

where $\alpha|k := (n_1, \ldots, n_k)$ for $\alpha = (n_1, n_2, \ldots) \in \mathbb{N}^\mathbb{N}$. Every Borel set in a metric space is a Suslin-$\mathcal{F}$-set [45, Theorem 44]. A map between metric spaces is called *analytic* if the preimage of every open set is a Suslin-$\mathcal{F}$-set. Bearing in mind the seminal results by R. Hansell in [21] we can now recall the following:

Corollary 2.7 (Lemma 5.9 in [23]). *Every analytic* (*in particular, Borel measurable*) *map from a complete metric space* (T, d) *into a metric space* (E, ϱ) *is* σ*-fragmented.*

Proof. Hansell's Theorem 3 in [21] states that every analytic map, in particular every Borel map, from a complete metric space into a metric space has a σ-discrete function base. Since every discrete family of sets is clearly scattered, the desired conclusion follows directly from Theorem 2.5. ∎

An important property of Borel maps from complete metric spaces into metric spaces is that they send separable subsets into separable ones (see for instance [47, Theorem 4.3.8]). The fact that σ-fragmented maps enjoy the same property is stated in the next theorem whose proof can be found in [37, Theorem 2.15], where the result has been used as an important tool for renorming in Banach spaces.

THEOREM 2.8. *Let (T,d) and (E,ϱ) be metric spaces and let $f:T\to E$ be a σ-fragmented map. Then for every $t\in T$ there exists a countable set $W_t\subset T$ such that*

$$f(t)\in\overline{\bigcup\{f(W_{t_n}):n=1,2,\ldots\}}^{\varrho}$$

whenever $\{t_n\}$ is a sequence converging to t in (T,d). In particular, $f(S)$ is separable whenever S is a separable subset of T.

We will use the above precise way of sending separable sets into separable ones by σ-fragmented maps in the proof of Theorem 4.1. It should be noted that σ-fragmented maps are not necessarily Borel measurable: for instance, every map between metric spaces with separable range is σ-fragmented. Let us remark that a map with domain a metric space and with values in a normed space is Baire one if, and only if, it is σ-fragmented and the preimages of open sets are $\mathcal{F}_\sigma$ sets (see [32, Chapter 2] and [25]).

The σ-fragmentability of maps is not only preserved by countable partitions. Indeed, if a map f is σ-fragmented when restricted to the sets of a scattered partition, then f is σ-fragmented. To be more precise, we have the following result.

PROPOSITION 2.9. *Let f be a map from a topological space (T,τ) into a metric space (E,ϱ). If there exists a well ordered family of open sets $\{G_\gamma:\gamma<\Gamma\}$ covering T such that f is σ-fragmented when restricted to every atom*

$$M_\gamma=G_\gamma\setminus\bigcup\{G_\beta:\beta<\gamma\},$$

then f is σ-fragmented on the whole of T.

Proof. Let us fix $\varepsilon>0$ and split M_γ as

$$M_\gamma=\bigcup\{M^n_{\gamma,\varepsilon}:n=1,2,\ldots\}$$

in such a way that $f|_{M^n_{\gamma,\varepsilon}}$ is ε-fragmented. If we set $T^n_\varepsilon:=\bigcup\{M^n_{\gamma,\varepsilon}:\gamma<\Gamma\}$, then it is clear that $T=\bigcup\{T^n_\varepsilon:n=1,2,\ldots\}$. We now prove that $f|_{T^n_\varepsilon}$ is ε-fragmented. If we fix some non-empty set $C\subset T^n_\varepsilon$ and choose γ_0 to be the first ordinal such that $C\cap M_{\gamma_0}$ is non-empty, then

$$C\cap G_{\gamma_0}=C\cap\Big(G_{\gamma_0}\setminus\bigcup\{G_\beta:\beta<\gamma_0\}\Big)=C\cap M_{\gamma_0}.$$

On the other hand, since $C \subseteq T_\varepsilon^n$ we have $\emptyset \neq C \cap M_{\gamma_0} = C \cap M_{\gamma_0,\varepsilon}^n$, and therefore the ε-fragmentability of $f|_{M_{\gamma_0,\varepsilon}^n}$ provides an open set W in (T,τ) such that $W \cap C \cap M_{\gamma_0,\varepsilon}^n$ is non-empty and

$$\varrho\text{-diam}(f(W \cap C \cap M_{\gamma_0,\varepsilon}^n)) < \varepsilon.$$

Since $W \cap C \cap G_{\gamma_0} = W \cap C \cap M_{\gamma_0,\varepsilon}^n$ we have

$$\varrho\text{-diam}(f((W \cap G_{\gamma_0}) \cap C)) < \varepsilon,$$

and therefore $f|_{T_\varepsilon^n}$ is ε-fragmented. ∎

The previous result leads to the following one (see [29, Theorem 4.1] for the identity map). We use the following terminology: a subset A of a metric space (E,ϱ) is said to be ε-*separable* ($\varepsilon > 0$) if there exists some countable subset H in E such that

$$A \subset \bigcup\{B_\varrho(h,\varepsilon) : h \in H\}.$$

PROPOSITION 2.10. *Let f be a map from a topological space (T,τ) into a metric space (E,ϱ). If for every $\varepsilon > 0$ there exists a countable family of subsets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ that covers T such that for every $n \in \mathbb{N}$ and every non-empty subset $C \subset T_n^\varepsilon$ there exists a τ-open subset V in T with $V \cap C$ non-empty and with $f(V \cap C)$ ε-separable, then f is σ-fragmented.*

Proof. For a fixed $\varepsilon > 0$ we shall construct a sequence

$$\{f_n^\varepsilon : (T,\tau) \to (E,\varrho) : n = 1,2,\ldots\}$$

of σ-fragmented maps such that

$$\varrho\text{-dist}(f(t), \{f_n^\varepsilon(t) : n = 1,2,\ldots\}) < \varepsilon.$$

Therefore $f(t) \in \overline{\{f_n^{1/p}(t) : n,p = 1,2,\ldots\}}^{\varrho}$ for every $t \in T$, and an appeal to Proposition 2.3 will ensure that f is σ-fragmented. For a fixed $\varepsilon > 0$ let us split $T = \bigcup_{n=1}^\infty T_n^\varepsilon$ as in our hypothesis. It is easily proved that in every T_n^ε we can produce a well ordered family $\{G_\gamma^n : \gamma < \Gamma_{n,\varepsilon}\}$ of open sets with

$$\bigcup\{G_\gamma^n : \gamma < \Gamma_{n,\varepsilon}\} = T_n^\varepsilon,$$

and for every atom

$$M_\gamma^n = G_\gamma^n \setminus \bigcup\{G_\beta^n : \beta < \gamma\}$$

there exists a countable set H_γ^n in E such that

$$f(M_\gamma^n) \subset \bigcup\{B_\varrho(h,\varepsilon) : h \in H_\gamma^n\}.$$

If $H_\gamma^n = \{h_\gamma^n(j) : j = 1,2,\ldots\}$ we now define the maps $f_{n,j}^\varepsilon : T_n^\varepsilon \to E$ by

$$f_{n,j}^\varepsilon(t) := h_\gamma^n(j) \quad \text{if } t \in M_\gamma^n.$$

Since $f_{n,j}^\varepsilon$ is constant on the atoms of a well ordered family of open sets in T_n^ε, it is σ-fragmented on the whole piece T_n^ε by Proposition 2.9. We now define $f^{\varepsilon}_{n,j}$ to be arbitrary but constant on $T \setminus T^{\varepsilon}_{n}$. The new $f^{\varepsilon}_{n,j}$ defined on the whole T is still σ-fragmented. On the other hand, it is clear that

$$\varrho\text{-dist}(f(t), \{f^{\varepsilon}_{n,j}(t) : j, n = 1, 2, \dots\}) < \varepsilon$$

for every $t \in T$, and the proof is complete. ■

We stress that the previous result for $f = \mathrm{id}$ has been used in [28] where it is proved that $C_{\mathrm{p}}(K)$ is σ-fragmented when K is a Rosenthal compactum of functions with at most countably many discontinuities.

3. σ-fragmented normed spaces. Let $(E, \|\cdot\|)$ be a normed space, τ a topology on E coarser than the norm topology, and H a subset of X. (H, τ) is said to be *fragmented* (resp. *σ-fragmented*) *by the norm of* E if the inclusion $i : (H, \tau) \to (E, \|\cdot\|)$ is fragmented (resp. σ-fragmented); when $\tau = w$ we simply say that E is *σ-fragmented* [29]. Recall that a subspace F of E^* is said to be *norming* if the function p on E given by

$$p(x) = \sup\{|x^*(x)| : x^* \in F \cap B_{E^*}\}$$

is a norm equivalent to $\|\cdot\|$; if this is the case then $\overline{F}^{w^*} = E^*$.

The next result is the counterpart for σ-fragmentability of [36, Theorem 8] that has been proved for LUR renorming.

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a normed space and $F \subset X^*$ a norming subspace. The following statements are equivalent*:

(i) *$(E, \sigma(E, F))$ is σ-fragmented by the norm*;

(ii) *the identity* id $: E \to E$ *is the uniform limit for the norm of a sequence $\{I_n : E \to E : n = 1, 2, \dots\}$ of maps which are piecewise barely constant for the topology $\sigma(E, F)$*;

(iii) *there exists a sequence $\{I_n : E \to E : n = 1, 2, \dots\}$ of maps which are piecewise barely constant for $\sigma(E, F)$ such that*

$$x \in \overline{\{I_n(x) : n = 1, 2, \dots\}}^{w} \quad \text{for every } x \in E;$$

(iv) *there exists a sequence $\{I_n : E \to E : n = 1, 2, \dots\}$ of maps which are piecewise barely constant for $\sigma(E, F)$ such that*

$$x \in \overline{\operatorname{span}\{I_n(x) : n = 1, 2, \dots\}}^{\|\cdot\|} \quad \text{for every } x \in E.$$

Proof. The equivalence between (i) and (ii) follows directly from Corollary 2.2. Clearly (ii)⇒(iii). The rest of the proof uses ideas that already appeared in Corollary 2.4.

(iii)⇒(iv). If $x \in \overline{\{I_n(x) : n = 1, 2, \dots\}}^{w}$ for every $x \in E$, then the Hahn–Banach theorem [13, Theorem 3.19] implies that

$$x \in \overline{\operatorname{span}\{I_n(x) : n = 1, 2, \dots\}}^{\|\cdot\|} \quad \text{for every } x \in E$$

and thus condition (iv) is satisfied.

(iv)⇒(i). If condition (iv) holds then the set of all rational linear combinations of $\{I_n : E \to E : n = 1,2,\ldots\}$ is a countable set of maps that we denote by $\{J_n : E \to E : n = 1,2,\ldots\}$, which are σ-fragmented for the $\sigma(E,F)$-topology, and such that for every $x \in E$ we have

$$x \in \overline{\{J_n(x) : n = 1,2,\ldots\}}^{\|\cdot\|}.$$

Proposition 2.3 now allows us to conclude that (i) is satisfied. ∎

4. σ-fragmented ε-selectors for the duality mapping. This section is devoted to proving our main results. The notion below will be used repeatedly.

DEFINITION 5. Let F be a set-valued map from a set T into the subsets of a metric space (E,ϱ) and $\varepsilon > 0$. An *ε-selector* for F is a function $f : T \to E$ such that

$$\varrho\text{-dist}(f(t), F(t)) < \varepsilon \quad \text{for every } t \in T.$$

Each selector for F is clearly an ε-selector for every $\varepsilon > 0$, but not vice versa. Note that ε-selectors have appeared in Theorem 2.1. Sometimes ε-selectors are the first step when finding a real selector (see for instance [30, 32, 49]).

Godefroy's result quoted in Theorem 1.1 can be rephrased in the following way that suits well our purposes.

LEMMA 1. *Let $(X,\|\cdot\|)$ be a Banach space, $J : X \to 2^{B_{X^*}}$ the duality mapping, and let f be an ε-selector of J, $0 < \varepsilon < 1$. If $Z \subset X$ is a subspace such that $f(Z)$ is separable for the norm of X^*, then*

$$Z^* = \overline{\operatorname{span} f(Z)|_Z}^{\|\cdot\|_{Z^*}}, \tag{4.1}$$

and consequently Z^* is norm separable.

Proof. See [9, Proposition 3.2, p. 50]. ∎

Our main result below is proved using a reduction to separable Banach spaces similar to the one used in [30, Theorem 26]; this argument goes back to [12]. Nonetheless, we note that our situation here is more complicated that the one in [30, Theorem 26], because we now deal with maps which are only σ-fragmented instead of Baire one maps used in [30]; to overcome the extra difficulties we will make use of the precise description of how separable sets are sent into separable ones via σ-fragmented maps (see Theorem 2.8 above).

THEOREM 4.1. *Let $(X,\|\cdot\|)$ be a Banach space and let $J : X \to 2^{B_{X^*}}$ denote the duality mapping. If for some fixed $0 < \varepsilon < 1$ there exists a* *σ-fragmented ε-selector $f : X \to X^*$ of J, then*

$$\overline{\operatorname{span} f(X)}^{\|\cdot\|_{X^*}} = X^*.$$

Proof. Take any linear form $g \in X^*$. The idea is to construct a subspace $Z \subset X$ satisfying the assumption in Lemma 1, i.e., with $f(Z) \subset X^*$ norm separable, in such a way that from the condition

$$g|_Z \in \overline{\operatorname{span}\{f(Z)|_Z\}}^{\|\cdot\|_{Z^*}} \ (= Z^*)$$

it follows that

$$g \in \overline{\operatorname{span}\{f(X)\}}^{\|\cdot\|_{X^*}}.$$

For every $x \in X$ we use Theorem 2.8 to pick a countable subset W_x in X such that

$$f(x) \in \overline{\{f(W_{x_n}) : n = 1, 2, \ldots\}}^{\|\cdot\|_{X^*}}$$

whenever $(x_n)_n$ converges to x in the Banach space $(X, \|\cdot\|)$.

Let us choose a countable $\mathbb{Q}$-linear subspace $\{0\} \neq Z_1 \subset X$. Define $D_1 := \bigcup\{W_x : x \in Z_1\}$, which is also a countable set, and write

$$C_1 := \mathbb{Q}\text{-span}\{f(W_x) : x \in D_1\} =: \{h_{1,j} : j \in \mathbb{N}\}.$$

We now find vectors $\{v_{1,j} : j \in \mathbb{N}\} \subset B_X$ such that

$$\langle g - h_{1,j}, v_{1,j}\rangle \geq \|g - h_{1,j}\| - 1,$$

and we collect the v's as $F_1 := \{v_{1,j} : j \in \mathbb{N}\}$.

An induction process produces, for every $n \in \mathbb{N}$, countable sets $C_n \subset X^*$, $Z_n, D_n \subset X$ and

$$F_n := \{v_{n,j} : j = 1, 2, \ldots,\} \subset B_X$$

such that

(i) each Z_n is $\mathbb{Q}$-linear subspace with $Z_n \cup F_n \subset Z_{n+1}$;
(ii) $D_n := \bigcup\{W_x : x \in Z_n\}$;
(iii) if we enumerate $C_n := \mathbb{Q}\text{-span}\{f(W_x) : x \in D_n\} := \{h_{n,j} : j \in \mathbb{N}\}$, then

$$\langle g - h_{n,j}, v_{n,j}\rangle \geq \|g - h_{n,j}\| - 1/n \quad \text{for every } j \in \mathbb{N}. \tag{4.2}$$

Indeed, if Z_i, D_i and F_i have been constructed for $1 \leq i \leq n$, then we define the countable sets

$$Z_{n+1} := \mathbb{Q}\text{-span}\{Z_n \cup F_n\}, \qquad D_{n+1} := \bigcup\{W_x : x \in Z_{n+1}\},$$

and once we have enumerated

$$C_{n+1} := \mathbb{Q}\text{-span}\{f(W_x) : x \in D_{n+1}\} =: \{h_{n+1,j} : j \in \mathbb{N}\}$$

we simply find vectors

$$F_{n+1} := \{v_{n+1,j} : j = 1, 2, \ldots\}$$

satisfying the corresponding inequality (4.2).

Define $Z := \overline{\bigcup\{Z_n : n = 1, 2, \ldots\}}^{\|\cdot\|}$. Our construction tells us that $f(\bigcup_{n=1}^{\infty} D_n) = \bigcup\{f(W_x) : x \in \bigcup_{n=1}^{\infty} Z_n\}$. Given $z \in Z$ there exists a sequence $(z_m)_m$ in $\bigcup\{Z_n : n = 1, 2, \ldots\}$ such that $\lim_m z_m = z$. Hence, by the choice of the sets W_x we conclude that

$$f(z) \in \overline{\bigcup\{f(W_{z_m}) : m = 1, 2, \ldots\}}^{\|\cdot\|_{X^*}} \subset \overline{f\Big(\bigcup\{D_n : n = 1, 2, \ldots\}\Big)}^{\|\cdot\|_{X^*}}.$$

In other words, $f(Z) \subset \overline{f(\bigcup\{D_n : n = 1, 2, \ldots\})}^{\|\cdot\|_{X^*}}$ and we can apply Lemma 1 to Z to conclude that

$$g|_Z \in Z^* = \overline{\operatorname{span}\{f(Z)|_Z\}}^{\|\cdot\|_{Z^*}} \subset \overline{\operatorname{span} f\Big(\bigcup\{D_n : n = 1, 2, \ldots\}\Big)\Big|_Z}^{\|\cdot\|_{Z^*}}.$$

To prove that $g \in \overline{\operatorname{span}\{f(X)\}}^{\|\cdot\|_{X^*}}$, fix $\delta > 0$ and pick $h \in \operatorname{span} f(\bigcup\{D_n : n = 1, 2, \ldots\})$ such that

$$\|g|_Z - h|_Z\|_{Z^*} < \delta/2.$$

On the one hand, since $D_j \subset D_{j+1}$, $j \in \mathbb{N}$, we can write

$$h = \sum_{i=1}^{p} q_i f(d_i)|_Z, \qquad q_i \in \mathbb{Q},\ d_i \in D_n,\ i = 1, \ldots, p,$$

for some $n \in \mathbb{N}$. On the other hand, since $h = \sum_{i=1}^{p} q_i f(d_i) \in C_n$ we have $h = h_{n,j}$ for some $j \in \mathbb{N}$ and without loss of generality we can and do assume that n is large enough to have $\delta > 2/n$. All things considered, we conclude that

$$\begin{aligned}\|g - h\|_{X^*} = \|g - h_{n,j}\|_{X^*} &\overset{(4.2)}{\leq} 1/n + \langle g - h_{n,j}, v_{n,j}\rangle \\ \leq \delta/2 + \langle g - h_{n,j}, v_{n,j}\rangle &\overset{v_{n,j}\in Z}{\leq} \delta/2 + \|(g - h_{n,j})|_Z\|_{Z^*} \\ = \delta/2 + \|g|_Z - h|_Z\|_{Z^*} &\leq \delta/2 + \delta/2 = \delta.\end{aligned}$$

Letting $\delta \to 0$ shows that $g \in \overline{\operatorname{span} f(X)}^{\|\cdot\|^{X^*}}$. ■

Another consequence of Lemma 1 is the following improvement of a result by C. Stegall (see, for instance, [9, Theorem I.5.9 and Remark I.5.11], [49, Corollary 9] and [51]). Unless otherwise explicitly stated, the notions of Baire one, Borel measurability, fragmentability, etc. when used for selectors $f : X \to X^*$ always refer to the norm in X and X^*.

THEOREM 4.2. *Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \to 2^{B_{X^*}}$ be the duality mapping. The following statements are equivalent*:

(i) *X is Asplund*;
(ii) *J has a Baire one selector*;
(iii) *J has a σ-fragmented selector*;
(iv) *for some $0 < \varepsilon < 1$, J has a σ-fragmented ε-selector*;

(v) *for some $0 < \varepsilon < 1$, J has an ε-selector that sends norm separable subsets of X into norm separable subsets of X^*.*

Proof. For (i)⇒(ii) we use the fact that if X is Asplund, then Theorem 8 in [31] provides a Baire one selector for J. For (ii)⇒(iii) refer for instance to Proposition 2.3. The implication (iii)⇒(iv) is clear and (iv)⇒(v) follows from Theorem 2.8.

To finish, if we assume that (v) holds, then Lemma 1 shows that each separable subspace of $Z \subset X$ has separable dual Z^*. Therefore (i) holds. ∎

We note that to prove (i)⇒(iv) there is no need to use the full power of the Jayne–Rogers theorem, [31, Theorem 8]. Indeed, if X is an Asplund space, then Proposition 11 in [30] implies that $J : X \to 2^{B_{X^*}}$ is σ-fragmented as a set-valued map; to obtain now a σ-fragmented ε-selector for J (as desired in (iv)) we can just apply Theorem 2.1 of this paper.

Corollary 4.3. *Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \to 2^{B_{X^*}}$ be the duality mapping. The following statements are equivalent*:

(i) *X is Asplund*;
(ii) *J has a Borel measurable selector*;
(iii) *J has an analytic selector.*

Proof. This is a consequence of Theorem 4.2 and Corollary 2.7. ∎

The equivalences in the above corollary can also be obtained from [6, Theorem A], where the techniques used come from vector integration.

We next prove that if X is a Banach space then the unit ball (B_{X^*}, w^*) is fragmented by the norm of X^* if, and only if, it is ε-fragmented for some fixed $0 < \varepsilon < 1$.

Corollary 4.4. *The following conditions are equivalent for a Banach space X*:

(i) *X is an Asplund space*;
(ii) *there exists $0 < \varepsilon < 1$ such that (B_{X^*}, w^*) is ε-fragmented*, *i.e.*, *for every non-empty subset $C \subset B_{X^*}$ there exists some w^*-open set V in B_{X^*} such that $C \cap V \neq \emptyset$ and $\|\cdot\|$-diam$(C \cap V) < \varepsilon$*;
(iii) *there exists $0 < \varepsilon < 1$ such that the duality mapping J is ε-σ-fragmented.*

Proof. The implication (i)⇒(ii) is classical: see for instance [39, 48]. The implication (ii)⇒(iii) follows from [30, Proposition 11]. To prove that (iii)⇒(i) we read again the proof of (i)⇒(ii) in Theorem 2.1 and observe that for our given ε and J we can construct a σ-fragmented map $f_\varepsilon : (X, \|\cdot\|) \to (X^*, \|\cdot\|_{X^*})$ such that $\|\cdot\|$-dist$(f_\varepsilon(x), J(x)) < \varepsilon$ for every $x \in X$; now Theorem 4.2 completes the proof of (iii)⇒(i). ∎

As far as we know, the above corollary appeared first in [38]; results in the same spirit have been proved in [14, 18].

5. Approximation of boundaries by descriptive sets. Let us recall now the combinatorial principle that underlies the James compactness theorem as it was found by S. Simons [46], and described in the famous lemma:

LEMMA 2 (Simons). *Let $(z_n)_n$ be a uniformly bounded sequence in $\ell^\infty(C)$ and let W be its convex hull. If B is a subset of C such that for every sequence $(\lambda_n)_n$ of positive numbers with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b \in B$ such that*

$$\sup\Big\{\sum_{n=1}^{\infty} \lambda_n z_n(y) : y \in C\Big\} = \sum_{n=1}^{\infty} \lambda_n z_n(b), \tag{5.1}$$

then

$$\sup_{b\in B}\{\limsup_{n\to\infty} z_n(b)\} \geq \inf\{\sup_C w : w \in W\}. \tag{5.2}$$

A topological space T is said to be *angelic* if, whenever A is a relatively countably compact subset of T, its closure $\overline{A}$ is compact and each element of $\overline{A}$ is a limit of a sequence in A; good references for angelic spaces are [15] and [41]. $C_{\mathrm{p}}(T)$ stands for the space of real continuous functions endowed with the topology of pointwise convergence on T.

LEMMA 3. *Let X be a Banach space, B a James boundary for B_{X^*}, and $\varepsilon \geq 0$ and $T \subset X^*$ such that $B \subset \bigcup_{t\in T} B(t,\varepsilon)$. Assume that for each $y^* \in X^*$ all compact subsets of $C_p(T \cup \{y^*\}, w)$ are angelic. The following statements hold*:

(i) *if* $\varepsilon < 1$, *then* $X^* = \overline{\operatorname{span} T}^{\|\cdot\|}$;

(ii) *if* $\varepsilon = 0$, *then* $X^* = \overline{\operatorname{span} T}^{\|\cdot\|}$ *and* $B_{X^*} = \overline{\operatorname{co}(B)}^{\|\cdot\|}$.

Proof. Statement (ii) follows directly from Theorem I.2 in [19]. Statement (i) can be proved exactly with the same ideas of Lemma 4 in [20]. The proof is by contradiction. Fix $\varepsilon < \varepsilon' < 1$. If $\overline{\operatorname{span} T}^{\|\cdot\|} \subsetneq X^*$, then there exists $x^{**} \in X^{**}$ with $\|x^{**}\| = 1$ and $x^{**}|_T = 0$. Take $y^* \in B_{X^*}$ such that $x^{**}(y^*) > (1+\varepsilon')/2$. Consider the restrictions

$$B_X|_{T\cup\{y^*\}} \subset B_{X^{**}}|_{T\cup\{y^*\}} \subset C_{\mathrm{p}}(T \cup \{y^*\}, w).$$

Since B_X is w^*-dense in $B_{X^{**}}$, our hypothesis ensures the existence of a sequence $(x_k)_k$ in B_X such that

$$\lim_k x^*(x_k) = x^{**}(x^*) = 0 \tag{5.3}$$

for every $x^* \in T$, and

$$\lim_k y^*(x_k) = x^{**}(y^*) > \frac{1+\varepsilon'}{2}.$$

We can assume that

$$y^*(x_k) > \frac{1+\varepsilon'}{2} \quad \text{for every } k \in \mathbb{N}. \tag{5.4}$$

It follows from (5.3) and the inclusion $B \subset \bigcup_{t\in T} B(t,\varepsilon)$ that for every $b^* \in B$ we have

$$\limsup_{k\to\infty} b^*(x_k) \le \varepsilon.$$

Simon's inequality (Lemma 2) applied to $C := B_{X^*}$, B and the sequence $(x_k)_k$ ensures the existence of $x \in \mathrm{co}(\{x_k : k \in \mathbb{N}\})$ with $\|x\| < \varepsilon'$. On the other hand,

$$\varepsilon' > \|x\| \ge y^*(x) \overset{(5.4)}{>} \frac{1+\varepsilon'}{2}.$$

The inequality $\varepsilon' > (1+\varepsilon')/2$ implies that $\varepsilon' > 1$, a contradiction that finishes the proof. ∎

Recall that a topological space (T,τ) is said to be *countably K-determined* (resp. *K-analytic*) if there exists an upper semicontinuous set-valued map $F : M \to 2^T$ for some separable metric space (resp. Polish space) M such that $F(M) = T$ and $F(m)$ is compact for each $m \in M$. Notice that this class of spaces properly contains the classes of K-analytic and (so) σ-compact spaces. The paper [50] is a milestone in the study of Banach spaces which are countably K-determined when endowed with their weak topologies. The main result in [41] states that if T is a countably K-determined space then $C_{\mathrm{p}}(T)$ is angelic.

The next theorem is the outcome of the previous preparations.

Theorem 5.1. *Let X be a Banach space, B a James boundary for B_{X^*}, and $0 \le \varepsilon < 1$ and $T \subset X^*$ such that $B \subset \bigcup_{t\in T} B(t,\varepsilon)$. If (T,w) is countably K-determined (resp. K-analytic) then*:

(i) *$X^* = \overline{\mathrm{span}\, T}^{\|\cdot\|}$ and X^* is weakly countably K-determined (resp. weakly K-analytic).*
(ii) *Every James boundary for B_{X^*} has property (S). In particular, $B_{X^*} = \overline{\mathrm{co}(B)}^{\|\cdot\|}$.*

Proof. The equality in (i) follows from Lemma 3 if we bear in mind that $T \cup \{y^*\}$ is countably K-determined for every $y^* \in X^*$ and therefore the space $C_{\mathrm{p}}(T\cup\{y^*\})$ is angelic [41]. If (T,w) is countably K-determined (resp. K-analytic) then $\overline{\mathrm{span}\, T}^{\|\cdot\|}$ is again weakly countably K-determined (resp.

K-analytic) by a result of [50] and thus (i) is proved. Statement (ii) follows from Lemma 3(ii) applied for $T = X^*$. ∎

We stress that weakly countably K-determined spaces are weakly Lindelöf. Furthermore, if X is a Banach space such that X^* is weakly Lindelöf, then X is Asplund (see [10, Proposition 1.8]). In particular, *a fortiori* the Banach spaces we deal with in Theorem 5.1 are Asplund spaces. We might think that for an Asplund space X and any boundary B of B_{X^*} we must have $B_{X^*} = \overline{\mathrm{co}(B)}^{\|\cdot\|}$. This is not true in general as the following example taken from [19] shows. Let ω_1 be the first uncountable ordinal and let $X = C([0,\omega_1])$ be the space of continuous functions on $[0,\omega_1]$ equipped with the supremum norm. Then X is an Asplund space and if δ_α denotes the Dirac measure at α then the set $B := \{\pm\delta_\alpha : 0 \le \alpha < \omega_1\}$ is a boundary for B_{X^*} for which $\delta_{\omega_1} \in B_{X^*} \setminus \overline{\mathrm{co}(B)}^{\|\cdot\|}$. The best positive results in this setting that we include below are due to Haydon and Godefroy.

Theorem 5.2 (Haydon, [27]). *Let X be a Banach space. The following statements are equivalent*:

(i) $\ell^1 \not\subset X$;
(ii) *for every w^*-compact convex subset C of X^*,*
$$C = \overline{\mathrm{co}(\mathrm{Ext}\, C)}^{\|\cdot\|};$$
(iii) *for every w^*-compact subset K of X^*,*
$$\overline{\mathrm{co}(K)}^{w^*} = \overline{\mathrm{co}(K)}^{\|\cdot\|}.$$

Theorem 5.3 (Godefroy, [19]). *Let X be a separable Banach space. The following statements are equivalent*:

(i) $\ell^1 \not\subset X$;
(ii) *for every w^*-compact subset K of X^* and every James boundary B of K we have $\overline{\mathrm{co}(K)}^{w^*} = \overline{\mathrm{co}(B)}^{\|\cdot\|}$;*
(iii) *for every w^*-compact convex subset C of X^* and every James boundary B of C we have $C = \overline{\mathrm{co}(B)}^{\|\cdot\|}$.*

The example given on $C([0,\omega_1])$ above shows that neither Theorem 5.2 holds for boundaries different from the *extreme points* ((i)⇒(ii) fails) nor Theorem 5.3 holds for all boundaries when X is not separable ((i)⇒(ii) fails). Nonetheless, it is possible to have the best of the above two theorems for general Banach spaces and arbitrary James boundaries if we replace $\|\cdot\|$ in X^* by the topology γ of uniform convergence on bounded and countable subsets of X.

Theorem 5.4. *Let X be a Banach space. The following statements are equivalent*:

(i) $\ell^1 \not\subset X$;

(ii) *for every w^*-compact subset K of X^* and any James boundary B of K we have* $\overline{\mathrm{co}(K)}^{w^*} = \overline{\mathrm{co}(B)}^{\gamma}$;

(iii) *for every w^*-compact subset K of X^*,* $\overline{\mathrm{co}(K)}^{w^*} = \overline{\mathrm{co}(K)}^{\gamma}$.

Proof. (i)⇒(ii). We have to prove that for each $\varepsilon > 0$, $x^* \in \overline{\mathrm{co}(K)}^{w^*}$, and $D \subset B_X$ bounded and countable, there exists $b^* \in \mathrm{co}(B)$ such that

$$|x^*(d) - b^*(d)| < \varepsilon \quad \text{for every } d \in D. \tag{5.5}$$

Define $Y := \overline{\operatorname{span} D}$. Let $r : X^* \to Y^*$ be the restriction map. Then $r(B)$ is a boundary for the w^*-compact set $r(K) \subset Y^*$. Since r is linear and w^*-w^*-continuous we find that

$$x^*|_Y = r(x^*) \in r(\overline{\mathrm{co}(K)}^{w^*}) \subset \overline{\mathrm{co}(r(K))}^{w^*}.$$

Since Y is separable and does not contain ℓ^1, Theorem 5.3 shows that $\overline{\mathrm{co}(r(K))}^{w^*} = \overline{\mathrm{co}(r(B))}^{\|\cdot\|_Y}$. Therefore

$$x^*|_Y = r(x^*) \in \overline{\mathrm{co}(r(B))}^{\|\cdot\|_Y},$$

which clearly implies (5.5).

The implication (ii)⇒(iii) is obvious. Our proof by contradiction for (iii)⇒(i) is almost the same, with a little extra remark, as the proof for (iii)⇒(i) in Theorem 5.2 as presented in [27, Theorem 3.3]. Assume that there exists an isomorphism $T : \ell^1 \to X$ onto its image. Then the adjoint map $S := T^* : X^* \to (\ell^1)^*$ is onto. If we let $(e_n)_n$ denote the canonical basis in ℓ^1 and we identify $(\ell^1)^* = \ell^\infty$ then S is nothing else than the map

$$S : X^* \to \ell^\infty, \qquad x^* \mapsto (x^*(Te_n))_n.$$

Notice that S is w^*-w^*-continuous and also γ-$\|\cdot\|_\infty$-continuous. Choose a w^*-compact subset $C \subset \ell^\infty$ such that

$$\overline{\mathrm{co}(C)}^{\|\cdot\|_\infty} \subsetneq \overline{\mathrm{co}(C)}^{w^*} \tag{5.6}$$

(see the proof of [27, Proposition 3.2]). We now take a w^*-compact subset K of X^* such that $S(K) = C$. Then $\overline{\mathrm{co}(K)}^{\gamma} \subsetneq \overline{\mathrm{co}(K)}^{w^*}$, because otherwise the equality would imply

$$\begin{aligned}\overline{\mathrm{co}(C)}^{w^*} &= \overline{\mathrm{co}(S(K))}^{w^*} = \overline{S(\mathrm{co}(K))}^{w^*} = S(\overline{\mathrm{co}(K)}^{w^*}) \\ &= S(\overline{\mathrm{co}(K)}^{\gamma}) \subset \overline{S(\mathrm{co}(K))}^{\|\cdot\|_\infty} = \overline{\mathrm{co}(C)}^{\|\cdot\|_\infty},\end{aligned}$$

which contradicts (5.6) and finishes the proof. ■

We note that the implication (i)⇒(ii) in the last result is indeed the James compactness theorem for the w^*-topology.

Corollary 5.5. *Let X be a Banach space such that $\ell^1 \not\subset X$. If $K \subset X^*$ is bounded, γ-closed, convex and for every $x \in X$ there exists some $k^* \in K$ such that*

$$k^*(x) = \sup\{y^*(x) : y^* \in K\}$$

then K is a w^*-compact subset of X^*.

Proof. Since K is a James boundary for $\overline{K}^{w^*}$, Theorem 5.4 applies to yield $\overline{\mathrm{co}(\overline{K}^{w^*})}^{w^*} = \overline{\mathrm{co}(K)}^{\gamma} = K$, which implies that K is w^*-compact. ∎

A Banach space X or more generally a convex subset M of X is said to have *property* $\mathcal{C}$ (after Corson) if each collection of relatively closed convex subsets of M with empty intersection has a countable subcollection with empty intersection. Since closed convex sets in X are also weakly closed, if (M, w) is Lindelöf then M has property $\mathcal{C}$. A good reference for property $\mathcal{C}$ is [43].

The following lemma follows from [4, Lemma 9] and also from the main result in [43].

Lemma 4. *Let X be a Banach space. If X^* has property $\mathcal{C}$, then γ is stronger than the weak topology of X^*.*

When X^* has property $\mathcal{C}$ the above results lead to the following consequence.

Corollary 5.6. *Let X be a Banach space such that X^* has property $\mathcal{C}$. Then for every w^*-compact subset K of X^* and any James boundary B of K we have*

$$\overline{\mathrm{co}(K)}^{w^*} = \overline{\mathrm{co}(B)}^{\|\cdot\|}.$$

In particular, every boundary for B_{X^*} has property (S).

Proof. On the one hand, if X^* has property $\mathcal{C}$ then $\ell^1 \not\subset X$. On the other hand, if X^* has property $\mathcal{C}$, Lemma 4 implies that the dual of (X^*, γ) is X^{**}, and therefore for any convex set $C \subset X^*$ we have $\overline{C}^{\gamma} = \overline{C}^{\|\cdot\|}$. The corollary now follows from Theorem 5.4. ∎

Theorem 5.7. *Let X be a Banach space. Then for every w^*-compact weakly Lindelöf subset K of X^* and any James boundary B of K we have*

$$\overline{\mathrm{co}(K)}^{w^*} = \overline{\mathrm{co}(B)}^{\|\cdot\|}.$$

Proof. Theorem 4.5 in [5] ensures that

$$\overline{\mathrm{co}(K)}^{w^*} = \overline{\mathrm{co}(K)}^{\|\cdot\|_{X^*}}.$$

Now, we apply [5, Corollary 6.4] to find that $\overline{\mathrm{span}\, K}^{\|\cdot\|_{X^*}}$ contains $\overline{\mathrm{co}(K)}^{w^*}$ and it is a weakly Lindelöf determined Banach space, i.e., its dual unit ball is Corson compact and in particular angelic when endowed with the w^*

topology. Therefore, every element in the bidual unit ball $B_{X^{**}}$ is the limit in the topology of pointwise convergence on $\overline{\mathrm{co}(K)}^{\sigma(X^*,X)}$ of a sequence in B_X. The conclusion now follows from Theorem I.2 in [19]. ∎

Some open problems

P.1. *The boundary problem.* The main question in this area is the so-called boundary problem that despite significant efforts is still open (see [19, question V.2] and [9, Problem I.2]): *Given a Banach space X and a boundary $B \subset B_{X^*}$ let $\sigma(X, B)$ be the topology on X defined by the pointwise convergence on B. If H is any norm bounded and $\sigma(X, B)$ compact subset of X, is it always true that H is weakly compact?* For boundaries with property (S) it is easily seen that the answer to the boundary problem is positive. It is also known that the answer is positive in the following cases:

(i) if H is convex [46];
(ii) if $B = \mathrm{Ext}(B_{X^*})$ [1];
(iii) if X does not contain an isomorphic copy of $\ell^1(\Gamma)$ with $|\Gamma| = \mathfrak{c}$ [3, 7];
(iv) if K compact and $X = C(K)$ with its natural norm $\|\cdot\|_\infty$ [2].

We observe that the solution to the boundary problem in full generality without using James' theorem about weak compactness would imply an alternative proof of the following version of James' theorem itself: a Banach space X is reflexive if, and only if, each element $x^* \in X^*$ attains its maximum in B_X.

P.2. *Topological properties versus property* (S). Let X be a Banach space, let K be a w^*-compact convex subset of X^* and B any James boundary of K. *If B is weakly Lindelöf, does it have property* (S)? We know that the answer is yes when B is w^*-compact (see [5, Theorem 4.5]). More generally: *Is there a topological property for B that characterizes property* (S)*?*

P.3. *Selectors of prescribed Borel class.* Let X be a separable Asplund space. What can be said of the complexity (in the sense of Kechris–Louveau [33]) of the first Baire class selectors (provided by Theorem 4.2) of duality mappings of equivalent norms on X?

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