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Subspaces of $\ell_2(X)$ and $\mathrm{Rad}(X)$ without local unconditional structure

by

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Abstract. It is shown that if a Banach space X is not isomorphic to a Hilbert space then the spaces $\ell_2(X)$ and $\mathrm{Rad}(X)$ contain a subspace Z without local unconditional structure, and therefore without an unconditional basis. Moreover, if X is of cotype $r < \infty$, then a subspace Z of $\ell_2(X)$ can be constructed without local unconditional structure but with 2-dimensional unconditional decomposition, hence also with basis.

1. Introduction. In this paper we continue the study of constructions of subspaces without an unconditional basis, or even without a local unconditional structure, in general Banach spaces, that we began in [K-T.1]. Our main result provides a characterization of a Hilbert space in terms of unconditionality: a Banach space X is isomorphic to a Hilbert space if and only if every subspace of $\ell_2(X)$ and of $\mathrm{Rad}(X)$ has an unconditional basis.

This result is based on an abstract approach that works in a direct sum of several Banach spaces with unconditional bases, such that the bases are badly comparable with one another. In the present case we work with tensor product spaces, which can then be found inside $\ell_2(X)$ and $\mathrm{Rad}(X)$, for an arbitrary Banach space X. Subspaces so constructed do not have an unconditional basis, but retain many regularities in their linear topological structure.

For concrete Banach spaces, this type of construction was first studied in [J-L-S] for Kalton–Peck's space, then used in [Ke] and [B] for subspaces of L_p, and in [K] for subspaces of an ℓ_2-sum of appropriately chosen Tsirelson-type spaces. Then a general method was developed in [K-T.1] (see also [K-T.2], [T.2]) to prove, among other results, that every Banach space either contains ℓ_2 or a subspace without an unconditional basis. In particular, this

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latter theorem was used by Gowers in the solution to the homogeneous space problem [G] (see also [T.2]): An infinite-dimensional Banach space isomorphic to all of its infinite-dimensional closed subspaces is necessarily isomorphic to ℓ_2.

It was pointed out by Casazza and Kalton [C-K] that the general method of [K-T.1] cannot be used in an arbitrary Banach space X without additional assumptions (see Remark 4.6 for the details). In the present paper we remove all the assumptions on X, and we consider instead $\ell_2(X)$ or $\mathrm{Rad}(X)$, which have enough extra structure for the technique to work, thus providing the aforementioned characterization of a Hilbert space. Let us also recall that it is still an open question whether ℓ_2 is the only Banach space (up to an isomorphism) all of whose subspaces have an unconditional basis.

Let us now describe the content of the paper in more detail. Section 2 contains some preliminaries and several definitions and facts related to unconditionality. We recall the main criterion for recognizing that a space with a special structure does not have local unconditional structure. In Section 3 we discuss a finite-dimensional quantitative version of our construction in a general tensor product setting. Due to the presence of the additional structure, the proof here is much clearer than in [K-T.1].

The final Section 4 contains the main results for $\ell_2(X)$ and $\mathrm{Rad}(X)$, for an arbitrary Banach space X. The results are stated in quantative forms, a finite-dimensional version in Theorem 4.1 and general in Theorem 4.4. This leads to a characterization of spaces isomorphic to ℓ_2. We also consider finite-dimensional quantitative constructions inside ℓ_p^N.

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2. Notation and preliminaries. We use the standard terminology from the Banach space theory (cf. e.g. [L-T.1], [L-T.2] and [T.1]) and we refer the reader to these books for all notation not explained here. In particular, the fundamental concepts related to bases and Schauder decompositions can be found in [L-T.1], 1.a.1 and 1.g.1, respectively.

Let us recall some notation related to unconditionality. For $\lambda \geq 1$, a sequence $\{x_i\}_i$ in a Banach space X is called λ-*unconditional* if for every $x = \sum_i t_i x_i \in X$ one has $\|\sum_i \varepsilon_i t_i x_i\| \leq \lambda \|x\|$ for all $\varepsilon_i = \pm 1$. A sequence is called *unconditional* if it is λ-unconditional for some $\lambda \geq 1$. The infimum of the constants λ is denoted by $\mathrm{unc}\{x_i\}_i$.

A Schauder decomposition $\{Z_i\}_i$ of a Banach space X is called λ-*unconditional*, for some $\lambda \geq 1$, if for all finite sequences $\{z_i\}_i$ with $z_i \in Z_i$ for all i, one has $\|\sum_i \varepsilon_i z_i\| \leq \lambda \|\sum_i z_i\|$ for all $\varepsilon_i = \pm 1$.

Definition 2.1. A Banach space X has *local unconditional structure* if there is $C \geq 1$ such that for every finite-dimensional subspace $X_0 \subset X$ there exist a Banach space F with a 1-unconditional basis and operators $u_0 : X_0 \to F$ and $w_0 : F \to X$ such that the natural embedding $j : X_0 \to X$ admits a factorization $j = w_0 u_0$ and $\|u_0\| \cdot \|w_0\| \leq C$. The infimum of the constants C is denoted by $\operatorname{lust}(X)$.

Clearly, if a space X has an unconditional basis then it has local unconditional structure. Having local unconditional structure passes to complemented subspaces. In particular, if X admits a sequence of finite-dimensional subspaces Y_n such that $\sup_n \operatorname{lust}(Y_n) = \infty$ and there exist projections P_n from X onto Y_n satisfying $\sup_n \|P_n\| < \infty$, then X does not have local unconditional structure.

The following proposition from [K-T.1] and [K-T.2] is our fundamental criterion for recognizing that a space with a special structure does not have local unconditional structure.

Proposition 2.2. *Let Z be a Banach space of cotype $r < \infty$, with cotype constant $C_r(Z)$. Suppose that Z has local unconditional structure and that Z has a λ-unconditional decomposition $\{Z_k\}_k$ with $\dim Z_k = 2$ for all k. Then there exists an operator $T : Z \to Z$ such that*

(i) $T(Z_k) \subset Z_k$ *for each* k,
(ii) $\|T\| \leq \lambda^2 M \operatorname{lust}(Z)$, *where* $M = M(r, C_r(Z)) \geq 1$ *depends on* r *and* $C_r(Z)$,
(iii) $\|(T - \theta \mathrm{Id})|_{Z_k}\| \geq 1/8$ *for each* $\theta \in \mathbb{R}$ *and* $k = 1, 2, \ldots$

Results and arguments of this paper gain on clarity in the tensor product presentation. More than anything else, this plays a role of a notational device, and only very basic general properties of tensor products and their simplest examples will be used.

If X_i are Banach spaces for $i = 1, \ldots, m$, a norm $\|\cdot\|$ on $X_1 \otimes \ldots \otimes X_m$ is called a *cross-norm* if

$$\|x_1 \otimes \ldots \otimes x_m\| = \|x_1\|_{X_1} \ldots \|x_n\|_{X_m}$$

for all $x_i \in X_i$, $i = 1, \ldots, m$.

Let F, G and H be finite-dimensional Banach spaces with (fixed) algebraic bases $\{f_i\}$, $\{g_j\}$ and $\{h_k\}$, respectively. Then $\{f_i \otimes g_j \otimes h_k\}$ is an algebraic basis in $F \otimes G \otimes H$, which we shall call the *natural tensor basis*. For many classical cross-norms, like projective and injective tensor products, and other cross-norms induced by (most) spaces of operators, the natural tensor basis is not well unconditional (cf. e.g. [P.2] and [T.1] and references therein). There are however particular cross-norms on tensor products for which the natural tensor basis is indeed unconditional if the basis in F is. An obvious example is the space $\ell_2(F)$ which, together with $\mathrm{Rad}(X)$, is a starting point for our applications of the general theorem from Section 3. Let us note that if the natural tensor basis is unconditional then, by a general property of bases, for an arbitrary order of $\{f_i \otimes g_j \otimes h_k\}$, its basis constant is well bounded and satisfies $\mathrm{bc}\{f_i \otimes g_j \otimes h_k\} \le \mathrm{unc}\{f_i \otimes g_j \otimes h_k\}$.

In this paper we shall consider tensor products of two or three factors, with all but one factor being the space ℓ_p^n for some $1 \le p < \infty$. In the space ℓ_p^n the standard unit vector basis is denoted by $\{e_i\}$.

Let X be a Banach space. We say that a system $\{x_{i,j,k}\}_{i,j,k}$ of vectors in X is λ-*tensor-unconditional*, for some $\lambda \ge 1$, if for every vector $x = \sum_{i,j,k} t_{i,j,k} x_{i,j,k} \in X$ one has the estimate $\|\sum_{i,j,k} \varepsilon_i \varepsilon'_j \varepsilon''_k t_{i,j,k} x_{i,j,k}\| \le \lambda \|x\|$ for all $\varepsilon_i, \varepsilon'_j, \varepsilon''_k = \pm 1$. Pisier proved in [P.1] that in the presence of the local unconditional structure there is a close relation between tensor-unconditionality and unconditionality. We shall state his result in a form convenient for our use.

PROPOSITION 2.3. *Let X be a Banach space of cotype $r < \infty$ which has local unconditional structure. A system $\{x_{i,j,k}\}_{i,j,k}$ in X which is λ-tensor-unconditional for some $\lambda \ge 1$ is automatically unconditional and*

$$\mathrm{unc}\{x_{i,j,k}\} \le a\lambda^2 \,\mathrm{lust}(X),$$

where $a = a(r, C_r(X)) \ge 1$ depends on r and the cotype constant $C_r(X)$.

A version of this proposition for two-fold tensor products follows from Propositions 2.1 and 2.2 of [P.1], and for three-fold tensor product the proof requires an obvious modification (and gives the same constant). For the sake of the reader not specializing in the local theory of Banach spaces, it might be worth mentioning that the results in [P.1] are stated for spaces not containing ℓ_∞^n's uniformly, so one should remember a fundamental result (which became a special case of the Maurey–Pisier theorem) that this class of spaces coincides with the class of Banach spaces X such that X is of cotype r for some $r < \infty$ (cf. e.g. , [T.1], [M.2]). In particular, our estimates in Proposition 2.3 follow from the arguments in [P.1] rather than from the statements themselves.

3. Construction of subspaces without local unconditional structure in tensor products. We will now present an abstract setting in which it is possible to construct subspaces of tensor product spaces without local unconditional structure.

THEOREM 3.1. *Let $\lambda \geq 1$ and $D \geq \sqrt{12}$. Let F be an n-dimensional Banach space with a normalized λ-unconditional basis $\{f_i\}_{i=1}^n$.*

(i) *Suppose that $\|\sum_{i=1}^n f_i\| \geq n^{1/2}D$. Consider three tensor product spaces $X_1 = F \otimes \ell_2^n \otimes \ell_2^n$, $X_2 = \ell_2^n \otimes F \otimes \ell_2^n$ and $X_3 = \ell_2^n \otimes \ell_2^n \otimes F$, each endowed with a cross-norm. Suppose further that there exists $C_\lambda \geq 1$ such that the natural tensor basis in each X_i is C_λ-unconditional. Set $X = X_1 \oplus X_2 \oplus X_3$. Let $2 \leq r < \infty$ and let $C_r(X)$ denote the cotype r constant of X. Then there exists a subspace Z of X such that* $\mathrm{lust}(Z) \geq a\lambda^{-1}C_\lambda^{-2}D$.

(ii) *Suppose that $\|\sum_{i=1}^n f_i\| \leq n^{1/2}/D$. Consider two tensor product spaces $X_1 = F \otimes \ell_2^n$ and $X_2 = \ell_2^n \otimes F$, each endowed with a cross-norm. Suppose further that there exists $C_\lambda \geq 1$ such that the natural tensor basis in each X_i is C_λ-unconditional. Set $X = X_1 \oplus X_2 \oplus \ell_2^{n^2}$. Let $2 \leq r < \infty$ and let $C_r(X)$ denote the cotype r constant of X. Then there exists a subspace Z of X such that* $\mathrm{lust}(Z) \geq aC_\lambda^{-2}D^{1/2}$.

Here $a = a(r, C_r(X)) > 0$ depends on r and $C_r(X)$ only. Moreover, the space Z has a basis with basis constant less than or equal to cC_λ, where $c \geq 1$ is a numerical constant, and Z admits a 2-dimensional C_λ-unconditional decomposition.

A choice of a particular norm on the space $X = X_1 \oplus X_2 \oplus X_3$ (where $X_3 = \ell_2^{n^2}$, in case (ii)) may affect only constants a and c by numerical factors. To fix ideas we may take for example the norm $\|(x_1, x_2, x_3)\| = \|x_1\| + \|x_2\| + \|x_3\|$ for $(x_1, x_2, x_3) \in X$.

REMARK 3.2. The same construction works, giving exactly the same estimates, if the space ℓ_2^n is replaced by ℓ_p^n (for $1 \leq p < \infty$) and the main assumptions in cases (i) and (ii) are replaced by the inequalities $\|\sum_{i=1}^n f_i\| \geq n^{1/p}D$ and $\|\sum_{i=1}^n f_i\| \leq n^{1/p}/D$, respectively.

The proof of Theorem 3.1 requires two lemmas.

LEMMA 3.3. *Let $\delta \in (0, 1)$.*

(i) *Let F be an n-dimensional space with a normalized λ-unconditional basis $\{f_i\}_{i=1}^n$ such that $\|\sum_{i=1}^n f_i\| \geq n^{1/2}D$ for some $\lambda \geq 1$ and $D \geq 1$. Then there exist $D^2 \leq n_0 \leq n$ and a subset $I \subset \{1, \ldots, n\}$ with $|I| = n_0$ such that for an arbitrary sequence $\{c_i\}_{i\in I}$ of real numbers there exists a subset $S \subset I$ with $|S| \geq [\delta n_0]$ such that*

$$\Big\|\sum_{i\in I} c_i f_i\Big\| \geq (1 - \sqrt{\delta}) \max_{i\in S} |c_i| \sqrt{n_0}\, \lambda^{-1} D. \tag{1}$$

Moreover, $\|\sum_{i\in I} f_i\| \geq \sqrt{n_0}\, D$.

(ii) *For every sequence of real numbers* $c_1, \dots, c_n$ *there exists a subset* $S \subset \{1, \dots, n\}$ *with* $|S| \ge [\delta n]$ *such that*

$$\Big(\sum_{i=1}^n c_i^2\Big)^{1/2} \ge \sqrt{1-\delta}\max_{i\in S}|c_i| n^{1/2}.$$

Proof. (i) We first show that there exist $n_0 \ge D^2$ and a subset $I \subset \{1, \dots, n\}$ with $|I| = n_0$ such that for every $J \subset I$ with $|J| \ge (1-\delta)n_0$,

$$\Big\|\sum_{i\in J} f_i\Big\| \ge (1-\sqrt{\delta})\sqrt{n_0}\, D. \tag{2}$$

Indeed, for $1 \le m \le n$ let

$$\varphi(m) = \sup \Big\|\sum_{i\in L} f_i\Big\|/m^{1/2}$$

where the supremum runs over all subsets $L \subset \{1, \dots, n\}$ with $|L| = m$. Pick $1 \le n_0 \le n$ such that $\varphi(n_0) = \max_{1\le m\le n}\varphi(m)$. Clearly, $\varphi(n_0) \ge D$. Pick $I \subset \{1, \dots, n\}$ with $|I| = n_0$ such that

$$\Big\|\sum_{i\in I} f_i\Big\|/n_0^{1/2} = \varphi(n_0).$$

Observe that $n_0 \ge \|\sum_{i\in I} f_i\| \ge n_0^{1/2} D$, hence $n_0 \ge D^2$; and the "moreover" part of the statement is trivially satisfied.

Let $J \subset I$ with $|J| \ge (1-\delta)n_0$. Then

$$\begin{aligned}\Big\|\sum_{i\in J} f_i\Big\| &\ge \Big\|\sum_{i\in I} f_i\Big\| - \Big\|\sum_{i\in I\setminus J} f_i\Big\| \\ &\ge (1-\delta^{1/2})n_0^{1/2}\varphi(n_0) \ge (1-\delta^{1/2})n_0^{1/2} D,\end{aligned}$$

as required.

Now to show (1), let $\{c_i\}_{i\in I}$ and a subset $S \subset I$ be such that $|S| = [\delta n_0]$ and $\max_{i\in S}|c_i| \le \min_{j\in I\setminus S}|c_j|$. Then by the λ-unconditionality and by (2) we get

$$\begin{aligned}\Big\|\sum_{i\in I} c_i f_i\Big\| &\ge \lambda^{-1}\sup_{\varepsilon_i=\pm1}\Big\|\sum_{i\in I}\varepsilon_i c_i f_i\Big\| \ge \lambda^{-1}\sup_{\varepsilon_i=\pm1}\Big\|\sum_{i\in I\setminus S}\varepsilon_i c_i f_i\Big\| \\ &\ge \lambda^{-1}\min_{i\in I\setminus S}|c_i| \quad \Big\|\sum_{i\in I\setminus S} f_i\Big\| \ge \max_{i\in S}|c_i|(1-\delta^{1/2})n_0^{1/2} D\lambda^{-1}.\end{aligned}$$

(ii) Assuming, without loss of generality, that $|c_1| \ge \dots \ge |c_n|$, it is easy to check that $S = \{[(1-\delta)n], \dots, n\}$ is the required set. ∎

LEMMA 3.4. *Let* $\delta \in [3/4, 1)$, $n \ge 12$ *and let* $I = \{1, \dots, n\}$.

(i) *Let* $A_{j,k}$, $B_{i,k}$ *and* $C_{i,j}$ *be subsets of* I *be such that* $|A_{j,k}| \ge [\delta n]$, $|B_{i,k}| \ge [\delta n]$ *and* $|C_{i,j}| \ge [\delta n]$ *for all* $i, j, k \in I$. *Then there exist* $i_0, j_0, k_0 \in I$

such that

$$i_0 \in A_{j_0,k_0}, \quad j_0 \in B_{i_0,k_0}, \quad k_0 \in C_{i_0,j_0}.$$

(ii) *Let A_j and B_i be subsets of I such that $|A_j| \geq [\delta n]$ and $|B_i| \geq [\delta n]$ for all $i, j \in I$, and let $C \subset I \times I$ be such that $|C| \geq [\delta n^2]$. Then there exist $i_0, j_0 \in I$ such that*

$$i_0 \in A_{j_0}, \quad j_0 \in B_{i_0}, \quad (i_0, j_0) \in C.$$

Proof. The proof of case (ii) is very similar to case (i), and therefore we will show only (i).

Consider the following three subsets of $I \times I \times I$:

$$A = \bigcup_{j,k \in I} A_{j,k} \times \{j\} \times \{k\}, \ B = \bigcup_{i,k \in I} \{i\} \times B_{i,k} \times \{k\}, \ C = \bigcup_{i,j \in I} \{i\} \times \{j\} \times C_{i,j}.$$

Observe that the cardinality of each of these sets is larger than or equal to $[\delta n] n^2$. Since for $\delta \geq 3/4$ and $n \geq 12$ we have $[\delta n] > \frac{2}{3} n$, it follows that $A \cap B \cap C \neq \emptyset$. For $(i_0, j_0, k_0) \in A \cap B \cap C$ we have

$$i_0 \in A_{j_0,k_0}, \quad j_0 \in B_{i_0,k_0}, \quad k_0 \in C_{i_0,j_0},$$

as required. ■

Now we are ready to pass to the proof of Theorem 3.1.

Proof of Theorem 3.1. Fix $\delta = 3/4$. Let $M = M(r, C_r(X))$ be the function appearing in Proposition 2.2. To simplify the notation we shall use the same letter $a = a(r, C_r(X))$ for all functions which depend on r and $C_r(X)$ only.

(i) Let $I \subset \{1, \ldots, n\}$ be the subset constructed in Lemma 3.3(i). Then $F' = \operatorname{span}\{f_i\}_{i \in I}$ is an n_0-dimensional subspace of F, and $X_1' = F' \otimes \ell_2^{n_0} \otimes \ell_2^{n_0}$ is a subspace of X_1. Let X_2' and X_3' be defined analogously. Let $X' = X_1' \oplus X_2' \oplus X_3' \subset X$. It is clearly sufficient to construct the required space Z as a subspace of X'. Note that $\| \sum_{i \in I} f_i \| \geq \sqrt{n_0}\, D$ and all primed spaces satisfy our further assumptions as well. Thus without loss of generality we may assume that F' is the original space F (and in particular $n = n_0 \geq D^2$). Moreover, for any sequence $c_1, \ldots, c_n$ of real numbers there exists a subset $S \subset \{1, \ldots, n\}$ with $|S| \geq [\delta n]$ such that

$$\Big\| \sum_{i=1}^{n} c_i f_i \Big\| \geq \max_{i \in S} |c_i| (1 - \delta^{1/2}) n^{1/2} D \lambda^{-1}. \tag{3}$$

Let $\{e_i\}_{i=1}^n$ denote the standard unit vector basis in ℓ_2^n. By our assumptions, the natural tensor bases $\{f_i \otimes e_j \otimes e_i\}_{i,j,k}$, $\{e_i \otimes f_j \otimes e_i\}_{i,j,k}$ and $\{e_i \otimes e_j \otimes f_i\}_{i,j,k}$ are C_λ-unconditional in X_1, X_2 and X_3, respectively.

For $i, j, k = 1, \ldots, n$, consider vectors in X defined by

$$\begin{aligned} x_{i,j,k} &= \quad e_i \otimes f_j \otimes e_k + e_i \otimes e_j \otimes f_k, \\ y_{i,j,k} &= f_i \otimes e_j \otimes e_k \quad + e_i \otimes e_j \otimes f_k. \end{aligned}$$

Observe that for all $i, j, k = 1, \ldots, n$ one has $1 \leq \|x_{i,j,k}\|, \|y_{i,j,k}\| \leq 2$; furthermore, for arbitrary scalars s and t we have

$$\max(|s|, |t|) \leq \|sx_{i,j,k} + ty_{i,j,k}\| \leq 2(|s| + |t|). \tag{4}$$

Let $Z = \operatorname{span}\{Z_{i,j,k}\}_{i,j,k}$, where $Z_{i,j,k} = \operatorname{span}\{x_{i,j,k}, y_{i,j,k}\}$ for $i, j, k = 1, \ldots, n$. Then $\{Z_{i,j,k}\}_{i,j,k}^n$ forms a C_λ-unconditional decomposition of $Z \subset X$. Since the basis constants of $\{f_i \otimes e_j \otimes e_i\}_{i,j,k}$, $\{e_i \otimes f_j \otimes e_i\}_{i,j,k}$ and $\{e_i \otimes e_j \otimes f_i\}_{i,j,k}$ are less than or equal to the unconditional basis constants, each of them is bounded above by C_λ, and hence it is easy to see by (4) that $\{x_{i,j,k}, y_{i,j,k}\}_{i,j,k}$ forms a basis in Z with basis constant less than or equal to $2C_\lambda$.

Let T be the operator obtained in Proposition 2.2. Write

$$T(x_{i,j,k}) = a_{i,j,k}x_{i,j,k} + c_{i,j,k}y_{i,j,k}, \qquad T(y_{i,j,k}) = b_{i,j,k}x_{i,j,k} + d_{i,j,k}y_{i,j,k},$$

for all $i, j, k = 1, \ldots, n$.

First observe that for all $i, j, k = 1, \ldots, n$ one has

$$\max(|a_{i,j,k} - d_{i,j,k}|, |b_{i,j,k}|, |c_{i,j,k}|) \geq 2^{-6}. \tag{5}$$

Indeed, for arbitrary $i, j, k = 1, \ldots, n$, estimates (4) imply that whenever $z = sx_{i,j,k} + ty_{i,j,k} \in Z_{i,j,k}$, then

$$\begin{aligned} \|Tz - d_{i,j,k}z\| &\leq 2(|(a_{i,j,k} - d_{i,j,k})s + b_{i,j,k}t| + |c_{i,j,k}s|) \\ &\leq 2\max(|s|, |t|)(|a_{i,j,k} - d_{i,j,k}| + |b_{i,j,k}| + |c_{i,j,k}|) \\ &\leq 6\max(|a_{i,j,k} - d_{i,j,k}|, |b_{i,j,k}|, |c_{i,j,k}|)\|z\|. \end{aligned}$$

Thus

$$\|(T - d_{i,j,k}\,\mathrm{id})|_{Z_{i,j,k}}\| \leq 6\max(|a_{i,j,k} - d_{i,j,k}|, |b_{i,j,k}|, |c_{i,j,k}|),$$

and (5) follows from Proposition 2.2(iii).

We have

$$\begin{aligned} \Big\| \sum_{i=1}^n x_{i,j,k} \Big\| &\leq 2n^{1/2} \quad \text{for } j, k = 1, \ldots, n, \\ \Big\| \sum_{j=1}^n y_{i,j,k} \Big\| &\leq 2n^{1/2} \quad \text{for } i, k = 1, \ldots, n, \\ \Big\| \sum_{k=1}^n (x_{i,j,k} - y_{i,j,k}) \Big\| &\leq 2n^{1/2} \quad \text{for } i, j = 1, \ldots, n. \end{aligned} \tag{6}$$

We show only the first inequality in (6):

$$\Big\|\sum_{i=1}^n x_{i,j,k}\Big\| \le \Big\|\sum_{i=1}^n e_i\otimes f_j\otimes e_k\Big\| + \Big\|\sum_{i=1}^n e_i\otimes e_j\otimes f_k\Big\|$$
$$= \Big\|\sum_{i=1}^n e_i\Big\|\,\|f_j\otimes e_k\| + \Big\|\sum_{i=1}^n e_i\Big\|\,\|e_j\otimes f_k\| = 2n^{1/2}.$$

For notational convenience, for $i,j,k=1,\dots,n$ define $\gamma_{i,j,k}=a_{i,j,k}-b_{i,j,k}+c_{i,j,k}-d_{i,j,k}$. Using (6), we get

$$\tag{7}\begin{aligned}\Big\|\sum_{i=1}^n c_{i,j,k}f_i\Big\| &\le 2n^{1/2}\|T\| \quad \text{for } j,k=1,\dots,n,\\ \Big\|\sum_{j=1}^n b_{i,j,k}f_j\Big\| &\le 2n^{1/2}\|T\| \quad \text{for } i,k=1,\dots,n,\\ \Big\|\sum_{k=1}^n \gamma_{i,j,k}f_k\Big\| &\le 2n^{1/2}\|T\| \quad \text{for } i,j=1,\dots,n.\end{aligned}$$

Again, we show only the first inequality:

$$2n^{1/2}\|T\| \ge \Big\|T\Big(\sum_{i=1}^n x_{i,j,k}\Big)\Big\| = \Big\|\sum_{i=1}^n (a_{i,j,k}x_{i,j,k}+c_{i,j,k}y_{i,j,k})\Big\|$$
$$\ge \Big\|\sum_{i=1}^n c_{i,j,k}f_i\otimes e_j\otimes e_k\Big\| = \Big\|\sum_{i=1}^n c_{i,j,k}f_i\Big\|\,\|e_j\otimes e_k\|$$
$$= \Big\|\sum_{i=1}^n c_{i,j,k}f_i\Big\|.$$

Hence by (3) one can choose subsets $A_{j,k}$ (for $j,k=1,\dots,n$), $B_{i,k}$ (for $i,k=1,\dots,n$) and $C_{i,j}$ (for $i,j=1,\dots,n$) of $\{1,\dots,n\}$ such that the cardinality of each set is at least $[\delta n]$ and

$$\tag{8}\begin{aligned}\max_{i\in A_{j,k}}|c_{i,j,k}|Dn^{1/2}(1-\delta^{1/2})\lambda^{-1} &\le 2n^{1/2}\|T\| \quad \text{for } j,k=1,\dots,n,\\ \max_{j\in B_{i,k}}|b_{i,j,k}|Dn^{1/2}(1-\delta^{1/2})\lambda^{-1} &\le 2n^{1/2}\|T\| \quad \text{for } i,k=1,\dots,n,\\ \max_{k\in C_{i,j}}|\gamma_{i,j,k}|Dn^{1/2}(1-\delta^{1/2})\lambda^{-1} &\le 2n^{1/2}\|T\| \quad \text{for } i,j=1,\dots,n.\end{aligned}$$

Using Lemma 3.4(i) (note that $n\ge D^2\ge 12$) we get i_0, j_0 and k_0 such that $i_0\in A_{j_0,k_0}$, $j_0\in B_{i_0,k_0}$ and $k_0\in C_{i_0,j_0}$. Thus

$$|c_{i_0,j_0,k_0}| \le 2D^{-1}(1-\delta^{1/2})^{-1}\lambda\|T\|,$$
$$|b_{i_0,j_0,k_0}| \le 2D^{-1}(1-\delta^{1/2})^{-1}\lambda\|T\|,$$

$$|\gamma_{i_0,j_0,k_0}| \le 2D^{-1}(1-\delta^{1/2})^{-1}\lambda\|T\|.$$

In particular, by the definition of γ_{i_0,j_0,k_0} this yields

$$|a_{i_0,j_0,k_0} - d_{i_0,j_0,k_0}| \le 6D^{-1}(1-\delta^{1/2})^{-1}\lambda\|T\|.$$

Recall that, by Proposition 2.2(ii), $\|T\| \le C_\lambda^2 M \operatorname{lust}(Z)$. Therefore, by (5) we finally get

$$2^{-6} \le 6D^{-1}(1-\delta^{1/2})^{-1}\lambda C_\lambda^2 M \operatorname{lust}(Z).$$

Thus $\operatorname{lust}(Z) \ge cD/(C_\lambda^2 M\lambda)$, where $c>0$ is a numerical constant, and this completes the proof of case (i).

(ii) The beginning of the proof is similar to case (i). Fix $\delta = 3/4$. Let $\{e_i\}_{i=1}^n$ be the standard unit vector basis in ℓ_2^n. For $i,j=1,\dots,n$ put

$$\begin{aligned} x_{i,j} &= \quad e_i \otimes f_j + D^{-1/2} e_i \otimes e_j, \\ y_{i,j} &= f_i \otimes e_j \quad + D^{-1/2} e_i \otimes e_j. \end{aligned}$$

Let $Z = \operatorname{span}\{Z_{i,j}\}_{i,j}$ where $Z_{i,j} = \operatorname{span}\{x_{i,j}, y_{i,j}\}$ for $i,j=1,\dots,n$. Then Z has the basis $\{x_{i,j}, y_{i,j}\}_{i,j}$ and the natural C_λ-unconditional decomposition analogous to those in case (i).

Let T be an operator from Proposition 2.2, and write

$$T(x_{i,j}) = a_{i,j}x_{i,j} + c_{i,j}y_{i,j}, \qquad T(y_{i,j}) = b_{i,j}x_{i,j} + d_{i,j}y_{i,j},$$

for all $i,j=1,\dots,n$. We have an estimate analogous to (5),

$$\max(|a_{i,j}-d_{i,j}|, |b_{i,j}|, |c_{i,j}|) \ge 2^{-6}, \tag{9}$$

valid for all $i,j=1,\dots,n$.

We have

$$\begin{aligned} \Big\|\sum_{j=1}^n x_{i,j}\Big\| &\le 2n^{1/2}D^{-1/2} \quad \text{for } i=1,\dots,n, \\ \Big\|\sum_{i=1}^n y_{i,j}\Big\| &\le 2n^{1/2}D^{-1/2} \quad \text{for } j=1,\dots,n, \\ \Big\|\sum_{i,j=1}^n (x_{i,j}-y_{i,j})\Big\| &\le 2nD^{-1}. \end{aligned} \tag{10}$$

We show the first and the last inequality in (10):

$$\begin{aligned} \Big\|\sum_{j=1}^n x_{i,j}\Big\| &\le \Big\|\sum_{j=1}^n e_i \otimes f_j\Big\| + D^{-1/2}\Big\|\sum_{j=1}^n e_i \otimes e_j\Big\| \\ &\le \Big\|\sum_{j=1}^n f_j\Big\| + D^{-1/2}n^{1/2} \\ &\le n^{1/2}D^{-1} + n^{1/2}D^{-1/2} \le 2n^{1/2}D^{-1/2}, \end{aligned}$$

and

$$\Big\| \sum_{i,j=1}^{n} x_{i,j} - y_{i,j} \Big\| \le 2 \Big\| \sum_{i,j=1}^{n} e_i \otimes f_j \Big\|$$
$$= 2 \Big\| \Big(\sum_{i=1}^{n} e_i \Big) \otimes \Big(\sum_{j=1}^{n} f_j \Big) \Big\| = 2n^{1/2} \Big\| \sum_{j=1}^{n} f_j \Big\| \le 2nD^{-1}.$$

Using (10), the boundedness of the operator T and Lemma 3.3(ii) as in (7) and (8), we see that there exist subsets $A_j, B_i \subset \{1, \ldots, n\}$ with $|A_j|, |B_i| \ge [\delta n]$ for $i, j = 1, \ldots, n$, and $C \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$ with $|C| \ge [\delta n^2]$ such that

$$\max_{j \in B_i} |c_{i,j}| (1-\delta)^{1/2} n^{1/2} \le 2n^{1/2} D^{-1/2} \|T\| \quad \text{for } i = 1, \ldots, n,$$
$$\max_{i \in A_j} |b_{i,j}| (1-\delta)^{1/2} n^{1/2} \le 2n^{1/2} D^{-1/2} \|T\| \quad \text{for } j = 1, \ldots, n,$$
$$D^{-1/2} \max_{(i,j) \in C} |a_{i,j} - d_{i,j} + c_{i,j} - b_{i,j}| (1-\delta)^{1/2} n \le 2nD^{-1} \|T\|.$$

Indeed, for example the first inequality follows from

$$2n^{1/2} D^{-1/2} \|T\| \ge \Big\| \sum_{j=1}^{n} T x_{i,j} \Big\| = \Big\| \sum_{j=1}^{n} (a_{i,j} x_{i,j} + c_{i,j} y_{i,j}) \Big\|$$
$$\ge \Big\| \sum_{j=1}^{n} c_{i,j} f_i \otimes e_j \Big\| = \Big(\sum_{j=1}^{n} c_{i,j}^2 \Big)^{1/2}$$
$$\ge \max_{j \in B_i} |c_{i,j}| (1-\delta)^{1/2} n^{1/2}.$$

And for the third inequality we have

$$2nD^{-1} \|T\| \ge \Big\| \sum_{i,j=1}^{n} T x_{i,j} - \sum_{i,j=1}^{n} T y_{i,j} \Big\|$$
$$= \Big\| \sum_{i,j=1}^{n} ((a_{i,j} - b_{i,j}) x_{i,j} + (c_{i,j} - d_{i,j}) y_{i,j}) \Big\|$$
$$\ge D^{-1/2} \Big\| \sum_{i,j=1}^{n} (a_{i,j} - b_{i,j} + c_{i,j} - d_{i,j}) e_i \otimes e_j \Big\|$$
$$\ge D^{-1/2} \max_{(i,j) \in C} |a_{i,j} - b_{i,j} + c_{i,j} - d_{i,j}| (1-\delta)^{1/2} n.$$

Therefore by Lemma 3.4(ii) one can find $i_0 \in A_{j_0}$ and $j_0 \in B_{i_0}$ such that $(i_0, j_0) \in C$. Hence

$$|c_{i_0,j_0}| \le 2D^{-1/2}(1-\delta)^{-1/2}\|T\|,$$
$$|b_{i_0,j_0}| \le 2D^{-1/2}(1-\delta)^{-1/2}\|T\|,$$
$$|a_{i_0,j_0} - d_{i_0,j_0} + c_{i_0,j_0} - b_{i_0,j_0}| \le 2D^{-1/2}(1-\delta)^{-1/2}\|T\|.$$

Again, as in the proof of case (i), using (9) and Proposition 2.2 we get

$$2^{-6} \le 6D^{-1/2}(1-\delta)^{-1/2}C_\lambda^2 M \,\mathrm{lust}(Z).$$

Thus $\mathrm{lust}(Z) \ge c\sqrt{D}/(C_\lambda^2 M)$, where $c > 0$ is a numerical constant, and this completes the proof of case (ii). ∎

Remark 3.5. As an immediate consequence of the proof of Theorem 3.1 note that the same conclusion as in case (ii) holds whenever $X_1 = F_1 \otimes \ell_2^n$ and $X_2 = \ell_2^n \otimes F_2$, and each F_ν is an n-dimensional Banach space with a normalized λ-unconditional basis $\{f_i^{(\nu)}\}_{i=1}^n$ such that $\|\sum_{i=1}^n f_i^{(\nu)}\| \le n^{1/2}/D$, for $\nu = 1, 2$.

In case (i) we want that $X_1 = F_1 \otimes \ell_2^n \otimes \ell_2^n$, $X_2 = \ell_2^n \otimes F_2 \otimes \ell_2^n$ and $X_3 = \ell_2^n \otimes \ell_2^n \otimes F_3$, and each F_ν is an n-dimensional Banach space with a normalized λ-unconditional basis $\{f_i^{(\nu)}\}_{i=1}^n$ such that $\|\sum_{i=1}^n f_i^{(\nu)}\| \ge n^{1/2}D$ and additionally, condition (3) is satisfied, for $\nu = 1, 2, 3$.

4. Subspaces of $\ell_2(X)$ and $\mathrm{Rad}(X)$ without local unconditional structure. In this section we consider spaces $\ell_2(X)$ and $\mathrm{Rad}(X)$ and we prove our main result that if X is not isomorphic to a Hilbert space then each of these spaces contains a subspace without local unconditional structure. As all these spaces have a natural structure of tensor products, the result will follow easily from our abstract scheme.

Recall that if X is a Banach space, $\ell_2(X)$ and $\mathrm{Rad}(X)$ are spaces of all sequences (x_i) with $x_i \in X$ for $i = 1, 2, \ldots$ such that the following expressions representing the respective norms are finite:

$$\|(x_i)\|_{\ell_2(X)} = \Big(\sum_{i=1}^{\infty} \|x_i\|^2\Big)^{1/2} < \infty,$$
$$\|(x_i)\|_{\mathrm{Rad}(X)} = \Big(\int_0^1 \Big\|\sum_{i=1}^{\infty} r_i(t)x_i\Big\|^2\,dt\Big)^{1/2} < \infty.$$

(Here $\{r_i\}_i$ denotes the sequence of Rademacher functions on $[0,1]$, defined by $r_i(t) = \operatorname{sign}\sin(2^i\pi t)$ for $t \in [0,1]$ and $i = 1, 2, \ldots$)

Let $\{e_i\}_i$ denote the standard unit vector basis in ℓ_2. For a natural number n, we denote by $\ell_2^n(X)$ and $\mathrm{Rad}_n(X)$ the spaces of all n-tuples $(x_i)_{i=1}^n$, endowed with the corresponding norms. These spaces can be (algebraically) identified with $\ell_2^n \otimes X$, via the map

$$(x_1, \ldots, x_n) \;\leftrightarrow\; e_1 \otimes x_1 + \ldots + e_n \otimes x_n \in \ell_2^n \otimes X. \tag{11}$$

We shall also identify $\ell_2^{k^2}(X)$ and $\mathrm{Rad}_{k^2}(X)$ to $\ell_2^k \otimes \ell_2^k \otimes X$ via the map

$$(x_{i,j})_{\substack{i=1,\dots,k\\ j=1,\dots,k}} \leftrightarrow \sum_{i,j} e_i \otimes e_j \otimes x_{i,j} \in \ell_2^k \otimes \ell_2^k \otimes X. \tag{12}$$

We shall consider two norms on $\ell_2^n \otimes X$, induced by the spaces $\ell_2^n(X)$ and $\mathrm{Rad}_n(X)$, respectively. By an obvious algebraic identification, $\ell_2^n \otimes X = X \otimes \ell_2^n$, the same spaces also induce norms on $X \otimes \ell_2^n$. Analogously, the spaces $\ell_2^{k^2}(X)$ and $\mathrm{Rad}_{k^2}(X)$ provide, via (12), three induced norms on the tensor products $X \otimes \ell_2^k \otimes \ell_2^k$, $\ell_2^k \otimes X \otimes \ell_2^k$ and $\ell_2^k \otimes \ell_2^k \otimes X$.

We shall require several remarks and well known easy facts about the above Banach spaces.

Clearly, $\mathrm{Rad}_n(X)$ contains X as a subspace and it can be identified with a subspace of $\ell_2^{2^n}(X)$, for $n \geq 1$. It is well known that if X has cotype r then so does $\ell_2(X)$, and it can be checked by direct calculation that $C_r(\ell_2(X)) = C_r(X)$. Here the cotype r constant is defined by means of the L_2-norms of Rademacher averages (cf. e.g. [T.1], [M.2]). Thus $C_r(\mathrm{Rad}_n(X)) = C_r(X)$ as well ($n \geq 1$).

(I) The norms on $\ell_2^n \otimes X$ induced by $\ell_2^n(X)$ and $\mathrm{Rad}_n(X)$ via (11) are cross-norms, as are the norms on $\ell_2^n \otimes \ell_2^n \otimes X$ induced by $\ell_2^{n^2}(X)$ and $\mathrm{Rad}_{n^2}(X)$ via (12).

This can be easily checked directly from the definitions. Note that if $u = \sum_i a_i e_i \in \ell_2^n$, $w = \sum_j b_j e_j \in \ell_2^n$ and $x \in X$ then to the element $u \otimes x \in \ell_2^n \otimes X$, (11) assigns the n-tuple $(a_1 x, \dots, a_n x)$; and to the element $u \otimes w \otimes x \in \ell_2^n \otimes \ell_2^n \otimes X$, (12) assigns the n^2-tuple $(a_i b_j x)_{i,j}$.

(II) Let X have cotype $r < \infty$. If $\{f_1, \dots, f_m\}$ is a 1-unconditional sequence in X then $\{e_i \otimes f_j\}_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$ is 1-unconditional in $\ell_2^n \otimes X$ with the norm induced by $\ell_2^n(X)$ and λ-unconditional in the norm induced by $\mathrm{Rad}_n(X)$. Furthermore, the sequence $\{e_i \otimes e_j \otimes f_k\}$, where $i = 1, \dots, n$, $j = 1, \dots, n$, $k = 1, \dots, m$, is 1-unconditional in $\ell_2^n \otimes \ell_2^n \otimes X$ with the norm induced by $\ell_2^{n^2}(X)$, and λ-unconditional in the norm induced by $\mathrm{Rad}_{n^2}(X)$. Here $\lambda = \lambda(r, C_r(X))$.

For the norms induced by $\ell_2^t(X)$ ($t = n, n^2$) the statement is obvious. It is clear that it is sufficient to prove the remaining statements for the two-fold tensor product only.

This is an easy consequence of the Maurey–Khinchin inequality, which for completeness we state in the form convenient for our present use (for the proof cf. e.g. [L-T.2], 1.d.6 and 1.f.9). Let Y be a Banach space with a 1-unconditional basis $\{y_j\}$ and cotype $r < \infty$. Then for any $w_i \in Y$ for $i = 1, 2, \dots$ we have

$$c\Big\|\Big(\sum_i |w_i|^2\Big)^{1/2}\Big\|_Y \le \Big(\int_0^1 \Big\|\sum_i r_i(t)w_i\Big\|_Y^2\,dt\Big)^{1/2} \le M\Big\|\Big(\sum_i |w_i|^2\Big)^{1/2}\Big\|_Y,$$

where $c > 0$ is a universal constant and $M = M(r, C_r(Y))$. Here, if $w_i = \sum_j w_i(j)y_j$ for all i, then the vector $w = (\sum_i |w_i|^2)^{1/2}$ is defined by the pointwise operation $w = \sum_j(\sum_i |w_i(j)|^2)^{1/2}y_j$.

Returning to the unconditionality in $\mathrm{Rad}_n(X)$, let $x = \sum_{i,j} a_{i,j}r_i f_j \in \mathrm{Rad}_n(X)$. Applying the Maurey–Khinchin inequality to the space $F = \mathrm{span}\{f_j\}$ one sees that $\|x\|_{\mathrm{Rad}(X)}$ is equivalent, up to a factor depending on r and $C_r(X)$, to the expression

$$\Big\|\sum_j \Big(\sum_i |a_{i,j}|^2\Big)^{1/2}\Big\|_F = \Big\|\sum_j \Big(\sum_i |a_{i,j}|^2\Big)^{1/2} f_j\Big\|_X,$$

which is obviously 1-unconditional.

(III) If a space X has cotype $r < \infty$ and local unconditional structure then the natural map from $\mathrm{Rad}_{n_1}(\mathrm{Rad}_{n_2}(X))$ to $\mathrm{Rad}_{n_1 n_2}(X)$ is an isomorphism with constant $C(r, C_r(X))\,\mathrm{lust}(X)$.

This is the content of Definition 2.1 and Proposition 2.1 of [P.1] (see the comments after Proposition 2.3 above). In the case of a Banach lattice see also the proof of Proposition 2.d.7 of [L-T.2].

Recall a standard convention that for a Banach space X, d_X denotes the Banach–Mazur distance from X to a Hilbert space. So $d_X = d(X, \ell_2^n)$ if $\dim X = n$, and $d_X = d(X, \ell_2)$ if X is infinite-dimensional; in particular, $d_X = \infty$ if X is not isomorphic to a Hilbert space.

The finite-dimensional quantitative version of our main result is:

THEOREM 4.1. *Let X be an n-dimensional Banach space with cotype r constant $C_r(X)$ for some $2 \le r < \infty$, and let $d_X = d(X, \ell_2^n)$. If Y is one of the finite-dimensional spaces listed below*, *then there exists a subspace $Z \subset Y$ such that* $\mathrm{lust}(Z) \ge a d_X^{1/8}$, *where $a = a(r, C_r(X))$ depends on r and $C_r(X)$ only.*

(i) $Y = \mathrm{Rad}_N(\mathrm{Rad}_n(X))$, *where* $N = 3n^2$;
(ii) $Y = \ell_2^M(X)$, *where* $M = 3n^2 2^n$.

Moreover, *the space Z admits a 2-dimensional decomposition which is a-unconditional*, *where $a = a(r, C_r(X)$.*

Before passing to the proof of the theorem, let us recall the notion of property (H), which will play an important role in our discussion. It was introduced by Pisier in [P.3] (see also [P.4]), and studied by Nielsen and Tomczak-Jaegermann in [N-T].

DEFINITION 4.2. Let X be a Banach space. For $m = 1, 2, \dots$, let $\kappa_m(X) \geq 1$ be the smallest constant κ such that for every 1-unconditional normalized sequence $\{g_i\}_{i=1}^{l}$ of vectors in X with $1 \leq l \leq m$, one has

$$\kappa^{-1} l^{1/2} \leq \Big\| \sum_{i=1}^{l} g_i \Big\| \leq \kappa l^{1/2}.$$

We say that X has *property* (H) if $\kappa(X) = \sup_m \kappa_m(X) < \infty$.

The following proposition is taken from [N-T], Proposition 1.2. It is a finite-dimensional version of [P.3], Proposition 4.3. We state it here in the form in which it was proved in [N-T] (although the actual formulation was slightly weaker).

PROPOSITION 4.3. *Let X be an n-dimensional Banach space. Then*

$$d_X \leq C \kappa_n(\mathrm{Rad}_n(X))^4,$$

where C is a universal constant.

Proof of Theorem 4.1. By Proposition 4.3, we have $\kappa_n(\mathrm{Rad}_n(X)) \geq c d_X^{1/4}$, where $c > 0$ is a universal constant. Thus there exist normalized 1-unconditional vectors $f_1, \dots, f_m$ in $\mathrm{Rad}_n(X)$, with $1 \leq m \leq n$, such that either

$$\Big\| \sum_{i=1}^{m} f_i \Big\| \geq c m^{1/2} d_X^{1/4},$$

or

$$\Big\| \sum_{i=1}^{m} f_i \Big\| \leq (1/c) m^{1/2} d_X^{-1/4}.$$

We may additionally assume that $c d_X^{1/4} \geq \sqrt{12}$, otherwise the theorem is true by adjusting $a(r, C_r(X))$.

Let $F = \mathrm{span}\{f_1, \dots, f_m\} \subset \mathrm{Rad}_n(X)$. In particular, F has a 1-unconditional basis and $\dim F = m \leq n$. Set $W_1 = (F \otimes \ell_2^m \otimes \ell_2^m) \oplus (\ell_2^m \otimes F \otimes \ell_2^m) \oplus (\ell_2^m \otimes \ell_2^m \otimes F)$ and $W_2 = (F \otimes \ell_2^m) \oplus (\ell_2^m \otimes F) \oplus \ell_2^{m^2}$.

(i) Consider the space $\ell_2^m \otimes \ell_2^m \otimes F$ with the norm induced by $\mathrm{Rad}_{m^2}(F)$. This is a cross-norm, and the vectors $\{e_i \otimes e_j \otimes f_k\}$ form a λ-unconditional basis, where $\lambda = \lambda(r, C_r(X))$. Moreover, for the cotype r constants we have $C_r(\mathrm{Rad}_{m^2}(F)) = C_r(F) \leq C_r(X)$. Similarly, consider $\ell_2^m \otimes F$ and identify it, via (11), with $\mathrm{Rad}_m(F)$.

For each term entering the definitions of W_1 and W_2 we can make analogous identifications to get the cross-norm in which the natural tensor basis is λ-unconditional. Moreover, the cotype r constants of W_1 and W_2 are less than or equal to $C_r(X)$.

Thus Theorem 3.1 yields that either there exists a subspace Z of W_1 with $\mathrm{lust}(Z) \geq a\lambda^{-2} d_X^{1/4}$, or there exists a subspace $Z \subset W_2$ with $\mathrm{lust}(Z) \geq a\lambda^{-2} d_X^{1/8}$, where $a = a(r, C_r(X)) > 0$. Since each of W_1 and W_2 (under the cross-norms considered above) can be identified with a subspace of $\mathrm{Rad}_N(F)$, (i) follows.

(ii) On the tensor products $\ell_2^m \otimes \ell_2^m \otimes F$ and $\ell_2^m \otimes F$ consider the cross-norms induced by $\ell_2^{m^2}(F)$ and $\ell_2^m(F)$, respectively; thus identify W_1 with $\ell_2^{3m^2}(F)$ and W_2 with $\ell_2^{2m}(F) \oplus \ell_2^{m^2}$. The natural tensor bases in these spaces are now 1-unconditional. It is easy to check that Theorem 3.1 again yields the existence of a subspace Z of W_1 or of W_2 admitting an estimate $\mathrm{lust}(Z) \geq a d_X^{1/8}$. Since $F \subset \mathrm{Rad}_n(X)$ and $\mathrm{Rad}_n(X)$ can be identified with a subspace of $\ell_2^{2^n}(X)$, the conclusion follows. ∎

As an easy consequence we obtain the following theorem which is stated in a quantitative form.

THEOREM 4.4. *Let X be a Banach space of cotype $r < \infty$. Then*

(i) *there exists a subspace Z in $\ell_2(X)$ such that* $\mathrm{lust}(Z) \geq a d_X^{1/8}$,

(ii) *there exists a subspace Z in* $\mathrm{Rad}(\mathrm{Rad}(X))$ *such that* $\mathrm{lust}(Z) \geq a d_X^{1/8}$.

Here $a = a(r, C_r(X)) > 0$ depends on r and $C_r(X)$ only.

The proof of this theorem is completely standard; we give a short outline for the convenience of the reader.

Outline of the proof. Recall that $d_X = \sup d_E$, where the supremum is taken over all finite-dimensional subspaces E of X. Thus if $d_X < \infty$ then the result follows immediately from Theorem 4.1.

If $d_X = \infty$ pick a sequence of finite-dimensional subspaces $E_k \subset X$ with $d_{E_k} \to \infty$ as $k \to \infty$. For $k = 1, 2, \ldots$, let $Z^k \subset \mathrm{Rad}_{N_k}(\mathrm{Rad}_{n_k}(E_k))$ be a subspace constructed in Theorem 4.1(i) such that $\mathrm{lust}(Z^k) \geq a d_{E_k}^{1/8}$ (here $n_k = \dim E_k$ and $N_k = 3n_k^2$).

In case (ii) partition the set $\mathbb{N}$ of all natural numbers as $\bigcup J_k = \bigcup I_k$ where J_k's (resp., I_k's) are successive intervals of natural numbers with cardinality $|J_k| = N_k$ (resp., $|I_k| = n_k$) for all k. For each k, consider the subspace $V_k = \mathrm{Rad}_{J_k}(\mathrm{Rad}_{I_k}(X))$ of $\mathrm{Rad}(\mathrm{Rad}(X))$ defined in the natural way, and identify Z^k with a subspace of V_k. Observe that by symmetry of Rademacher functions, the V_k's form a monotone Schauder decomposition of $\mathrm{Rad}(\mathrm{Rad}(X))$ (which is in fact 1-unconditional). Let Z be the subspace of $\mathrm{Rad}(\mathrm{Rad}(X))$ spanned by the Z^k's, $Z = \mathrm{span}\{Z^k\}_k$. Denote by Q_k the natural projection from $\mathrm{Rad}(\mathrm{Rad}(X))$ onto V_k and observe that $Q_k|Z$ is a (norm 1) projection from Z onto Z^k. Thus $\mathrm{lust}(Z) \geq \mathrm{lust}(Z^k) \to \infty$, hence $\mathrm{lust}(Z) = \infty$.

In case (i) we let Z be the ℓ_2-sum of the Z^k's, $Z = (\sum \oplus Z^k)_{\ell_2}$, so that Z is a subspace of $\ell_2(\ell_2(X)) = \ell_2(X)$. The rest of the proof is similar to case (ii). ∎

Case (i) of the above proof also shows that if X is not isomorphic to a Hilbert space and is of cotype $r < \infty$, then the subspace $Z \subset \ell_2(X)$ without local unconditional structure is of the form $Z = (\sum \oplus Z^k)_2$, where each space Z^k has a 2-dimensional unconditional decomposition. It easily follows that Z itself has a 2-dimensional unconditional decomposition as well, thus proving the "moreover" part of the abstract.

We now easily get an isomorphic characterization of a Hilbert space in terms of local unconditional structure.

COROLLARY 4.5. *For any Banach space X the following conditions are equivalent*:

(i) *X is isomorphic to a Hilbert space.*
(ii) *Every subspace of $\ell_2(X)$ has local unconditional structure.*
(iii) *Every subspace of* $\mathrm{Rad}(X)$ *has local unconditional structure.*

Proof. It is easy to see by Theorem 4.4 and by (III) above that the only case to prove is when X does not have a finite cotype, which is equivalent to X containing ℓ_∞^n's uniformly.

Recall the well known fact that for any Banach space and a finite-dimensional subspace E there exists a finite-codimensional subspace Y such that $\|e\| \le 2\|e+y\|$ for all $e \in E$ and $y \in Y$ (cf. e.g. the proof of [L-T.1], 1.a.6). Thus in our case we can construct by induction a sequence of subspaces $E_n \subset X$ such that $\dim E_n = n$, $d(E_n, \ell_\infty^n) \le 2$ for all n, and if $X_0 = \mathrm{span}\{E_n\}_n \subset X$ then the natural projection Q_n from X_0 onto $\mathrm{span}\{E_k\}_{k\le n}$ has norm ≤ 2. Then $P_n = Q_n - Q_{n-1}$ is the natural projection from X_0 onto E_n with $\|P_n\| \le 4$.

Recall also that a "random" $[n/2]$-dimensional subspace of ℓ_∞^n has the Gordon–Lewis constant, hence also the lust-constant of maximal order (cf. e.g. [T.1]). This implies that for every n, there is a subspace $Z^n \subset E_n$ with $\dim Z^n = [n/2]$ and $\mathrm{lust}(Z^n) \ge \mathrm{GL}(Z^n) \ge c\sqrt{n}$, where $c > 0$ is an absolute constant. Let $Z = \mathrm{span}\{Z^n\}_n \subset X_0 \subset X$. Then $P_n|Z$ is a projection from Z onto Z^n with norm $\|P_n|Z\| \le 4$. This implies that $\mathrm{lust}(Z) = \infty$, and for the same reason, Z does not have the Gordon–Lewis property. ∎

Let us recall that it is still an open question whether the condition that every subspace of X has an unconditional basis, or merely local unconditional structure, implies that X is isomorphic to a Hilbert space. In this connection the following remark is of interest.

REMARK 4.6. Casazza and Kalton [C-K] proved that the general method from [K-T.1] does not work in an arbitrary Banach space X. Just as in Theorems 3.1 and 4.1, subspaces $Z \subset X$ constructed by this method which fail to have an unconditional basis, still have a 2-dimensional Schauder decomposition which is $a = a(r, C_r(X))$-unconditional. It is shown in [C-K] that there is an Orlicz sequence space $\ell_F \neq \ell_2$ with the property that whenever Y is a closed subspace of ℓ_F with an unconditional finite-dimensional decomposition $\{Z_k\}_k$ such that $\sup \dim Z_k < \infty$, then Y has an unconditional basis. Clearly, ℓ_F has a 1-unconditional basis and it is easy to check that it is of cotype 2. It is also easy to check that every subspace of ℓ_F contains an isomorphic copy of ℓ_2.

The following theorem is a finite-dimensional quantitative result for subspaces of ℓ_p^N. It was proved by B. Maurey [M.1], who on this occasion suggested the use of tensor products in the context of [K-T.1].

THEOREM 4.7. (i) *For* $1 \leq p < 2$ *the space* ℓ_p^N *contains a subspace* Z *of dimension* $\geq N/2$ *with* $\mathrm{lust}(Z) \geq cN^{\alpha}$ *which has a basis with basis constant less than or equal to* $1/c$. *Here* $\alpha = (1/3)(1/p - 1/2)$ *and* $c > 0$ *is an absolute constant.*

(ii) *For* $2 < p < \infty$ *the space* ℓ_p^N *contains a subspace* Z *of dimension* $\geq a_p N^{\alpha}$ *with* $\mathrm{lust}(Z) \geq a_p N^{\beta}$ *which has a basis with basis constant less than or equal to* c. *Here* $\alpha = 3(2 + p/2)^{-1}$, $\beta = (2 + p/2)^{-1}(1/2 - 1/p)$, $c \geq 1$ *is an absolute constant and* $a_p > 0$ *depends on* p *only.*

Proof. For $p \neq 2$, denote by $\{e_j\}$ the standard unit vector basis in ℓ_p. On the tensor product $\ell_p^{n_1} \otimes \ell_p^{n_2} \otimes \ell_p^{n_3}$ consider the norm induced by ℓ_p^N, where $N = n_1 n_2 n_3$. Obviously, it is a cross-norm.

If $\{g_i\}$ is a λ-unconditional sequence in $\ell_p^{n_1}$ for some $\lambda \geq 1$, a straightforward calculation shows that $\{g_i \otimes e_j \otimes e_k\}_{i,j,k}$ is a λ-unconditional sequence in $\ell_p^{n_1} \otimes \ell_p^{n_2} \otimes \ell_p^{n_3}$ with the above cross-norm. An analogous statement is true for sequences obtained by permuting the place of the g_i's in the tensor product.

More generally, it follows from Proposition 2.3 that if $\{f_i\}$, $\{g_j\}$ and $\{h_k\}$ are λ-unconditional sequences in $\ell_p^{n_\nu}$ for $\nu = 1, 2, 3$, respectively, then $\{f_i \otimes g_j \otimes h_k\}$ is C_λ-unconditional in ℓ_p^N with $C_\lambda \leq c_1 \lambda^2$, where c_1 is a universal constant if $1 \leq p < 2$, and c_1 depends on p if $p \geq 2$.

Denote by $\{h_i\}$ the standard unit vector basis in ℓ_2.

(i) First consider arbitrary cross-norms on the tensor products $X_1 = \ell_p^n \otimes \ell_2^n \otimes \ell_2^n$, $X_2 = \ell_2^n \otimes \ell_p^n \otimes \ell_2^n$ and $X_3 = \ell_2^n \otimes \ell_2^n \otimes \ell_p^n$ such that the natural tensor bases are λ-unconditional for some $\lambda \geq 1$. (So $\{e_i \otimes h_j \otimes h_k\}$ is λ-unconditional in X_1, etc.) Set $X = X_1 \oplus X_2 \oplus X_3$. Applying Theorem 3.1(i) with $F = \ell_p^n$ and $D = \| \sum_{i=1}^n e_i \| \, n^{-1/2} = n^{1/p - 1/2}$ we get a $2n^3$-dimensional subspace Z of X such that $\mathrm{lust}(Z) \geq c_2 \lambda^{-2} D$, where $c_2 > 0$ is an absolute constant.

Now recall that for any k, the space ℓ_p^k contains an n-dimensional subspace with $n \geq \sqrt{3}k/2$ which is C-isomorphic to ℓ_2^n, where $C \geq 1$ is a universal constant. (In fact this is true for any proportion $0 < \xi < 1$ and $n \geq \xi k$, with the constant C depending on ξ. We refer the reader e.g. to [M-S] and [P.4] for the deep general theory of Euclidean subspaces of finite-dimensional spaces.) Let $k = [2n/\sqrt{3}] + 1$ and let $E \subset \ell_p^k$ be an n-dimensional C-Euclidean subspace as above. Let $\{h_j'\} \subset E$ be a normalized basis C-equivalent to the basis $\{h_j\}$ in ℓ_2^n (i.e., there are $a, b > 0$ with $ab = C$ such that $1/b \leq \|\sum t_j h_j'\| \leq a$ for all sequences (t_j) of scalars with $\sum |t_j|^2 = 1$). In particular, $\{h_j'\} \subset E$ is C-unconditional. Consider the tensor product $\widetilde{X}_1 = \ell_p^n \otimes E \otimes E$ as the subspace of $\ell_p^n \otimes \ell_p^k \otimes \ell_p^k$, with the cross-norm from $\ell_p^{nk^2}$, as described at the beginning of the proof. The discussion at the beginning of the proof also shows that the natural tensor basis $\{e_i \otimes h_j' \otimes h_k'\}$ is λ'-unconditional, where $\lambda' \leq c_1 C^2$. An analogous construction can be done for $\widetilde{X}_2$ and $\widetilde{X}_3$, and for $\widetilde{X}$, which makes $\widetilde{X}$ a subspace of ℓ_p^N for $N = 3nk^2 = 4n^3$. Using Theorem 3.1(i) in the same way as in the previous paragraph, and noting that only the upper bound a enters in the proof of this theorem, we get a $2n^3$-dimensional subspace $\widetilde{Z} \subset \widetilde{X} \subset \ell_p^N$ such that $\operatorname{lust}(\widetilde{Z}) \geq c_2 a^{-1} \lambda'^{-2} D \geq c_3 D$, where $c_3 > 0$ is an absolute constant.

(ii) We use Theorem 3.1(i), with the modification indicated in Remark 3.2. First, letting $F = \ell_2^n$ with the basis $\{h_i\}$, we get a $2n^3$-dimensional subspace Z of $X = (\ell_2^n \otimes \ell_p^n \otimes \ell_p^n) \oplus (\ell_p^n \otimes \ell_2^n \otimes \ell_p^n) \oplus (\ell_p^n \otimes \ell_p^n \otimes \ell_2^n)$ with $\operatorname{lust}(Z) \geq aD$, where $D = n^{1/2-1/p}$ and $a > 0$ depends on cotype properties of X.

Recall that ℓ_2^n is 2-isomorphic to a subspace E of ℓ_p^k with $k = C_p n^{p/2}$, where $C_p \geq 1$ depends on p only (cf. e.g. [M-S]). Let $\widetilde{X}_1 = E \otimes \ell_p^n \otimes \ell_p^n$ and define $\widetilde{X}_2$ and $\widetilde{X}_3$ by analogous formulas. An argument similar to (i) shows that on each $\widetilde{X}_\nu$ there is a cross-norm which makes this space into a subspace of ℓ_p^N (where $N = C_p n^{2+p/2}$). Let $\{h_i'\}$ be a basis in E 2-equivalent to the basis $\{h_i\}$ in ℓ_2^n. Then the natural tensor basis $\{h_i' \otimes e_j \otimes e_k\}$ is 2-unconditional in $\widetilde{X}_1$, and a similar calculation is valid in $\widetilde{X}_2$ and $\widetilde{X}_3$. Finally, note that the cotype p constant of ℓ_p satisfies $C_p(\ell_p) \leq c\sqrt{p}$, where c is an absolute constant. Thus, by Theorem 3.1(i), the space $\widetilde{X} = \widetilde{X}_1 \oplus \widetilde{X}_2 \oplus \widetilde{X}_3$ contains a $2n^3$-dimensional subspace $\widetilde{Z}$ with $\operatorname{lust}(\widetilde{Z}) \geq a_p' n^{1/2-1/p}$, where $a_p' > 0$ depends on p only. In both cases (i) and (ii) the statement about the basis constant follows directly from Theorem 3.1. ∎

Let us conclude by several comments about subspaces of ℓ_p (and ℓ_p^N). We start with $1 \leq p \leq 2$. It is well known that in this case a random $N/2$-dimensional subspace of ℓ_p^N is nearly Euclidean (see the proof of Theorem 4.7(i) above), and therefore, the subspace constructed in Theorem 4.7(i) is very far from being random. In this range of p, every subspace of ℓ_p has the Gordon–Lewis property (this is true for subspaces of any Banach lattice of cotype 2, cf. e.g. [P.2]). It follows that all subspaces of ℓ_p^N have the GL-constant uniformly bounded above. The lower estimate for $\mathrm{lust}(Z)$ obtained here is of the largest order known to date. Still, it is quite likely that there exists a subspace $\widetilde{Z} \subset \ell_p$ with $\dim(\widetilde{Z}) = n$ such that $\mathrm{lust}(\widetilde{Z}) \geq cn^{1/p-1/2}$.

For $2 \leq p \leq \infty$, a "random" $[N/2]$-dimensional subspace $\widetilde{Z} \subset \ell_p^N$ satisfies $\mathrm{lust}(\widetilde{Z}) \geq \mathrm{GL}(\widetilde{Z}) \geq cN^{1/2-1/p}$; and this gives an asymptotically maximal order, up to a numerical factor. This in particular means that although our method seems to require the assumption of finite cotype, Banach spaces which do not satisfy this assumption, hence contain ℓ_∞^n's uniformly, automatically have subspaces without the Gordon–Lewis property, hence without local unconditional structure (see the proof of Corollary 4.5). However, an $[N/2]$-dimensional subspace $\widetilde{Z} \subset \ell_p^N$ as above has basis constant also of maximal order (i.e., the basis constant of every basis in $\widetilde{Z}$ admits an appropriate lower estimate). On the other hand, although the example in Theorem 4.7(ii) obviously does not produce the worse possible behaviour for the lust-constant, it sharply contrasts the bad behaviour of this constant with the good behaviour of the basis constant.

References

[B] A. Borzyszkowski, *Unconditional decomposition and local unconditional structure in some subspaces of $L_p, 1 \leq p < 2$*, Studia Math. 76 (1983), 267–278.

[C-K] P. G. Casazza and N. J. Kalton, *Unconditional bases and unconditional finite-dimensional decompositions in Banach spaces*, Israel J. Math. 95 (1996), 349–373.

[G] W. T. Gowers, *A new dichotomy for Banach spaces*, Geom. Funct. Anal. 6 (1996), 1083–1093.

[J-L-S] W. B. Johnson, J. Lindenstrauss and G. Schechtman, *On the relation between several notions of unconditional structure*, Israel J. Math. 37 (1980), 120–129.

[Ke] T. Ketonen, *On unconditionality in L_p spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 35 (1983).

[K] R. Komorowski, *Weak Hilbert spaces without unconditional basis*, Proc. Amer. Math. Soc. 120 (1994), 101–107.

[K-T.1] R. Komorowski and N. Tomczak-Jaegermann, *Banach spaces without local unconditional structure*, Israel J. Math. 89 (1995), 205–226.

[K-T.2] —, —, *Erratum to*: *"Banach spaces without local unconditional structure"*, ibid. 105 (1998), 85–92.

[L-T.1] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer, Berlin, 1977.

[L-T.2] —, —, *Classical Banach Spaces II*, Springer, Berlin, 1979.

[M.1] B. Maurey, private communication.

[M.2] B. Maurey, *Type*, *cotype and K-convexity*, in: Handbook in Banach Spaces, W. B. Johnson and J. Lindenstrauss (eds.), Elsevier, to appear.

[M-S] V. Milman and G. Schechtman, *Asymptotic Theory of Normed Spaces*, Lecture Notes in Math. 1200, Springer, 1987.

[N-T] N. J. Nielsen and N. Tomczak-Jaegermann, *Banach lattices with property* (H) *and weak Hilbert spaces*, Illinois J. Math. 36 (1992), 345–371.

[P.1] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compositio Math. 37 (1978), 3–19.

[P.2] —, *Factorization of Linear Operators and Geometry of Banach Spaces*, CBMS Regional Conf. Ser. in Math. 60, Amer. Math. Soc., Providence, RI, 1986.

[P.3] —, *Weak Hilbert spaces*, Proc. London Math. Soc. 56 (1988), 547–579.

[P.4] —, *Volume Inequalities in the Geometry of Banach Spaces*, Cambridge Tracts in Math. 94, Cambridge Univ. Press, Cambridge, 1989.

[T.1] N. Tomczak-Jaegermann, *Banach–Mazur Distances and Finite-Dimensional Operator Ideals*, Pitman Monographs 38, Longman Sci. Tech., Harlow, 1989.

[T.2] —, *A solution of the homogeneous Banach space problem*, in: Canadian Mathematical Society 1945–95. Anniversary Volume, III, CMS, Ottawa, 1996, 267–286.

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