STUDIA MATHEMATICA 149 (1) (2002)

Domination properties in ordered Banach algebras

by

H. DU T. MOUTON and S. MOUTON (Stellenbosch)

Abstract. We recall from [9] the definition and properties of an algebra cone C of a real or complex Banach algebra A. It can be shown that C induces on A an ordering which is compatible with the algebraic structure of A. The Banach algebra A is then called an ordered Banach algebra. An important property that the algebra cone C may have is that of normality. If C is normal, then the order structure and the topology of A are reconciled in a certain way. Ordered Banach algebras have interesting spectral properties. If A is an ordered Banach algebra with a normal algebra cone C, then an important problem is that of providing conditions under which certain spectral properties of a positive element b will be inherited by positive elements dominated by b. We are particularly interested in the property of b being an element of the radical of A. Some interesting answers can be obtained by the use of subharmonic analysis and Cartan's theorem.

1. Introduction. In [9] and [8] some spectral theory of positive elements in ordered Banach algebras was developed. An interesting problem in this theory is that of finding conditions under which properties of a positive element b will be inherited by any positive element "smaller than", i.e. dominated by, b. This problem has originally been studied in the context of Banach lattices; i.e. if E is a Banach lattice and S and T are bounded linear operators on E, which properties of T are inherited by S if we know that $0 \leq S \leq T$? Topological properties (e.g. compactness ([3] and [6]), weak compactness ([2]) and the property of being Dunford–Pettis ([1])) as well as spectral properties ([5]) have been considered. A survey of some of these results is given in ([10], Chapter 18). The problem was introduced in the context of ordered Banach algebras in ([9], Section 6), where some complementary results to the Aliprantis–Burkinshaw theory for positive operators were obtained. In this paper we shall consider the following problem: Let A be an ordered Banach algebra (see Section 3). Under which conditions does it follow from $0 \leq a \leq b$ in A and b being in the radical of A that a

2000 *Mathematics Subject Classification*: 46H05, 47A10, 47B65, 06F25.

Key words and phrases: ordered Banach algebra, positive element, subharmonic analysis.

is in the radical of A? Some interesting answers will be given by the use of subharmonic analysis.

2. Preliminaries. Throughout, A will be a Banach algebra with unit. Unless otherwise stated, A will be over $\mathbb{C}$. The spectrum of an element a in A will be denoted by $\sigma(a)$ and the spectral radius of a in A by $r(a)$ (or by $\sigma(a, A)$ and $r(a, A)$ if necessary to avoid confusion). We denote the set of quasinilpotent elements in A by $\mathrm{QN}(A)$ and the radical of A by $\mathrm{Rad}(A)$. Recall that $\mathrm{Rad}(A) = \{a \in A : aA \subset \mathrm{QN}(A)\}$. A Banach algebra is called *semisimple* if its radical consists of zero only. We shall denote the linear span of a set B in A by $\operatorname{span} B$.

Let $\mathcal{D}$ be a domain of $\mathbb{C}$. A function $\phi : \mathcal{D} \to \mathbb{R} \cup \{-\infty\}$ is said to be *subharmonic* on $\mathcal{D}$ ([4], p. 52) if ϕ is upper semicontinuous on $\mathcal{D}$ and satisfies the mean inequality $\phi(\lambda_0) \leq (2\pi)^{-1} \int_0^{2\pi} \phi(\lambda_0 + re^{i\theta})\, d\theta$ for all closed disks $\overline{B}(\lambda_0, r)$ included in $\mathcal{D}$. For properties of subharmonic functions we refer to [7]. The following theorem by E. Vesentini has a huge number of applications in spectral theory and will also be an indispensable tool in this paper:

THEOREM 2.1 (E. Vesentini; [4], Theorem 3.4.7). *Let f be an analytic function from a domain $\mathcal{D}$ of $\mathbb{C}$ into a Banach algebra A. Then $\lambda \mapsto r(f(\lambda))$ and $\lambda \mapsto \log r(f(\lambda))$ are subharmonic on $\mathcal{D}$.*

Another concept that we shall need is that of *capacity* ([4], p. 179) of a compact set in the complex plane. Let $\mathcal{B}_n$ denote the set of polynomials of the form $p(z) = z^n + a_1 z^{n-1} + \ldots + a_n$. Let K be a compact set and denote by $t_n(K)$ the quantity $\inf_{p \in \mathcal{B}_n} \max_{z \in K} |p(z)|$. Since K is compact and $\mathcal{B}_n$ is finite-dimensional, $t_n(K) = \max_{z \in K} |p_n(z)|$ for some (unique) $p_n \in \mathcal{B}_n$. Let $\delta_n(K) := (t_n(K))^{1/n}$. Then the *capacity* $c(K)$ of K is defined by

$$c(K) = \lim_{n \to \infty} \delta_n(K).$$

It can be shown that closed balls ([4], Corollary A.1.26) and closed line segments ([4], Corollary A.1.27) have nonzero capacities.

The concept of capacity can be extended to bounded subsets of the complex plane. A subset of $\mathbb{C}$ is *locally of capacity zero* ([4], p. 180) if all its bounded subsets have zero capacity. Therefore open balls and closed line segments are not locally of capacity zero. Also, a subset of a set which is locally of capacity zero is also locally of capacity zero.

For further information regarding capacity we refer to [4]. We now formulate the important Cartan's Theorem, which, together with Theorem 2.1, will provide the basis for the results in this paper:

THEOREM 2.2 (H. Cartan; [4], Theorem A.1.29). *Let ϕ be subharmonic on a domain $\mathcal{D}$ of $\mathbb{C}$ and not identically $-\infty$. Then $\{\lambda \in \mathcal{D} : \phi(\lambda) = -\infty\}$ is a G_δ-set which is locally of capacity zero.*

For our purposes we provide the following corollary:

COROLLARY 2.3. *Let f be an analytic function from a domain $\mathcal{D}$ of $\mathbb{C}$ into a Banach algebra A. Suppose E is either an open ball or a closed line segment with $E \subset \{\lambda \in \mathcal{D} : r(f(\lambda)) = 0\}$. Then $r(f(\lambda)) = 0$ for all λ in $\mathcal{D}$.*

Proof. If f is analytic on $\mathcal{D}$, then by Theorem 2.1, $\phi = \log(r \circ f)$ is subharmonic on $\mathcal{D}$. Suppose there exists a $\lambda \in \mathcal{D}$ with $r(f(\lambda)) \neq 0$. Then $\phi(\lambda) \neq -\infty$ so that Cartan's Theorem shows that $\{\lambda \in \mathcal{D} : r(f(\lambda)) = 0\} = \{\lambda \in \mathcal{D} : \phi(\lambda) = -\infty\}$ is locally of capacity zero. Since E is contained in the first set, it follows that E is locally of capacity zero as well. However, as E is known to be either an open ball or a closed line segment, we have a contradiction. ∎

3. Ordered Banach algebras. In ([9], Section 3) we defined an algebra cone C of a complex Banach algebra A and showed that C induced on A an ordering which was compatible with the algebraic structure of A. Such a Banach algebra is called an ordered Banach algebra (OBA). We recall those definitions now and also the additional properties that C may have. Of these properties normality is the most significant one, as it reconciles the order structure and the topology of A.

Let A be a complex Banach algebra with unit 1. We call a nonempty subset C of A a *cone* of A if C satisfies the following:

(1) $C + C \subseteq C$,
(2) $\lambda C \subseteq C$ for all $\lambda \geq 0$.

If in addition C satisfies $C \cap -C = \{0\}$, then C is called a *proper* cone.

Any cone C of A induces an *ordering* "$\leq$" on A in the following way:

$$a \leq b \quad \text{if and only if} \quad b - a \in C \tag{3.1}$$

$(a, b \in A)$. It can be shown that this ordering is a partial order on A, i.e., for every $a, b, c \in A$,

(a) $a \leq a$ ($\leq$ is *reflexive*),
(b) if $a \leq b$ and $b \leq c$, then $a \leq c$ ($\leq$ is *transitive*).

Furthermore, C is proper if and only if this partial order has the additional property of being *antisymmetric*, i.e. if $a \leq b$ and $b \leq a$, then $a = b$. Considering the partial order that C induces we find that $C = \{a \in A : a \geq 0\}$ and therefore we call the elements of C *positive*.

A cone C of a Banach algebra A is called an *algebra cone* of A if C satisfies the following conditions:

(3) $C.C \subseteq C$,
(4) $1 \in C$.

Motivated by this concept we call a complex Banach algebra with unit 1 an *ordered Banach algebra* (OBA) if A is partially ordered by a relation "$\leq$" in such a manner that for every $a, b, c \in A$ and $\lambda \in \mathbb{C}$,

(1′) $a, b \geq 0 \Rightarrow a + b \geq 0$,
(2′) $a \geq 0, \lambda \geq 0 \Rightarrow \lambda a \geq 0$,
(3′) $a, b \geq 0 \Rightarrow ab \geq 0$,
(4′) $1 \geq 0$.

Therefore if A is ordered by an algebra cone C, then A, or more specifically (A, C), is an OBA.

An algebra cone C of A is called *proper* if C is a proper cone of A, and *closed* if it is a closed subset of A. Furthermore, C is said to be *normal* if there exists a constant $\alpha > 0$ such that it follows from $0 \leq a \leq b$ in A that $\|a\| \leq \alpha\|b\|$. It is well known that if C is a normal algebra cone, then C is proper.

If an algebra cone C has the property that $r(a) \leq r(b)$ whenever $0 \leq a \leq b$, then we say that the spectral radius is *monotone* (relative to C). It is always the case that if C is normal, then the spectral radius is monotone ([9], Theorem 4.1).

Let A and B be Banach algebras such that $1 \in B \subset A$. If C is an algebra cone of A, then $C \cap B$ is an algebra cone of B. Moreover, if C is a proper algebra cone of A, then $C \cap B$ is a proper algebra cone of B. In the case where B has a finer norm than A (i.e. $\|b\|_A \leq \|b\|_B$ for all $b \in B$), we have the additional fact that if the algebra cone C of A is closed in A, then the algebra cone $C \cap B$ of B is closed in B. If B is a closed subalgebra of A, then normality of C in A implies normality of $C \cap B$ in B.

4. Domination properties. Let A be an OBA with a normal algebra cone C. If $0 \leq a \leq b$, which properties of b are inherited by a? This problem has originally been investigated for bounded linear operators on a Banach lattice (see [1], [2], [3], [6], [5], [10]). It was introduced in the context of ordered Banach algebras in [9]. In this section we are specifically interested in the property of being an element of the radical of A. Hence the problem becomes: If A is an OBA with a normal algebra cone C, which conditions ensure that if $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$? We will show that a number of interesting results can be obtained by the use of Cartan's Theorem.

LEMMA 4.1. *Let A be an OBA with a normal algebra cone C. If $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $aC \subset \mathrm{QN}(A)$.*

Proof. If $b \in \mathrm{Rad}(A)$, then $bA \subset \mathrm{QN}(A)$, so that $bC \subset \mathrm{QN}(A)$. Since $0 \leq a \leq b$, it follows that $0 \leq ac \leq bc$ for all $c \in C$. The normality of C

implies that the spectral radius is monotone, so that $r(ac) \leq r(bc)$ for all $c \in C$, but since $bC \subset \mathrm{QN}(A)$, it follows that $aC \subset \mathrm{QN}(A)$. ∎

The above lemma will lead to a number of results (4.2, 4.3, 4.6, 4.9, 4.10 and 4.13) which give answers to the problem we posed.

THEOREM 4.2. *Let A be an OBA with a normal algebra cone C and such that for every $x \in A$ there is a $0 \neq \lambda \in \mathbb{C}$ such that $\lambda x \in C$. Then if $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

Proof. If $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$ then $aC \subset \mathrm{QN}(A)$, by the above lemma. We show that $a \in \mathrm{Rad}(A)$ by showing that $aA \subset \mathrm{QN}(A)$: Let $x \in A$. Then, by the assumption, $\lambda x \in C$ for some $0 \neq \lambda \in \mathbb{C}$. Hence $a(\lambda x) \in aC \subset \mathrm{QN}(A)$, so that $r(a(\lambda x)) = 0$, i.e. $|\lambda| r(ax) = 0$. It follows that $r(ax) = 0$. Therefore $ax \in \mathrm{QN}(A)$. Since x was arbitrary, we have shown that $aA \subset \mathrm{QN}(A)$. ∎

COROLLARY 4.3. *Let A be an OBA with a normal algebra cone C and such that for every $x \in A$ there is a line segment L in $\mathbb{C}$ such that $\lambda x \in C$ for all $\lambda \in L$. If $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

It is interesting to note that another (direct) proof of the above fact can be obtained by the use of subharmonic techniques:

Proof. We show that $aA \subset \mathrm{QN}(A)$: Let $x \in A$. Then $\lambda x \in C$ for all $\lambda \in L$, where L is some line segment in $\mathbb{C}$. By Lemma 4.1 it follows that $r(a(\lambda x)) = 0$ for all $\lambda \in L$. This, together with the fact that $f(\lambda) = a(\lambda x)$ is analytic on $\mathbb{C}$, implies by Corollary 2.3 that $r(a(\lambda x)) = 0$ for all $\lambda \in \mathbb{C}$, and hence for $\lambda = 1$. So $ax \in \mathrm{QN}(A)$. Since x was arbitrary, we have shown that $aA \subset \mathrm{QN}(A)$, i.e. $a \in \mathrm{Rad}(A)$. ∎

In our next theorems subharmonic analysis will be essential. We begin with the following lemma:

LEMMA 4.4. *Let A be an OBA with a normal algebra cone C. If $aC \subset \mathrm{QN}(A)$, then $a \operatorname{span} C \subset \mathrm{QN}(A)$.*

Proof. Take any $n \in \mathbb{N}$ and any $c_1, \ldots, c_n \in C$. Now take fixed positive real numbers $\lambda_2, \ldots, \lambda_n$ and let $f_1(\lambda_1) = a(\lambda_1 c_1 + \ldots + \lambda_n c_n)$, with $\lambda_1 \in \mathbb{C}$. Then f_1 is analytic on $\mathbb{C}$. For $\lambda_1 \in \mathbb{R}^+$ we have $f_1(\lambda_1) \in aC$, so that, by the assumption, $r(f_1(\lambda_1)) = 0$ for all $\lambda_1 \in \mathbb{R}^+$. By letting E in Corollary 2.3 be the interval $[0, 1]$, it follows from this corollary that $r(f_1(\lambda_1)) = 0$ for all $\lambda_1 \in \mathbb{C}$. The choices of $\lambda_2, \ldots, \lambda_n$ in $\mathbb{R}^+$ were arbitrary, so that we have shown that

$$r(a(\lambda_1 c_1 + \ldots + \lambda_n c_n)) = 0 \quad \text{for all } \lambda_1 \in \mathbb{C} \text{ and all } \lambda_2, \ldots, \lambda_n \in \mathbb{R}^+. \tag{4.5}$$

In the next step, take a fixed $\lambda_1 \in \mathbb{C}$ and fixed $\lambda_3, \ldots, \lambda_n \in \mathbb{R}^+$ and let $f_2(\lambda_2) = a(\lambda_1 c_1 + \ldots + \lambda_n c_n)$, with $\lambda_2 \in \mathbb{C}$. Again, f_2 is analytic on $\mathbb{C}$ and for $\lambda_2 \in \mathbb{R}^+$ we have $r(f_2(\lambda_2)) = 0$, by (4.5). Again it follows from Corollary 2.3 that $r(f_2(\lambda_2)) = 0$ for all $\lambda_2 \in \mathbb{C}$. Since the choices of λ_1 in $\mathbb{C}$ and $\lambda_3, \ldots, \lambda_n$ in $\mathbb{R}^+$ were arbitrary, we have shown that

$$r(a(\lambda_1 c_1 + \ldots + \lambda_n c_n)) = 0 \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C} \text{ and all } \lambda_3, \ldots, \lambda_n \in \mathbb{R}^+.$$

After n steps we get

$$r(a(\lambda_1 c_1 + \ldots + \lambda_n c_n)) = 0 \quad \text{for all } \lambda_1, \ldots, \lambda_n \in \mathbb{C},$$

i.e. $r(ax) = 0$ for all $x \in \operatorname{span} C$. ∎

We first consider the case where A is the linear span of C.

THEOREM 4.6. *Let A be an OBA with a normal algebra cone C and suppose that $A = \operatorname{span} C$. If $0 \leq a \leq b$ and $b \in \operatorname{Rad}(A)$, then $a \in \operatorname{Rad}(A)$.*

Proof. If $0 \leq a \leq b$ and $b \in \operatorname{Rad}(A)$, then, by Lemma 4.1, $aC \subset \operatorname{QN}(A)$. Since each $x \in A$ is in $\operatorname{span} C$, it follows from Lemma 4.4 that $aA \subset \operatorname{QN}(A)$. Therefore $a \in \operatorname{Rad}(A)$. ∎

The condition $A = \operatorname{span} C$ in the above theorem means that A has a Hamel basis consisting of positive elements. Referring to Theorem 4.2, Corollary 4.3 and Theorem 4.6, we give the following

EXAMPLE 4.7. *Let $A = \mathbb{C}$ and $C = \mathbb{R}^+$. Then (A, C) is an ordered Banach algebra, C is normal and*

(1) *for each $x \in A$ there is a $0 \neq \lambda \in \mathbb{C}$ with $\lambda x \in C$;*

(2) *for every $x \in A$ there is a line segment L in $\mathbb{C}$ such that $\lambda x \in C$ for all $\lambda \in L$;*

(3) $A = \operatorname{span}\{1\}$, *so that* $A = \operatorname{span} C$.

Note, however, that A is semisimple. This means that $\operatorname{Rad}(A) = \{0\}$, so that $0 \leq a \leq b$ and $b \in \operatorname{Rad}(A)$ implies that $0 \leq a \leq 0$. Since C is normal, C is proper, so that $\leq$ is antisymmetric. Hence $a = 0$ so that $a \in \operatorname{Rad}(A)$. It follows that in this case $b \in \operatorname{Rad}(A) \Rightarrow a \in \operatorname{Rad}(A)$ is trivial.

We proceed to give another example, one where A is not semisimple, so that the above-mentioned implication is not trivial and thus better illustrates the applicability of the previous theorem:

EXAMPLE 4.8. *Let A be the set of upper triangular 2×2 complex matrices and C the subset of A of matrices with only nonnegative entries. Then (A, C) is an ordered Banach algebra, C is normal and $A = \operatorname{span} C$, since A is the linear span of*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Proof. That (A, C) is an ordered Banach algebra with C normal follows either directly, or by using the properties of the algebra cone of the Banach algebra of *all* 2×2 complex matrices, together with the properties of the algebra cone of a subalgebra (in this case A), as mentioned in the last paragraph of Section 3. ∎

Note that A is not semisimple, since $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Rad}(A)$.

In the case where the linear span of C is dense in A we can say the following:

Theorem 4.9. *Let A be an OBA with a normal algebra cone C. Suppose that $A = \overline{\mathrm{span}\, C}$ and the spectral radius function r is continuous on A. If $0 \le a \le b$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

Proof. If $0 \le a \le b$ and $b \in \mathrm{Rad}(A)$, Lemma 4.1 yields $aC \subset \mathrm{QN}(A)$. For each $x \in A$ there is a sequence of the form $(\lambda_{n1} c_{n1} + \ldots + \lambda_{nm_n} c_{nm_n})$, with the $c_{nj} \in C$, which converges to x as $n \to \infty$. Hence

$$a(\lambda_{n1} c_{n1} + \ldots + \lambda_{nm_n} c_{nm_n}) \to ax \quad \text{as } n \to \infty.$$

The elements $\lambda_{n1} c_{n1} + \ldots + \lambda_{nm_n} c_{nm_n}$ are in $\mathrm{span}\, C$ and therefore, by Lemma 4.4, $r(a(\lambda_{n1} c_{n1} + \ldots + \lambda_{nm_n} c_{nm_n})) = 0$. Since r is continuous, $r(ax) = \lim_{n \to \infty} r(a(\lambda_{n1} c_{n1} + \ldots + \lambda_{nm_n} c_{nm_n}) = 0$, i.e. $ax \in \mathrm{QN}(A)$. Since x was arbitrary in A, we have shown that $aA \subset \mathrm{QN}(A)$, i.e. $a \in \mathrm{Rad}(A)$. ∎

Theorem 4.9 can be applied when A has a positive Schauder basis and a continuous spectral radius. The theorem is specifically applicable in the case of the *scattered Banach algebras*, i.e., the Banach algebras in which the spectrum of every element is finite or countable, since by ([4], Corollary 3.4.5) the spectral radius function is continuous at all elements having finite or countable spectrum.

In our main theorem we consider the case where $\mathrm{span}\, C$ has nonempty interior:

Theorem 4.10. *Let A be an OBA with a normal algebra cone C and suppose that $\mathrm{span}\, C$ contains an interior point. If $0 \le a \le b$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

Proof. Since $\mathrm{span}\, C$ contains an interior point, there is a $c_0 \in \mathrm{span}\, C$ and a $\delta > 0$ such that if $x \in A$ then

$$\|c_0 - x\| < \delta \;\Rightarrow\; x \in \mathrm{span}\, C. \tag{4.11}$$

If $0 \le a \le b$ and $b \in \mathrm{Rad}(A)$ then, by Lemma 4.1, $aC \subset \mathrm{QN}(A)$ and hence, by Lemma 4.4,

$$a\, \mathrm{span}\, C \subset \mathrm{QN}(A). \tag{4.12}$$

Now we show that $a \in \mathrm{Rad}(A)$ by showing that $aA \subset \mathrm{QN}(A)$: Let $x \in A$. Define $f_x : \mathbb{C} \to A$ by $f_x(\lambda) = a(c_0 + \lambda(x - c_0))$. Then f_x is analytic on $\mathbb{C}$.

Let $\varepsilon_x \leq \delta/(\|x\| + \|c_0\|)$ and let E be the open ball with centre 0 in $\mathbb{C}$ and radius ε_x. Then $\|c_0 - [c_0 + \lambda(x - c_0)]\| < \delta$ for all $\lambda \in E$. By (4.11), $c_0 + \lambda(x - c_0) \in \operatorname{span} C$ so that $f_x(\lambda) \in a\ \operatorname{span} C$, for all $\lambda \in E$. By (4.12), $r(f_x(\lambda)) = 0$ for all $\lambda \in E$. It follows from Corollary 2.3 that $r(f_x(\lambda)) = 0$ for all $\lambda \in \mathbb{C}$ and hence for $\lambda = 1$. This means that $r(ax) = 0$. Since x was arbitrary in A, we have shown that $aA \subset \mathrm{QN}(A)$. ∎

Corollary 4.13. *Let A be an OBA with a normal algebra cone C and suppose that C contains an interior point. If $0 \leq a \leq b$ and $b \in \mathrm{Rad}(A)$, then $a \in \mathrm{Rad}(A)$.*

As a first example we consider the infinite-dimensional but semisimple Banach algebra l^∞ of all bounded sequences of complex numbers:

Example 4.14. *Let $A = l^\infty$ and $C = \{(c_1, c_2, \ldots) \in l^\infty : c_i \geq 0$ for all $i \in \mathbb{N}\}$. Then (A, C) is an ordered Banach algebra, C is normal and $A = \operatorname{span} C$, so that $\operatorname{span} C$ has interior points.*

Proof. By defining multiplication coordinatewise, it follows that A is indeed a Banach algebra, with unit $(1, 1, \ldots)$. Direct calculation shows that C as defined is an algebra cone. Suppose $(0, 0, \ldots) \leq (x_1, x_2, \ldots) \leq (y_1, y_2, \ldots)$ in A. By definition of C this means that $0 \leq x_k \leq y_k$ for all $k \in \mathbb{N}$. Hence $\sup_{k \in \mathbb{N}} |x_k| \leq \sup_{k \in \mathbb{N}} |y_k|$, that is, $\|(x_1, x_2, \ldots)\| \leq \|(y_1, y_2, \ldots)\|$. Choosing $\alpha = 1$ in the definition of normality, we see that C is normal. The fact that each element of A can be written in the form $c_1 - c_2 + ic_3 - ic_4$, where c_1, c_2, c_3, c_4 are elements of C, ensures that $A = \operatorname{span} C$. ∎

We note that since l^∞ is commutative and $\sigma((x_1, x_2, \ldots)) = \{x_1, x_2, \ldots\}$ for $(x_1, x_2, \ldots) \in l^\infty$, it follows that $\mathrm{Rad}(l^\infty) = \mathrm{QN}(l^\infty) = \{0\}$, i.e. l^∞ is semisimple.

To get Banach algebras which are not semisimple we use the 2×2 upper triangular matrices, as in Example 4.8, obtaining a finite-dimensional but not semisimple Banach algebra:

Example 4.15. *Let A be the set of upper triangular 2×2 complex matrices and C the subset of A of matrices with only nonnegative entries. Then (A, C) is an ordered Banach algebra, C is normal and $A = \operatorname{span} C$, so that $\operatorname{span} C$ has interior points.*

Proof. We already know from Example 4.8 that (A, C) is an ordered Banach algebra with C normal. Each element of A can be written as $c_1 - c_2 + ic_3 - ic_4$, with c_1, c_2, c_3, c_4 in C, so that $A = \operatorname{span} C$. ∎

To improve this example by changing the finite dimensionality to the more general infinite dimensionality, we look at the set $l^\infty(A)$ consisting of all "bounded sequences of upper triangular 2×2 complex matrices":

EXAMPLE 4.16. *Let A be the set of upper triangular 2×2 complex matrices, $l^\infty(A)$ the set*

$$\{x=(x_1,x_2,\ldots): x_i\in A \text{ for all } i\in\mathbb{N} \text{ and } \|x_i\|_A\le K_x \text{ for all } i\in\mathbb{N}\},$$

and C the set

$$\{(c_1,c_2,\ldots)\in l^\infty(A): c_i \text{ has only nonnegative entries for all } i\in\mathbb{N}\}.$$

Then $(l^\infty(A),C)$ is an ordered Banach algebra, C is normal and $l^\infty(A)=\operatorname{span}C$, so that $\operatorname{span}C$ has interior points.

Proof. By defining addition, scalar multiplication and vector multiplication coordinatewise, and the norm to be $\|(x_1,x_2,\ldots)\|=\sup_{j\in\mathbb{N}}\|x_j\|$ (where $\|x_j\|$ is the norm of the matrix x_j in A), it can be shown that $l^\infty(A)$ is a normed algebra, with unit $\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\ldots\right)$. Completeness can be shown as for l^∞. Direct calculation shows that C is an algebra cone of $l^\infty(A)$.

Suppose $0\le x\le y$, where

$$0=\left(\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix},\ldots\right),\qquad x=\left(\begin{pmatrix}x_{11}&x_{12}\\0&x_{14}\end{pmatrix},\begin{pmatrix}x_{21}&x_{22}\\0&x_{24}\end{pmatrix},\ldots\right)$$

and

$$y=\left(\begin{pmatrix}y_{11}&y_{12}\\0&y_{14}\end{pmatrix},\begin{pmatrix}y_{21}&y_{22}\\0&y_{24}\end{pmatrix},\ldots\right).$$

By definition of C this means that $0\le x_{jk}\le y_{jk}$ for all $j\in\mathbb{N}$ and $k=1,2,4$. Therefore $\max\{|x_{j1}|+|x_{j2}|,|x_{j4}|\}\le\max\{|y_{j1}|+|y_{j2}|,|y_{j4}|\}$, i.e.

$$\left\|\begin{pmatrix}x_{j1}&x_{j2}\\0&x_{j4}\end{pmatrix}\right\|\le\left\|\begin{pmatrix}y_{j1}&y_{j2}\\0&y_{j4}\end{pmatrix}\right\|,$$

for all $j\in\mathbb{N}$. It follows that

$$\sup_{j\in\mathbb{N}}\left\|\begin{pmatrix}x_{j1}&x_{j2}\\0&x_{j4}\end{pmatrix}\right\|\le\sup_{j\in\mathbb{N}}\left\|\begin{pmatrix}y_{j1}&y_{j2}\\0&y_{j4}\end{pmatrix}\right\|,$$

i.e. $\|x\|\le\|y\|$. Choosing $\alpha=1$ in the definition of normality we deduce that C is normal.

As in the previous example, each element of $l^\infty(A)$ can be written as a linear combination of four algebra cone elements, using the scalars 1, -1, i and $-i$. Hence $l^\infty(A)=\operatorname{span}C$. ■

Since $\left(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\ldots\right)$ is an element of its radical, $l^\infty(A)$ is not semisimple. Furthermore, $l^\infty(A)$ is infinite-dimensional.

Finally we observe that, under each of the assumptions in the previous results, we have a characterization of the radical of A in terms of the algebra cone:

THEOREM 4.17. *Let A be an OBA with a normal algebra cone C and suppose that at least one of the following conditions holds*:

(1) *For every $x \in A$ there is a $0 \neq \lambda \in \mathbb{C}$ such that $\lambda x \in C$.*

(2) *For every $x \in A$ there is a line segment L in $\mathbb{C}$ such that $\lambda x \in C$ for all $\lambda \in L$.*

(3) $A = \operatorname{span} C$.

(4) $A = \overline{\operatorname{span} C}$ *and the spectral radius function r is continuous on A.*

(5) $\operatorname{span} C$ *contains an interior point.*

Then $\operatorname{Rad}(A) = \{a \in A : aC \subset \operatorname{QN}(A)\}$.

Proof. For the nontrivial implication, let $aC \subset \operatorname{QN}(A)$. Then Lemma 4.4 implies that $a \operatorname{span} C \subset \operatorname{QN}(A)$, from which $aA \subset \operatorname{QN}(A)$, i.e. $a \in \operatorname{Rad}(A)$, follows readily in cases (1)–(4).

To prove that $a \in \operatorname{Rad}(A)$ in case (5), suppose that $B(c_0, \delta) \subset \operatorname{span} C$. If $x \in A$, let $\varepsilon_x = \delta/(\|x\| + \|c_0\|)$ and $E = B(0, \varepsilon_x)$. Then $c_0 + \lambda(x - c_0) \in \operatorname{span} C$ for all $\lambda \in E$, so that $r(f_x(\lambda)) = 0$ for all $\lambda \in E$, where $f_x(\lambda) = a(c_0 + \lambda(x - c_0))$. It follows from Corollary 2.3 that $r(f_x(\lambda)) = 0$ for all $\lambda \in \mathbb{C}$. The case $\lambda = 1$ yields $ax \in \operatorname{QN}(A)$. Since $x \in A$ was arbitrary, it follows that $aA \subset \operatorname{QN}(A)$, i.e. $a \in \operatorname{Rad}(A)$. ∎

It is worth noting that all the results in this section will still be valid if the requirement that the algebra cone C is normal is replaced by the weaker condition that the spectral radius is monotone relative to C.

References

[1] C. D. Aliprantis and O. Burkinshaw, *Dunford–Pettis operators on Banach lattices*, Trans. Amer. Math. Soc. 274 (1982), 227–238.

[2] —, —, *On weakly compact operators on Banach lattices*, Proc. Amer. Math. Soc. 83 (1981), 573—587.

[3] —, —, *Positive compact operators on Banach lattices*, Math. Z. 174 (1980), 289–298.

[4] B. Aupetit, *A Primer on Spectral Theory*, Springer, New York, 1991.

[5] V. Caselles, *On the peripheral spectrum of positive operators*, Israel J. Math. 58 (1987), 144–160.

[6] P. G. Dodds and D. H. Fremlin, *Compact operators in Banach lattices*, ibid. 34 (1979), 287–320.

[7] W. K. Hayman and P. B. Kennedy, *Subharmonic Functions I*, London Math. Soc. Monogr. 9, Academic Press, London, 1976.

[8] S. Mouton (née Rode) and H. Raubenheimer, *More spectral theory in ordered Banach algebras*, Positivity 1 (1997), 305–317.

[9] H. Raubenheimer and S. Rode, *Cones in Banach algebras*, Indag. Math. (N.S.) 7 (4) (1996), 489–502.

[10] A. C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983.

H. du T. Mouton
Department of Electrical and Electronic Engineering
University of Stellenbosch
Private Bag X1
Matieland 7602, South Africa
E-mail: dtmouton@ing.sun.ac.za

S. Mouton
Department of Mathematics
University of Stellenbosch
Private Bag X1
Matieland 7602, South Africa
E-mail: smo@akad.sun.ac.za

Received November 20, 2000
Revised version April 17, 2001 (4640)