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Sequences of 0's and 1's

by

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Abstract. We investigate the extent to which sequence spaces are determined by the sequences of 0's and 1's that they contain.

1. Introduction. Given a sequence space E we denote by $\chi(E)$ the linear span of the sequences of 0's and 1's contained in E and we ask to what extent $\chi(E)$ determines E.

Several degrees of precision are possible, as described below, but the sharpest (and most interesting) formulation of our problem is to ask whether

$$\chi(E) \subseteq F \;\Rightarrow\; E \subseteq F \tag{1.1}$$

whenever F is an arbitrary FK-space. We say then that E has the *Hahn property* (or that E is a *Hahn space*). If E itself is an FK-space, then (1.1) implies that E is the smallest such space containing $\chi(E)$.

The theory of FK-spaces may be found in [36] or [39], but its essential features are described in Section 2. We merely recall here that the most commonly encountered sequence spaces of Analysis are all FK-spaces, so that (1.1) is indeed a sensible interpretation of our requirement that E be determined by $\chi(E)$.

The Hahn property is, in a sense, completely understood, at least for FK-spaces:

THEOREM 1.1 ([7], Theorem 1). *Let E be an FK-space. Then the following conditions are equivalent*:

(i) *E has the Hahn property*;
(ii) *$\chi(E)$ is a dense*, *barrelled subspace of E.*

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Yet there is only one application of the theorem in the literature:

Corollary 1.2 ([7], Corollary to Theorem 1). *ℓ^∞ has the Hahn property.*

The problem with Theorem 1.1 is that its hypotheses are difficult to check. The density of $\chi(E)$ in E is obvious in the corollary, but it may be far less transparent in other cases ($bs + c$, for example). The real difficulty, however, lies in checking that $\chi(E)$ is barrelled in E. This is a non-trivial task even when $E = \ell^\infty$. The solution in that case, due independently to Dieudonné [14, 15], Darst [13], Seever [32], and Rosenthal [30], [31], requires an extension to finitely additive set functions of a celebrated theorem of Nikodym ([37], 14.4.11). Nikodym's result, in turn, has been hailed by Dunford and Schwartz ([16], page 309) as a "striking improvement of the principle of uniform boundedness".

The basic properties of Hahn spaces are described in Section 2. In particular, it is shown there that any FK-space with the Hahn property must be non-separable.

Sections 3 and 4 provide examples of spaces with the Hahn property. Our results contain new versions of the uniform boundedness principle (courtesy of Theorem 1.1) and it is an intriguing exercise to try to prove these from scratch. See problems 8 and 9 in Section 7.

In Section 5 we investigate the so-called matrix Hahn and separable Hahn properties. These are weaker than the Hahn property itself, and hence are more generally applicable, yet they maintain the same philosophy: "if certain spaces contain many sequences of 0's and 1's, they must also contain...".

Applications are given in Section 6 and a list of problems is discussed in Section 7.

2. Notation and preliminary results. We denote by ω the space of all real-valued sequences, and any vector subspace of ω is called a *sequence space*. An *FK-space* is a sequence space endowed with a complete, metrizable, locally convex topology under which the coordinate mappings $x \mapsto x_k$ ($k = 1, 2, \ldots$) are all continuous.

Familiar examples of FK-spaces are ℓ^∞ (bounded sequences), with the sup norm $\| \|_\infty$, and its closed subspaces c (convergent sequences) and c_0 (null sequences); ℓ^p, $1 \le p < \infty$ (absolutely p-summable sequences), with its usual norm; and ω under the topology of coordinatewise convergence. The space bs of *bounded series*, defined by

$$bs = \Big\{ x \in \omega : \sup_n \Big| \sum_{k=1}^n x_k \Big| < \infty \Big\},$$

is another example, one that plays a decisive and unexpected role in our investigations.

Not all sequence spaces can be equipped with an FK-topology. The simplest such example is φ, the space of all sequences of *finite support*; $\chi(\ell^\infty)$ is another.

A fundamental property of FK-spaces (courtesy of the closed graph theorem) is that their topologies are *monotonic*: if $E \subseteq F$, then E is continuously embedded in F. This means that a sequence space can have at most one FK-topology, and we take advantage of this fact, on several occasions, by not actually specifying the topology under consideration.

PROPOSITION 2.1. *If E_α is a Hahn space for each α in some index set $\mathcal{A}$, then $\sum_{\alpha\in\mathcal{A}} E_\alpha$ is also a Hahn space.*

PROPOSITION 2.2. *If E is a Hahn space, then $E \subseteq \ell^\infty$ and $\chi(E)$ is dense in E with respect to the sup norm topology.*

Proof. To see that $E \subseteq \ell^\infty$, we apply (1.1) with $F = \ell^\infty$. To check the density, we apply (1.1) again, with F replaced by the closure of $\chi(E)$ in ℓ^∞. ■

There are various "algebraic" notions of duality in sequence space theory that do not depend on the presence of an underlying topology. We shall work with just one of these, the β-dual, though our results have obvious analogues for the others (α- and γ-duals).

The *β-dual* of a sequence space E is defined by

$$E^\beta = \Big\{y \in \omega : \sum_k x_k y_k \text{ converges for all } x \in E\Big\}.$$

If E is one-dimensional, say $E = \mathrm{Sp}\{x\}$, we shall write x^β in place of $(\mathrm{Sp}\{x\})^\beta$. Obviously, x^β is an FK-space with seminorms defined by

$$y \mapsto |y_k|, \quad k = 1, 2, \ldots, \quad \text{and} \quad y \mapsto \sup_n \Big|\sum_{k=1}^n x_k y_k\Big|.$$

PROPOSITION 2.3. *If E is a Hahn space, then $\chi(E)^\beta = E^\beta$.*

Proof. If $\chi(E)^\beta \neq E^\beta$, we may choose $x \in \chi(E)^\beta \setminus E^\beta$. But then x^β is an FK-space which contains $\chi(E)$ and not E, so that E cannot be a Hahn space. ■

LEMMA 2.4. *Let F be a sequence space containing φ and suppose that F is the linear span of a countable set of sequences. Then F is an intersection of FK-spaces.*

Proof. We may suppose that F has the form $F = \mathrm{Sp}\{a^{(1)}, a^{(2)}, \ldots\}$, where

$$a_k^{(n)} = \begin{cases} 0 & \text{if } k < n, \\ 1 & \text{if } k = n, \end{cases}$$

since $F \supseteq \varphi$. The matrix A defined by $a_{nk} = a_n^{(k)}$ is lower triangular with non-zero diagonal entries and so has a unique two-sided inverse. It follows that

$$F = A(\varphi) = A\Big(\bigcap_{x\in\omega} x^\beta\Big) = \bigcap_{x\in\omega} A(x^\beta),$$

and the last expression is an intersection of FK-spaces. ∎

THEOREM 2.5. *Let E be an FK-space containing φ. If E has the Hahn property*, *then E is non-separable.*

Proof. $E \subseteq \ell^\infty$ by Proposition 2.2, and it suffices, via the monotonicity of FK-topologies, to show that E is non-separable in the sup norm.

Suppose not. Then E contains only countably many sequences of 0's and 1's (since such sequences are pairwise separated by sup norm distance 1). It follows from Lemma 2.4 that $\chi(E)$ is an intersection of FK-spaces, and then, from (1.1), that $\chi(E) = E$. But this is impossible since

$$\begin{aligned} \chi(E) &\subseteq \chi(\ell^\infty) \cap E = \{x \in E : x \text{ has finite range}\} \\ &= \bigcup_{n=1}^{\infty} \{x \in E : \text{range of } x \text{ has cardinality } n\} \end{aligned} \tag{2.1}$$

and the last set is of first category in E. ∎

It is important to realize that the inclusion (2.1) may be strict. This is seen, for example, by taking $E = bs$, and by observing that the sequence $(1, -1, 1, -1, \ldots)$ belongs to $\chi(\ell^\infty) \cap E$ but not to $\chi(E)$. A partial converse to (2.1) is possible, however, and this is given in Lemma 3.2 below.

3. Big Hahn spaces. In this section we exhibit a large class of sequence spaces having the Hahn property. Such spaces, in view of Theorem 2.5, must be "big" subsets of ℓ^∞, and it makes sense, while searching for examples, to ask whether "bigness" implies "Hahn". Our main result, Theorem 3.4 below, gives a very satisfactory answer to this question.

PROPOSITION 3.1. *If E is a sequence space satisfying*

$$bs + \mathrm{Sp}\{e\} \subseteq E \subseteq \ell^\infty,$$

where $e = (1, 1, \ldots, 1)$, then, *in fact*,

$$E = bs + \chi(E).$$

Proof. Let $x \in E$ be given. We construct a sequence $y \in \chi(E)$ such that $z = x - y \in bs$. By adding a suitable constant sequence to x (and then to y), we may assume that $x_k \geq 0$ for $k = 1, 2, \ldots$ Furthermore, multiplying x by a suitable scalar (and doing the same for y and z) allows us to assume that $0 \leq x_k < 1$ for $k = 1, 2, \ldots$ Now take $y_1 = 0$ and define $z_1, y_2, z_2, y_3, \ldots$ inductively as follows:

$$z_k = x_k - y_k \quad \text{and} \quad y_{k+1} = \begin{cases} 1 & \text{if } \sum_{j=1}^k y_j < \sum_{j=1}^k x_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $z \in bs$, in fact $\|z\|_{bs} \leq 1$, and $y = x - z \in E + bs = E$, so that $y \in \chi(E)$. ■

Lemma 3.2. *If E is a sequence space containing bs, and $x \in E$ takes only the values $\{0, 1, \ldots, N\}$, then $x \in \chi(E)$.*

Proof. Let $U_j = \{\nu : x_\nu = j\}$ for $j = 0, 1, \ldots, N$ and let L be the least common multiple of $\{1, \ldots, N\}$. Define sets $V_1, \ldots, V_L$ as follows:

the nth member of U_j (in the usual ordering of the integers) goes into V_k if and only if $k \equiv n \bmod (L/j)$.

(Some members go into more than one V_k, and the process terminates if U_j is finite.) From the construction it follows that

$$\sum_{n=1}^m \chi_{U_j \cap V_k}(n) \quad \text{differs from} \quad \frac{j}{L} \sum_{n=1}^m \chi_{U_j}(n)$$

by no more than 1 ($m = 1, 2, \ldots$). Summing over j, we deduce that

$$\sum_{n=1}^m \chi_{V_k}(n) \quad \text{differs from} \quad \frac{1}{L} \sum_{n=1}^m x_n$$

by no more than N ($m = 1, 2, \ldots$), since $x = \sum_{j=1}^N j \chi_{U_j}$. In other words, $\chi_{V_k} - (1/L)x \in bs$, so that $\chi_{V_k} \in E$, and hence $x \in \chi(E)$, since $x = \sum_{k=1}^L \chi_{V_k}$. ■

Our next result provides a representation of the space $\chi(\ell^\infty)$ in terms of the *partial sums operator*, Σ:

$$\Sigma x = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots). \tag{3.1}$$

It is obvious that $\Sigma(E)$ is an FK-space precisely when E is.

Lemma 3.3. $\chi(\ell^\infty) \subseteq \Sigma(\chi(bs + \mathrm{Sp}\{e\}))$.

Proof. Suppose x is an arbitrary sequence of 0's and 1's. The sequence y defined by

$$y = \Sigma^{-1} x = (x_1, x_2 - x_1, x_3 - x_2, \ldots) \tag{3.2}$$

belongs to bs and takes only the values $-1,0,1$. It follows from Lemma 3.2 that

$$y+e\in\chi(bs+\mathrm{Sp}\{e\})$$

and consequently that

$$y\in\chi(bs+\mathrm{Sp}\{e\}).$$

Therefore

$$x\ (=\Sigma y)\in\Sigma\chi(bs+\mathrm{Sp}\{e\}).\ \blacksquare$$

THEOREM 3.4. *If E is a sequence space satisfying*

$$bs+\mathrm{Sp}\{e\}\subseteq E\subseteq\ell^{\infty},$$

then E has the Hahn property.

Proof. Proposition 3.1 shows that

$$E=bs+\mathrm{Sp}\{e\}+\chi(E).$$

Since $\chi(E)$ certainly has the Hahn property, it suffices, in view of Proposition 2.1, to show that so does $bs+\mathrm{Sp}\{e\}$. Suppose, then, that F is an FK-space containing $\chi(bs+\mathrm{Sp}\{e\})$; we must show that F contains all of $bs+\mathrm{Sp}\{e\}$. For this it is sufficient to check that $bs\subseteq F$, or, what is equivalent, that $\ell^{\infty}\subseteq\Sigma(F)$. But this last assertion follows from Corollary 1.2, it being plain, from Lemma 3.3, that $\chi(\ell^{\infty})\subseteq\Sigma(F)$. $\blacksquare$

As an immediate consequence of Theorems 1.1 and 3.4 we get:

COROLLARY 3.5. *Let E be any FK-space with $bs+\mathrm{Sp}\{e\}\subseteq E\subseteq\ell^{\infty}$. Then $\chi(E)$ is both dense and barrelled in E.*

An interesting illustration of Theorem 3.4 is provided by the space of all bounded *Cesàro limitable sequences*:

$$E_1=\left\{x\in\ell^{\infty}:\lim_{n\to\infty}\frac{x_1+\ldots+x_n}{n}\text{ exists}\right\}.\tag{3.3}$$

The sequences of 0's and 1's in E_1 form a particularly important class, for they may be identified in an obvious way with the subsets of the positive integers having a *natural density*. Theorem 3.4 asserts that E_1 is the smallest FK-space containing this class.

It is instructive to consider also the bounded *Cesàro-null sequences*:

$$E_2=\left\{x\in\ell^{\infty}:\lim_{n\to\infty}\frac{x_1+\ldots+x_n}{n}=0\right\}.\tag{3.4}$$

The space is not covered by Theorem 3.4, and, indeed, it *fails* to have the Hahn property. To see this, we have only to consider

$$F=\left\{x\in\ell^{\infty}:\lim_{n\to\infty}\frac{|x_1|+\ldots+|x_n|}{n}=0\right\},\tag{3.5}$$

the space of bounded *strongly Cesàro-null sequences*, which is an FK-space containing $\chi(E_2)$ but not E_2.

The two examples, E_1 and E_2, differ by only a single sequence, e, and they serve to illustrate the delicate nature of Theorem 3.4.

4. More Hahn spaces

THEOREM 4.1. *If E is a sequence space satisfying*

$$E = \ell^\infty \cdot \chi(E) + c_0, \tag{4.1}$$

then E has the Hahn property if and only if $\chi(E)^\beta = \ell^1$.

Proof. The necessity of $\chi(E)^\beta = \ell^1$ follows from Proposition 2.3, it being clear from (4.1) that $E^\beta = \ell^1$.

To prove sufficiency, we assume that an FK-space F with

$$\chi(E) \subseteq F \tag{4.2}$$

is given and we deduce that

$$E \subseteq F. \tag{4.3}$$

This is done by considering separately the two component pieces, $\ell^\infty \cdot \chi(E)$ and c_0, of E.

To check

$$\ell^\infty \cdot \chi(E) \subseteq F \tag{4.4}$$

we have only to observe that

$$\chi\big(\ell^\infty \cdot \chi(E)\big) \subseteq F \tag{4.5}$$

and that

$$\ell^\infty \cdot \chi(E) \text{ has the Hahn property.} \tag{4.6}$$

Observation (4.5) is a consequence of (4.1) and (4.2); (4.6) follows from Proposition 2.1 by expressing $\ell^\infty \cdot \chi(E)$ as a sum,

$$\ell^\infty \cdot \chi(E) = \sum \ell^\infty \cdot x,$$

the summation being taken over all sequences of 0's and 1's in E. (That each $\ell^\infty \cdot x$ is a Hahn space follows from Corollary 1.2 by restricting attention to sequence spaces "defined on the support of x".)

Our second inclusion is proved by a standard duality argument, it being known that

$$c_0 \subseteq F \tag{4.7}$$

is equivalent to

$$\sum_{k=1}^{\infty} |f(e^k)| < \infty \quad \text{whenever } f \in F', \tag{4.8}$$

where e^k $(k = 1, 2, \ldots)$ is the sequence $(0, \ldots, 0, 1, 0, \ldots)$ with the 1 in the kth position. (See [6], Proposition 5, [35], p. 598, or [36], p. 138.)

For each $y \in \chi(E)$, $\ell^\infty \cdot y$ is a closed subspace of ℓ^∞ and $\ell^\infty \cdot y \subseteq F$ by (4.4). It follows by the monotonicity of FK-topologies that the restriction to $\ell^\infty \cdot y$ of any $f \in F'$ is sup norm continuous. Hence

$$\Big|\sum_{k=1}^{n} \pm y_k f(e^k)\Big| = |f(\pm y_1, \ldots, \pm y_n, 0, \ldots)| \le \|f|_{\ell^\infty \cdot y}\| \cdot \|y\|_\infty.$$

The upper bound is independent of n, and of the choice of signs, so that

$$\sum_{k=1}^{\infty} |y_k f(e^k)| < \infty.$$

Thus $(f(e^k)) \in \chi(E)^\beta = \ell^1$, and it follows from (4.8) that (4.7) must hold. ■

By combining the following two beautiful results from classical summability theory we obtain an interesting illustration of Theorem 4.1. (Other examples are given in Section 6.)

BUCK'S THEOREM ([12], Theorem 2.4). *A bounded sequence x satisfies $(|x_1| + \ldots + |x_n|)/n \to 0$ precisely when there exists a subset, Z, of the positive integers, of zero density, such that*

$$\lim_{\substack{k\to\infty \\ k\notin Z}} x_k = 0.$$

AGNEW'S THEOREM ([2], Theorem 2). *If $\sum_{k=1}^{\infty} a_{n_k}$ converges whenever $k/n_k \to 0$, then $\sum_{n=1}^{\infty} |a_n| < \infty$.*

COROLLARY 4.2. *The space of bounded strongly Cesàro-null sequences,*

$$\Big\{x \in \ell^\infty : \frac{|x_1| + \ldots + |x_n|}{n} \to 0\Big\}, \tag{4.9}$$

has the Hahn property.

Proof. We denote by E the space (4.9). Each $x \in E$ may be represented in the form

$$x = x \cdot \chi_Z + x \cdot \chi_{\mathbb{N}\setminus Z},$$

and Buck's choice of Z shows that

$$x \in \ell^\infty \cdot \chi(E) + c_0.$$

Agnew's theorem, on the other hand, tells us that $\chi(E)^\beta = \ell^1$. ■

5. Weaker Hahn-type properties. We say that a sequence space E has the *separable Hahn property* if (1.1) holds whenever F is a *separable* FK-space.

This concept is less satisfactory than the Hahn property itself because it does not (seem to) admit a purely functional analytic interpretation like that exhibited in Theorem 1.1. The two concepts, however, share the same philosophy, and the new one, as we shall see, is more widely applicable.

Indeed, significant results are obtainable even when F is further restricted to summability domains. It matters not whether we consider *variational domains*, bv_A ([33], [4]), *absolute summability domains*, ℓ^p_A ([3], [4]), or ordinary domains, c_A, the results are the same. For the sake of definiteness, however, we shall restrict attention to c_A.

Given an infinite matrix $A = (a_{nk})$, the *convergence domain*, c_A, is defined by

$$c_A = \Big\{x \in \omega : Ax = \Big(\sum_{k=1}^{\infty} a_{nk}x_k\Big)_n \text{ and } \lim(Ax) \text{ exist}\Big\}. \tag{5.1}$$

The space c_A is an FK-space when topologized by means of the seminorms

$$\begin{aligned} &x \mapsto |x_k| && (k = 1, 2, \ldots), \\ &x \mapsto \sup_m \Big|\sum_{k=1}^{m} a_{nk}x_k\Big| && (n = 1, 2, \ldots), \\ &x \mapsto \sup_n \Big|\sum_{k=1}^{\infty} a_{nk}x_k\Big| \end{aligned}$$

([36], Theorem 4.3.13) and it is separable in this topology ([36], Theorem 16.2.1).

The *null domain* of A,

$$(c_0)_A = \{x \in c_A : Ax \in c_0\}, \tag{5.2}$$

and the *strong domain*

$$\Big\{x \in \omega : \exists l,\ \lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk}| \cdot |x_k - l| \text{ exists}\Big\} \tag{5.3}$$

will also play a role in what follows. (These spaces have already been encountered in Section 3, A being restricted therein to the *Cesàro matrix*,

$$a_{nk} = \begin{cases} 1/n & \text{if } 1 \le k \le n, \\ 0 & \text{if } k > n \end{cases} \qquad (n, k = 1, 2, \ldots).) \tag{5.4}$$

We say that a sequence space E has the *matrix Hahn property* if

$$\chi(E) \subseteq c_A \ \Rightarrow\ E \subseteq c_A \tag{5.5}$$

for every matrix A. It is clear that

Hahn property $\Rightarrow$ separable Hahn property $\Rightarrow$ matrix Hahn property and that Proposition 2.1 holds for all three. In fact, all the results of Section 2 are preserved, with the possible exception of the density clause of Proposition 2.2. These observations are summarized in

THEOREM 5.1. *If E is a matrix Hahn space, then $E \subseteq \ell^\infty$ and $\chi(E)^\beta = E^\beta$. If, in addition, E is an FK-space containing φ, then E must be non-separable.*

Proof. To see that $E \subseteq \ell^\infty$ we proceed as in Proposition 2.2, noting that ℓ^∞ is an intersection of convergence domains. To check that $\chi(E)^\beta = E^\beta$ we follow the proof of Proposition 2.3, noting that x^β is a convergence domain, $x^\beta = c_{B(x)}$ (say), where

$$B(x) = \begin{pmatrix} x_1 & & & \\ x_1 & x_2 & & \\ x_1 & x_2 & x_3 & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The last part of the theorem calls for an improvement in Lemma 2.4: that F, in fact, may be expressed as an intersection of convergence domains. We had

$$F = \bigcap_{x \in \omega} A(x^\beta).$$

But

$$A(x^\beta) = A(c_{B(x)}) = c_{B(x)A^{-1}},$$

the last identity being valid since A and $B(x)$ are both row-finite matrices. It follows that F is an intersection of convergence domains:

$$F = \bigcap_{x \in \omega} c_{B(x)A^{-1}}. \quad \blacksquare$$

Our next result shows that the matrix Hahn and separable Hahn properties are equivalent for a large class of spaces.

We recall that a sequence space E is said to be *solid* if $\ell^\infty \cdot E \subseteq E$, and *monotone* if just $\chi(\ell^\infty) \cdot E \subseteq E$. (The ℓ^p-spaces, for example, are all solid, while $\chi(\ell^\infty)$ is monotone, but not solid.)

THEOREM 5.2. *Suppose that E is a monotone sequence space containing φ. Then the following conditions are equivalent*:

(i) *E has the matrix Hahn property*;
(ii) *E has the separable Hahn property*;
(iii) $\chi(E)^\beta = E^\beta$.

Proof. We discuss only the implication (iii)$\Rightarrow$(ii). To do this, we suppose that a separable FK-space, F, with $\chi(E) \subseteq F$ is given, and we deduce that $E \subseteq F$.

Now $\chi(E)$ is a monotone sequence space (since E is) so that $\chi(E)^\beta$ is $\sigma(\chi(E)^\beta, \chi(E))$-sequentially complete by Proposition 3 of [4]. It follows from Kalton's closed graph Theorem ([22], Theorem 2.4) that the natural injection

$$i : (\chi(E), \tau(\chi(E), \chi(E)^\beta)) \to F$$

is continuous. But $(E, \tau(E, E^\beta))$ is an AK-space since E is monotone ([4], Proposition 2). In particular, $\chi(E)$ is dense in $(E, \tau(E, E^\beta))$ and therefore $\tau(E, E^\beta)|_{\chi(E)} = \tau(\chi(E), E^\beta) = \tau(\chi(E), \chi(E)^\beta)$. Thus for each $x \in E$ we have $(x_1, \ldots, x_n, 0, \ldots) \to x$, implying, because i is continuous, that $((x_1, \ldots, x_n, 0, \ldots))_n$ is a Cauchy sequence in F with only possible limit x. Consequently, $E \subseteq F$. ■

Theorem 5.2 enables us to show that the separable Hahn property does not imply the Hahn property.

THEOREM 5.3. *Let $s = (s_n)$ be a sequence of positive integers with $s_1 = 1$ and $(s_{n+1} - s_n)$ unbounded. Then the space*

$$E = \Big\{x \in \omega : \|x\|_E = \sup_n \sum_{k=s_n}^{s_{n+1}-1} |x_k| < \infty\Big\}$$

has the separable Hahn property, but not the Hahn property.

(These spaces are discussed in some detail in [4]; they have interesting applications in classical summability theory.)

Proof. E is a solid sequence space containing φ, and $\chi(E)^\beta = E^\beta$ since both dual spaces coincide with $\{y \in \omega : \sum_n \max_{s_n \le k < s_{n+1}} |y_k| < \infty\}$. It follows from Theorem 5.2 that E has the separable Hahn property.

On the other hand, E is an FK-space under the indicated norm, and $\chi(E)$, as we shall see, fails to be dense in E. Theorem 1.1 then implies that E cannot be a Hahn space. To see that $\chi(E)$ is not dense in $(E, \| \ \|_E)$, we consider the sequence x defined by

$$x_k = \frac{1}{s_{n+1} - s_n} \quad \text{if } s_n \le k < s_{n+1} \ (n = 1, 2, \ldots).$$

It is clear that $x \in E$ and we shall show that $\|x - y\|_E \ge 1$ for every $y \in \chi(E)$. If $y = 0$, then obviously

$$\|x - y\|_E = \|x\|_E = 1.$$

On the other hand, if $y \ne 0$, we set $\alpha = \min_{y_k \ne 0} |y_k| > 0$ and choose n so large that $s_{n+1} - s_n > 1/\alpha$. If $s_n \le k < s_{n+1}$, then

$$|x_k - y_k| \begin{cases} = |x_k| = \dfrac{1}{s_{n+1} - s_n} & \text{if } y_k = 0, \\ \geq |y_k| - |x_k| \geq \dfrac{1}{s_{n+1} - s_n} & \text{if } y_k \neq 0. \end{cases}$$

In either case we have

$$\|x - y\|_E \geq \sum_{s_n \leq k < s_{n+1}} |x_k - y_k| \geq 1. \ \blacksquare$$

REMARK 5.4. If the sequence s in Theorem 5.3 is *geometric*, say $s_n = r^{n-1}$ $(n = 1, 2, \ldots)$, then the space E does not depend on the actual value of r (> 0). The sequences of 0's and 1's in E may then be identified with the subsets of the positive integers that are *lacunary in the sense of Hadamard* ([40], pp. 131 ff.).

Theorem 5.3 contains several interesting summability results; the following corollary is typical.

COROLLARY 5.5. *A matrix* $A = (a_{nk})$ *satisfies*

$$\lim_{n\to\infty} \sum_{k\in H} a_{nk} = 0 \tag{5.6}$$

whenever H *is an Hadamard lacunary subset of* $\mathbb{N}$, *if and only if*

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} \max_{2^m \leq k < 2^{m+1}} |a_{nk}| = 0. \tag{5.7}$$

Proof. (5.6) is equivalent to the assertion that

$$\chi_H \in (c_0)_A$$

whenever H is lacunary in the sense of Hadamard. It follows from Theorem 5.3 that

$$E \subseteq (c_0)_A$$

where

$$E = \Big\{x \in \omega : \sup_m \sum_{k=2^m}^{2^{m+1}-1} |x_k| < \infty\Big\}.$$

The rows of A, $\{a^{(n)} : n \in \mathbb{N}\}$, are therefore $\sigma(E^\beta, E)$-convergent to zero in E^β. But E^β is a Schur space ([4], corollary to Theorem 16) so that $a^{(n)} \to 0$ in the norm of E^β, and this is equivalent to (5.7). $\blacksquare$

6. Examples. We list here several concrete illustrations of our results.

(A) *Hahn's theorem.* The prototypical example is due to Hahn ([20], Satz Vb):

$$\chi(\ell^\infty) \subseteq c_A \ \Rightarrow\ \ell^\infty \subseteq c_A, \tag{6.1}$$

and our terminology has been chosen in his honor. His result asserts that ℓ^∞ has the matrix Hahn property; it is superseded by Corollary 1.2.

(B) *Cesàro summability.* Kuttner and Maddox ([23], Theorem 2) show that the space of bounded Cesàro-convergent sequences, (3.3), has the matrix Hahn property. Their result is superseded by Theorem 3.4. It must be said, however, that the present paper owes much to the incisive analysis of [23].

(C) *Structure of certain sequence spaces.*

(6.2) *A bounded sequence may be expressed as a finite linear combination of sequences of* 0*'s and* 1*'s plus a bounded series.*

This is a special case of Proposition 3.1. The proposition itself gave us the first hint that *bs* had a significant role to play in the study of Hahn spaces.

(D) *Almost convergent sequences.* We recall that a sequence x is said to be *almost convergent* provided that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m}^{m+n-1} x_k \text{ exists uniformly in } m. \tag{6.3}$$

The space, *ac*, of almost convergent sequences was introduced by Lorentz [26]. He showed that all Banach limits coincide at x precisely when (6.3) holds. It is clear that

$$bs + \mathrm{Sp}\{e\} \subseteq ac \subseteq \ell^\infty \tag{6.4}$$

so that

(6.5) *ac has the Hahn property,*

courtesy of Theorem 3.4.

Lorentz [26] calls a summability matrix, A, *strongly regular* provided that $ac \subseteq c_A$ and $\lim(Ax) = l$ whenever x is almost convergent to l.

(E) *Strongly conservative matrices.* Bennett [5] and Kuttner–Parameswaran [24] show that the bounded convergence domain of any *strongly conservative* matrix, i.e., a matrix A with $ac \subseteq c_A$, has the matrix Hahn property:

$$\left.\begin{aligned} ac &\subseteq c_A \\ \chi(c_A) &\subseteq c_B \end{aligned}\right\} \Rightarrow \ell^\infty \cap c_A \subseteq c_B. \tag{6.6}$$

Theorem 3.4 allows us to weaken their hypotheses and strengthen their conclusion:

$$\left.\begin{aligned} &bs \subseteq c_A,\ e \in c_A \\ &\chi(c_A) \subseteq F \end{aligned}\right\} \Rightarrow \ell^\infty \cap c_A \subseteq F, \tag{6.7}$$

F being an arbitrary FK-space. In particular, we have

THEOREM 6.1. *The bounded convergence domain of any strongly conservative matrix is a Hahn space.*

One of the deepest results about general, regular matrix transformations is the celebrated *bounded consistency theorem* ([27], [28], [11]; for more details see, for instance, [9], p. 81):

(6.8) *If A and B are regular matrices with $\ell^\infty \cap c_A \subseteq c_B$, then* $\lim(Bx) = \lim(Ax)$ *whenever $x \in \ell^\infty \cap c_A$.*

An amusing consequence of (6.6) is the following result.

(F) *The* $0,1$*-consistency theorem.*

(6.9) *If A is a strongly regular matrix and B is consistent with A on the A-limitable sequences of 0's and 1's, then $\ell^\infty \cap c_A \subseteq c_B$ and B is consistent with A on all of $\ell^\infty \cap c_A$.*

Proof. We are assuming that

$$\chi(c_A) \subseteq c_B \tag{6.10}$$

and

$$\lim(Bx) = \lim(Ax) \quad \text{whenever } x \in \chi(c_A). \tag{6.11}$$

Since $\ell^\infty \cap c_A$ has the matrix Hahn property (by (6.6)), it follows from (6.10) that

$$\ell^\infty \cap c_A \subseteq c_B \tag{6.12}$$

and then from (6.11) and (6.12) that

$$B \text{ is regular.} \tag{6.13}$$

We deduce from (6.12), (6.13) and the bounded consistency theorem, (6.8), that

$$\lim(Bx) = \lim(Ax) \quad \text{whenever } x \in \ell^\infty \cap c_A. \ \blacksquare$$

(G) *Strong almost convergence.* Freedman and Sember ([34], Corollary 6.3) show that the space of *strongly almost-null* sequences,

$$|ac|_0 := \left\{ x \in \omega : \lim_{n\to\infty} \sup_m \frac{1}{n} \sum_{k=m}^{m+n-1} |x_k| = 0 \right\}, \tag{6.14}$$

has the matrix Hahn property. $|ac|_0$ is solid and so, by Theorem 5.2, it must also have the separable Hahn property. We have been unable to decide whether $|ac|_0$ is a Hahn space.

(H) *Strong summability.* Freedman and Sember ([34], Corollary 6.2) show that the space of bounded strongly Cesàro-null sequences, (4.9), has the matrix Hahn property. Their result is superseded by Corollary 4.2.

If A is any regular matrix with non-negative entries the *strong summability domain* of A is defined by

$$\Big\{x \in \omega : \exists l \in \mathbb{R},\ \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}|x_k - l| = 0\Big\}. \tag{6.15}$$

Hill and Sledd ([21], Theorem 4.1) have extended Buck's theorem (Section 4) to this general setting:

A bounded sequence x satisfies

$$\sum_{k=1}^{\infty} a_{nk}|x_k| \to 0 \qquad \textit{as } n \to \infty$$

precisely when there exists a subset Z of the positive integers, of zero A-density,

$$\lim_{n\to\infty} \sum_{k\in Z} a_{nk} = 0,$$

such that

$$\lim_{\substack{k\to\infty \\ k\notin Z}} x_k = 0.$$

Their theorem leads to the following generalization of Corollary 4.2.

THEOREM 6.2. *Suppose that A is a strongly regular matrix with non-negative entries. Then the space of bounded strongly A-null sequences*,

$$|A|_0 := \Big\{x \in \ell^\infty : \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}|x_k| = 0\Big\}, \tag{6.16}$$

has the Hahn property. The same is true of the bounded strongly A-limitable sequences,

$$|A| := \Big\{x \in \ell^\infty : \exists l \in \mathbb{R},\ \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}|x_k - l| = 0\Big\}. \tag{6.17}$$

Proof. We denote by E the space (6.16). Each $x \in E$ may be represented in the form

$$x = x \cdot \chi_Z + x \cdot \chi_{\mathbb{N}\setminus Z}$$

(since E is solid), and the Hill–Sledd choice of Z shows that

$$x \in \ell^\infty \cdot \chi(E) + c_0,$$

so that E is a "small" space,

$$E = \ell^\infty \cdot \chi(E) + c_0. \tag{6.18}$$

On the other hand, E is not too small, for it certainly contains enough sequences of zeros and ones to guarantee that

$$\chi(E)^\beta = \ell^1. \tag{6.19}$$

It follows from (6.18), (6.19) and Theorem 4.1 that E is a Hahn space. To check that (6.19) holds, we have only to observe that $|ac|_0 \subseteq E$ (since A is strongly regular) and that $\chi(|ac|_0)^\beta = \ell^1$ ([34], Proposition 1–3).

That $|A| = |A|_0 \oplus \langle e \rangle$ has the Hahn property now follows from Proposition 2.1. ∎

(I) *Ordinary summability.* Strong regularity is not needed in Theorem 6.1. This is seen most easily by the following construction due to Zeller [38]. He produces a regular matrix A with the property that $x \in c_A$ precisely when

$$\frac{1}{n}\sum_{k=1}^{n} |x_k - l| \to 0 \quad \text{as } n \to \infty, \tag{6.20}$$

where $l = \lim(Ax)$. It follows from Theorem 6.2 that $\ell^\infty \cap c_A$ has the Hahn property, yet A is not strongly regular. [The sequence $(x_n) = ((-1)^n)$, for instance, is absent from (6.20) and hence from c_A.]

(J) *A curious summability matrix.* One of the most remarkable results in all of summability theory (see [28]) is this:

A regular matrix A limits no bounded divergent sequences ($\ell^\infty \cap c_A = c$) or else it limits many ($\ell^\infty \cap c_A$ is a non-separable subspace of ℓ^∞).

There is thus the suggestion that any bounded convergence domain, if not trivial ($\ell^\infty \cap c_A \neq c$), might be a Hahn space. This idea is certainly reinforced by (I). We give here a counter-example of a rather extreme type: a non-trivial bounded convergence domain which contains *no* divergent sequences of 0's and 1's. Let A be the regular, tri-diagonal matrix given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ -1 & 1 & 0 & 0 & 0 & \dots \\ 1 & -1 & 1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A limits the sequence

$$x = (1, 2, 2, 1, 0, 0, 1, 2, 2, 1, 0, 0, \dots)$$

of period 6; indeed, $Ax = e$, so that

$$x \in \chi(\ell^\infty) \cap c_A.$$

On the other hand, A limits no divergent sequence of 0's and 1's. To see this, let y be any such sequence. Then y contains the patterns

$$0 \quad 1 \qquad \text{and} \qquad 1 \quad 0$$

infinitely often, and so also the patterns

$$0 \quad 1 \quad 0 \quad \text{or} \quad 0 \quad 1 \quad 1 \qquad \text{and} \qquad 1 \quad 0 \quad 0 \quad \text{or} \quad 1 \quad 0 \quad 1$$

infinitely often. Hence Ay contains

$$-1 \quad \text{or} \quad 0 \qquad \text{and} \qquad 1 \quad \text{or} \quad 2$$

infinitely often, so that $y \notin c_A$.

(K) *Convergent subseries.* Suppose that $a = (a_n)$ is a sequence of positive terms with

$$\sum_n a_n = \infty \quad \text{and} \quad a_n \to 0. \tag{6.21}$$

The *convergent subseries* of a, namely those sequences of positive integers $n_1 < n_2 < \ldots$ for which

$$\sum_k a_{n_k} < \infty, \tag{6.22}$$

are objects of considerable interest. Let us temporarily denote them by $cs(a)$. Several authors have shown that $cs(a)$ cannot be too big (else (6.21) fails). Thus

$$\sum_n a_n < \infty \quad \text{if} \quad \sum_k a_{n_k} < \infty \text{ whenever } \frac{k}{n_k} \to 0 \tag{6.23}$$

is proved in [2], [17] and [19], while the same conclusion is reached from a weaker hypothesis in [1] and [34]:

$$\sum_n a_n < \infty \quad \text{if} \quad \sum_k a_{n_k} < \infty \text{ whenever } n_{k+1} - n_k \to \infty. \tag{6.24}$$

(The convergent subseries of the harmonic series are studied in [29], but from a different viewpoint than the one adopted here.)

Our next result determines when b has more convergent subseries than a, b being a fixed but arbitrary sequence of real (or complex) numbers. It is surprising that this rather natural problem seems not to have been investigated till now.

PROPOSITION 6.3. *Suppose that* (6.21) *holds. Then*

$$\sum_k b_{n_k} \textit{ converges whenever } \sum_k a_{n_k} < \infty \tag{6.25}$$

if and only if

$$b \in \ell^1 + \ell^\infty \cdot a. \tag{6.26}$$

Proof. The sufficiency of (6.26) follows from the well known Comparison Test for series. To prove necessity, we first observe that (6.25) may be replaced by an apparently stronger condition

$$\sum_k |b_{n_k}| < \infty \quad \text{whenever} \quad \sum_k a_{n_k} < \infty. \tag{6.27}$$

[If $\sum_k a_{n_k} < \infty$, the same is true of all its subseries; thus, by (6.25), all subseries of $\sum_k b_{n_k}$ must converge, forcing $\sum |b_{n_k}| < \infty$.] This observation allows us to restrict attention to non-negative sequences [replace b by $|b|$]. We may further assume that b is a null sequence, $b_n \to 0$. [Otherwise, there would exists a subsequence (b_{n_k}) of b with $b_{n_k} \geq \delta > 0$ $(k \in \mathbb{N})$. Since $a_n \to 0$, a subsequence (m_k) of (n_k) could be chosen such that $\sum_k a_{m_k} < \infty$, and this would violate (6.25).] We may thus restrict attention to non-negative and bounded sequences b satisfying

$$\sup_n b_n = M \qquad (> 0). \tag{6.28}$$

[If $M = 0$, the proposition is trivial.] We complete the proof by deriving a contradiction from the supposition that (6.25) holds while (6.26) fails. The sets E_n $(n = 1, 2, \ldots)$, defined by

$$E_n = \{k : b_k \geq n a_k\} \tag{6.29}$$

are all infinite; indeed,

$$\sum_{k \in E_n} b_k = \infty. \tag{6.30}$$

[Otherwise (6.26) would be valid, via the representation $b = \chi_{E_n} \cdot b + \chi_{\mathbb{N} \setminus E_n} \cdot b$.] We construct finite disjoint subsets F_n of E_n satisfying

$$M \leq \sum_{k \in F_n} b_k < 3M \tag{6.31}$$

as follows. Let

$$F_n = E_n \cap \{m_{n-1} + 1, m_{n-1} + 2, \ldots, m_n\}, \tag{6.32}$$

where $m_0 = 0$ and m_n $(n = 1, 2, \ldots)$ is the smallest positive integer $r \in E_n$, $r > m_{n-1}$, such that

$$M \leq \sum_{k \in E_n \cap \{m_{n-1}+1, \ldots, r\}} b_k. \tag{6.33}$$

It follows from (6.32) that the F_n's are disjoint, and, from (6.28) and (6.33), that (6.31) holds. Now let

$$F = \bigcup_{n=1}^{\infty} F_{n^2}.$$

We see from (6.31) that

$$\sum_{k\in F} b_k = \sum_{n=1}^{\infty} \sum_{k\in F_{n^2}} b_k \geq \sum_{n=1}^{\infty} M = \infty.$$

On the other hand, it follows from (6.29) and (6.31) that

$$\sum_{k\in F} a_k = \sum_{n=1}^{\infty} \sum_{k\in F_{n^2}} a_k \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k\in F_{n^2}} b_k \leq 3M \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus (6.25) cannot be valid, and this contradiction finishes the proof. ∎

(L) *Köthe duals.* We recall that the *Köthe dual* of a sequence x is defined by

$$x^\alpha = \Big\{ y \in \omega : \sum_k |x_k y_k| < \infty \Big\}.$$

It is easy to see that x^α fails to have any of the Hahn properties. The subject springs to life again, however, when we restrict attention to the bounded sequences in x^α.

THEOREM 6.4. *$\ell^\infty \cap x^\alpha$ has*

(i) *the Hahn property if and only if $x \in \ell^1$;*
(ii) *the separable Hahn property if and only if $x \in c_0$;*
(iii) *the matrix Hahn property if and only if $x \in c_0$.*

Proof. If $x \in \ell^1$, then $\ell^\infty \cap x^\alpha$ coincides with ℓ^∞, and so has the Hahn property by Corollary 1.2.

If $x \notin c_0$, there exists a subsequence of x with $|x_{n_k}| \geq \delta > 0$. This forces

$$y_{n_k} \to 0 \quad \text{whenever } y \in x^\alpha;$$

in particular, there can be only finitely many exceptions to the assertion

$$y_{n_k} = 0 \quad \text{whenever } y \in \chi(\ell^\infty \cap x^\alpha).$$

It follows that

$$\chi(\ell^\infty \cap x^\alpha)^\beta \neq (\ell^\infty \cap x^\alpha)^\beta$$

and then from Theorem 5.1 that $\ell^\infty \cap x^\alpha$ fails even to have the matrix Hahn property.

If $x \in c_0 \setminus \ell^1$, Proposition 6.3 shows that

$$\chi(\ell^\infty \cap x^\alpha)^\beta = \ell^1 + \ell^\infty \cdot \{x\}.$$

It is easy to check that

$$\ell^1 + \ell^\infty \cdot \{x\} \subseteq (\ell^\infty \cap x^\alpha)^\beta$$

so that all three spaces coincide. Since $\ell^\infty \cap x^\alpha$ is a solid sequence space, Theorem 5.2 shows that it enjoys the separable Hahn property.

On the other hand, $\ell^\infty \cap x^\alpha$ is an FK-space under the norm

$$\|y\| = \sup_n |y_n| + \sum_n |x_n y_n|$$

and $\chi(\ell^\infty \cap x^\alpha)$ fails to be $\| \ \|$-dense in $\ell^\infty \cap x^\alpha$. It follows from Theorem 1.1 that $\ell^\infty \cap x^\alpha$ is not a Hahn space. ∎

(M) $bs + c_0$ *does not have any of the Hahn properties.* The result, at first sight, is hardly surprising. The only sequences of 0's and 1's to be found in either bs or c_0 are those having finite support, and the same would seem to be true of $bs + c_0$. But this is not the case at all, and it is important to keep in mind the space $bs + c$. Here again, the components, bs and c, contain few sequences of 0's and 1's, but their sum, $bs + c$, being a Hahn space, contains many.

We begin by characterizing $\chi(bs + c_0)$. A divergent sequence x of 0's and 1's will be called *thin* if

$$n_{k+1} - n_k \to \infty \quad \text{as } k \to \infty, \tag{6.34}$$

where n_k is the coordinate of the kth "one" in x. (Thin sequences have large gaps.) We denote by τ the linear span of the thin sequences. As C. Orhan pointed out, the following result is due to Freedman [18], Theorem 3.

PROPOSITION 6.5. $\chi(bs + c_0) = \tau$.

Proof. We show first that $\tau \subseteq \chi(bs + c_0)$. If x is a thin sequence of 0's and 1's, say

$$x = (\ldots, 0, \overset{\overset{\scriptstyle n_k}{\downarrow}}{1}, 0, \ldots, 0, \overset{\overset{\scriptstyle n_{k+1}}{\downarrow}}{1}, 0 \ldots), \tag{6.35}$$

we set

$$z = (\ldots, -\varepsilon_{k-1}, \overset{\overset{\scriptstyle n_k}{\downarrow}}{1}, -\varepsilon_k, \ldots, -\varepsilon_k, \overset{\overset{\scriptstyle n_{k+1}}{\downarrow}}{1}, -\varepsilon_{k+1}, \ldots), \tag{6.36}$$

where

$$\varepsilon_k = \frac{1}{n_{k+1} - n_k}, \tag{6.37}$$

and

$$y = x - z. \tag{6.38}$$

It follows from (6.37) that $z \in bs$, from (6.34) that $y \in c_0$, and from (6.38) that $x \in \chi(bs + c_0)$.

To prove the reverse inclusion, $\chi(bs + c_0) \subseteq \tau$, we use the following observation ([8], Theorem 3(ii)): if $x \in bs + c_0$, then

$$\sup_m \limsup_n \left| \sum_{k=n+1}^{n+m} x_k \right| < \infty. \tag{6.39}$$

If x is a divergent sequence of 0's and 1's in $bs + c_0$, the functional defined in (6.39) must be positive-integer-valued, say

$$\sup_m \limsup_n \left| \sum_{k=n+1}^{n+m} x_k \right| = r.$$

For each $m = 1, 2, \ldots$, there exists an integer $N(m)$ such that

$$\sup_{n \geq N(m)} \left| \sum_{k=n+1}^{n+m} x_k \right| \leq r.$$

Fixing m, we see that any "block" $x_{n+1}, \ldots, x_{n+m}$ of terms of x, of length not exceeding m, contains no more than r "ones". Therefore

$$n_{k+r} - n_k > m \quad \text{if } n_k \geq N(m)$$

so that

$$n_{k+r} - n_k \to \infty \quad \text{as } k \to \infty.$$

Thus x may be expressed as a sum of r thin sequences (or fewer). ∎

It is now easy to show that $bs + c_0$ does not have the matrix Hahn property. The convergence domain (6.20), for example, contains $\chi(bs + c_0)$, by Proposition 6.5, but not bs.

7. Problems

1. *Must the intersection of two Hahn spaces be a Hahn space?* What about arbitrary intersections?

The first problem persists if we consider the separable Hahn or matrix Hahn properties, while the second admits a negative solution in those cases. [The spaces $\ell^\infty \cap x^\alpha$ $(x \in c_0)$ all have the separable Hahn property by Theorem 6.4, whereas their intersection

$$\bigcap_{x \in c_0} (\ell^\infty \cap x^\alpha) = \ell^1$$

fails the matrix Hahn property because of Theorem 5.1.]

2. *Does the space $|ac|_0$ have the Hahn property?* (See (G), Section 6 and Problem 9 below.)

3. *Which bounded sequences x are such that $bs+\mathrm{Sp}\{x\}$ is a Hahn space?* We recall that $bs+\mathrm{Sp}\{e\}$ is a Hahn space (Theorem 3.4), whereas $bs+c_0$ is not (Proposition 6.5).

4. *What is the closed linear span of the set of sequences of* 0*'s and* 1*'s in* $\{x\in\omega : \|x\|=\sup_n \sum_{k=2^{n-1}}^{2^n-1} |x_k|<\infty\}$? (See Section 5.) Is it a Hahn space?

5. *Do the thin sequences have a smallest containing FK-space?* (See (M), Section 6.)

6. *Does the matrix Hahn property imply the separable Hahn property?* Theorem 5.2 shows that the answer is affirmative for solid (or monotone) sequence spaces. The following closely related problem has a negative solution.

> *Must a separable FK-space be the intersection of the convergence domains containing it?*

A counterexample has been constructed by M. Zeltser (student of Leiger) and it will be presented elsewhere.

7. *Which bounded domains have the Hahn property?* It follows from an observation of Kuttner and Parameswaran ([24], Theorem 2) that the converse of Theorem 6.1 is valid for conservative Hausdorff matrices. (See [9] for definitions.)

They show that the bounded convergence domain of such a matrix has the matrix Hahn property precisely when the diagonal entries converge to zero. The last condition forces the matrix to be strongly conservative ([26], Theorem 13) and permits Theorem 6.1 to be applied.

An analogue of the Kuttner/Parameswaran result has been given for weighted mean matrices ([25], Theorem 1), but the details are here much more troublesome (see [10]):

> *if A is a conservative weighted mean matrix, then $\ell^\infty\cap c_A$ has the matrix Hahn property precisely when* $\lim_n a_{nn}=0$.

Does $\ell^\infty\cap c_A$ have the Hahn property? What can be said about Nörlund matrices?

8. *Uniform Boundedness Principles.* Behind any FK-space, E, that is known to enjoy the Hahn property, there lies a uniform boundedness principle:

$$\chi(E) \text{ is barrelled in } E. \tag{7.1}$$

The principle may be difficult to enunciate, for sure, owing to the nonseparability of E and the attendant intractability of its dual space, E'. But if we suppose that E is a closed subspace of ℓ^∞, then E', just like $(\ell^\infty)'$, becomes manageable, and there is the enticing possibility that (7.1) may lead to new forms of the Uniform Boundedness Principle.

Such is the case with

$$E = \left\{ x \in \ell^\infty : \frac{|x_1| + \ldots + |x_n|}{n} \to 0 \right\}.$$

The sequences of 0's and 1's in E may be identified with the subsets of $\mathbb{N}$ having zero density. Let us denote this class of sets by Δ_0. Then Δ_0 is a *ring* (i.e. closed under finite unions and under differences), but not a σ-ring.

A direct proof of (7.1) demands that we find an extension of Nikodym's theorem and the following concept is exactly what is needed here.

We say that a collection $\mathcal{C}$ of subsets of $\mathbb{N}$ has the *σ-density property* if whenever

$$S_1, S_2, \ldots \text{ are pairwise disjoint members of } \mathcal{C},$$

there exists a subsequence of the S's, say $T_1, T_2, \ldots$, such that T_n admits a partition into 2^{n-1} subsets,

$$T_n = T_{n,1} \cup T_{n,2} \cup \ldots \cup T_{n,2^n-1} \quad \text{with} \quad \bigcup_{i \in I} T_{i,j(i)} \in \mathcal{C}$$

for every $I \subseteq \mathbb{N}$ and every choice of $j(i) \in \{1, 2, \ldots, 2^i - 1\}$.

We omit the proof that Δ_0 has the σ-density property as well as the proof of the following theorem.

THEOREM 7.1. *Suppose that $\mathcal{R}$ is a ring of subsets of $\mathbb{N}$ with the σ-density property. If $(\mu_n)_{n=1}^\infty$ is a sequence of bounded, finitely additive (real-valued) set functions on $\mathcal{R}$ that is "pointwise" bounded,*

$$\sup_n |\mu_n(S)| < \infty \qquad \textit{for each } S \in \mathcal{R},$$

then $(\mu_n)_{n=1}^\infty$ is "uniformly" bounded,

$$\sup_{S \in \mathcal{R}} \sup_n |\mu_n(S)| < \infty.$$

Nikodym's version of Theorem 7.1 requires that $\mathcal{R}$ be a σ-algebra and that the μ_n's be countably additive. The remarkable extension of Nikodym's theorem to *finitely additive* set functions is due to Dieudonné [14]. His result was rediscovered by Darst [13] and Seever [32]. Our contribution is to remove the restriction that $\mathcal{R}$ be a σ-algebra.

9. *What is Nikodym's theorem for Δ?* The space of bounded Cesàro limitable sequences, (3.3), has the Hahn property so there must exist an analogue of Theorem 7.1 for Δ, the subsets of $\mathbb{N}$ having density. Δ certainly has the σ-density property, but Theorem 7.1 does not apply, because Δ is not a ring. The same comments apply when the Cesàro matrix is replaced by any strongly regular matrix (see Theorem 6.2). Problem 2 remains open because the ring of subsets of $\mathbb{N}$ associated with $|ac|_0$ does not have the σ-density property.

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