STUDIA MATHEMATICA 190 (1) (2009)

Asymptotic behaviour of averages of k-dimensional marginals of measures on $\mathbb{R}^n$

by

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Abstract. We study the asymptotic behaviour, as $n \to \infty$, of the Lebesgue measure of the set $\{x \in K : |P_E(x)| \le t\}$ for a random k-dimensional subspace $E \subset \mathbb{R}^n$ and an isotropic convex body $K \subset \mathbb{R}^n$. For k growing slowly to infinity, we prove it to be close to the suitably normalised Gaussian measure in $\mathbb{R}^k$ of a t-dilate of the Euclidean unit ball. Some of the results hold for a wider class of probabilities on $\mathbb{R}^n$.

1. Preliminaries and notation. Let E be a k-dimensional subspace of $\mathbb{R}^n, 1 \le k \le n$, and denote by P_E the orthogonal projection onto E. For any Borel probability $\mathbb{P}$ on $\mathbb{R}^n$, its marginal probability on E is defined as $\mathbb{P}_E(A) := \mathbb{P}(A + E^\perp) = \mathbb{P}\{x \in \mathbb{R}^n : P_E(x) \in A\}$ for $A \subseteq E$. A Borel probability $\mathbb{P}$ is *isotropic* if $\int_{\mathbb{R}^n} x \, d\mathbb{P}(x) = 0$ and its covariance matrix is a multiple of the identity. A convex body K of volume 1 is *isotropic* if the uniform measure on K is. In this case, the above multiple of the identity is denoted by L_K^2.

In [Kl2] the author solved the so called *central limit problem for convex bodies* (posed in [ABP], [BV] for $k = 1$ and considered in [BK], [BHVV], [KL], [MM], [Mi], [Wo]). He showed that every isotropic convex body K (and more generally, every isotropic log-concave probability measure) has the property that *most* of its k-dimensional marginal distributions are *approximately* Gaussian, with respect to the total variation metric, provided that $k \ll \log n/\log\log n$.

In a more general probabilistic setting, the k-dimensional version of the problem goes back to [W] (see also [DF], [Bo], [Su]). In [NR], the authors studied proximity of k-marginals to the Gaussian measure with respect to the (weaker) T-distance, for a class of isotropic probabilities satisfying some concentration hypothesis. In [M], Gaussian approximation of k-marginals

2000 *Mathematics Subject Classification*: 52A20, 60F05.
Key words and phrases: isotropic bodies, central limit, concentration phenomena.
Research partially supported by DGA E64 (Spain), MEyC (Spain) MTM2007-61446 and Marie Curie RTN CT-2004-511953.

with respect to the Wasserstein distance is studied for isotropic probabilities with geometric symmetries.

A key tool in all those results is a kind of concentration property of the Euclidean norm with respect to the probability $\mathbb{P}$.

Let K be an isotropic convex body and consider the distribution function

$$F_K(t,E) := |\{x \in K : |P_E(x)| \le t\}|, \qquad t \ge 0, \tag{1.1}$$

where $|\cdot|$ denotes both the Euclidean norm and the Lebesgue measure on $\mathbb{R}^n$. The function $F_K(t,E)$ is the marginal measure (of the uniform measure on K) on E of the t-dilate of the Euclidean unit ball. Denote by $\Gamma_K^k(t)$ the k-dimensional Gaussian measure (centred with variance L_K^2) of $\{s \in \mathbb{R}^k : |s| \le t\}$.

We are interested in studying the closeness of $F_K(t,E)$ and $\Gamma_K^k(t)$. The results in [Kl2] give in particular some estimates of $|F_K(t,E) - \Gamma_K^k(t)|$. It was pointed out to the authors by V. Milman that it is of interest to consider the (stronger) comparison $|F_K(t,E)/\Gamma_K^k(t) - 1|$ in the spirit of [So] and we will address this question. With a concentration assumption on K (see (3.3) below) we will show

Theorem 3.11. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body satisfying condition (3.3), and $t_0 > 0$. Then for every $0 < \varepsilon < 1$ and $1 \le k \le c_1 \varepsilon \log n/(\log\log n)^2$ we have*

$$\nu\left\{E \in G_{n,k} : \sup_{t \ge t_0} \left|\frac{F_K(t,E)}{\Gamma_K^k(t)} - 1\right| \le \varepsilon\right\} \ge 1 - \exp(-c_2 n^{0.37})$$

where c_1 depends only on the constants appearing in (3.3), and c_2 depends only on t_0.

We follow a fairly standard procedure: first we show that the average of $F_K(t,E)$ on the grassmannian $G_{n,k}$ is close to the Gaussian measure. Then, by the concentration of measure phenomena on $G_{n,k}$, we show that for most subspaces E, $F_K(t,E)$ is close to its average.

It turns out that the average of $F_K(t,E)$ can be written in a way that admits generalisation to any probability $\mathbb{P}$. In the second and in the last sections of the paper we study properties of this averaging, including proximity to the Gaussian measure in the uniform distance.

The paper is organised as follows:

In Section 2 we introduce an average of k-dimensional marginals for any probability $\mathbb{P}$ on $\mathbb{R}^n$, compute the (radial) density $\varphi_{\mathbb{P}}^k(s)$, $s \in \mathbb{R}^k$, of its absolutely continuous part (Proposition 2.1), and explain its geometrical meaning (Proposition 2.3). For $\mathbb{P}$ the uniform measure on K, the relation with our problem is given by the formula

$$F_K^k(t) := \int_{G_{n,k}} F_K(t,E)\,d\nu(E) = \int_{\{|s|\le t\}} \varphi_K^k(s)\,ds \tag{1.2}$$

where

$$\varphi_K^k(s) = \int_{O(n)} |(s_1\xi_1 + \cdots + s_k\xi_k + \{\xi_1,\ldots,\xi_k\}^\perp) \cap K|_{n-k}\,dU,$$

with integration with respect to the Haar probability on the orthogonal group $O(n)$ and $U = (\xi_1 \ \ldots \ \xi_n) \in O(n)$. Moreover, each $F_K(t,E)$ is a certain average on $O(E)$ of marginal densities (see Remark 2.4).

In Section 3.1, we investigate the closeness of the average density $\varphi_K^k(s)$ to a suitably normalised Gaussian density $\gamma_K^k(s)$ and obtain estimates for

$$\left|\frac{\varphi_K^k(s)}{\gamma_K^k(s)} - 1\right|$$

(Theorem 3.5(1)). At this stage, it is still possible to state the result for general probabilities $\mathbb{P}$ satisfying (3.3) with no extra effort and we do so (Theorem 3.1). We extend the ideas in [So] (where $k = 1$) to estimates with s far from the origin. The study of estimates for s near the origin leads us to consider the parameter $M_{\mathbb{P}}$ (see definition below).

A simple integration yields relations between the *average* of $F_K(t,E)$ and $\Gamma_K^k(t)$, that is, an estimate for

$$\left|\frac{F_K^k(t)}{\Gamma_K^k(t)} - 1\right|$$

(Theorem 3.5(2)). In Section 3.2, the concentration of measure phenomenon on $G_{n,k}$ will be the key ingredient to show that for "most" subspaces E, $F_K(t,E)$ is close to its average $F_K^k(t)$. For that purpose we estimate the modulus of continuity of $F_K(t,E)$.

All the results in this section, valid for the uniform probability on isotropic convex bodies, can be stated and hold true for log-concave probabilities $\mathbb{P}$.

Finally, in Section 4 we return to the study of the average density $\varphi_{\mathbb{P}}^k(s)$. For a class of probability measures $\mathbb{P}$, we estimate

$$\sup_{s\in\mathbb{R}^k} |\varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s)| \quad \text{and} \quad \sup_{t\ge 0} |F_K^k(t) - \Gamma_{\mathbb{P}}^k(t)|$$

(Theorem 4.2) and show that such differences tend to 0 (as $n \to \infty$) provided that $k = O(\sqrt{\log n}/(\log\log n)^{1/2+\delta})$ for some $\delta > 0$. We extend the ideas in [BK] and solve the difficulties appearing in that paper for $s = 0$.

When k increases very fast to infinity, $k = n - \ell$ with ℓ fixed, or $k = (1-\lambda)n$, $0 < \lambda < 1$, we cannot expect a Gaussian behaviour. We obtain upper bounds for the average marginal density (Proposition 4.7), which, for some cases, are shown to be sharp. Such upper bounds are also needed in the first part of the section (Lemma 4.5).

Next we introduce some notation and definitions. We denote by D_n the Euclidean ball in $\mathbb{R}^n$ and by ω_n its Lebesgue measure. The area measure of the unit sphere S^{n-1} is $|S^{n-1}| = n\omega_n$. The letters $c, C, c_1, \dots$ will denote absolute numerical constants whose value may change from line to line.

The elements of the orthogonal group $O(n)$ are denoted by $U = (\xi_1 \ \dots \ \xi_n)$ so the columns (ξ_i) form an orthonormal basis in $\mathbb{R}^n$, and dU is the Haar probability on $O(n)$. The Haar probability on S^{n-1} is denoted by σ_{n-1}.

Let $\mathbb{P}$ be a Borel probability on $\mathbb{R}^n$. We introduce the following parameters:

$$M_2^2 = M_2^2(\mathbb{P}) := \frac{1}{n} \int\limits_{\mathbb{R}^n} |x|^2 \, d\mathbb{P}(x), \qquad M_{\mathbb{P}} := \sup_{t>0} \frac{\mathbb{P}\{tD_n\}}{|tD_n|}$$

and

$$\sigma_{\mathbb{P}}^2 := n\left(\frac{\int_{\mathbb{R}^n} |x|^4 \, d\mathbb{P}(x)}{(\int_{\mathbb{R}^n} |x|^2 \, d\mathbb{P}(x))^2} - 1 \right) = \frac{\mathrm{Var}(|x|^2)}{nM_2^4(\mathbb{P})},$$

When $\mathbb{P}$ is the uniform measure on K we change the notation accordingly, that is, $\sigma_{\mathbb{P}}$ to σ_K and so on.

Remark 1.1. $\sigma_{\mathbb{P}}$ is a concentration parameter. Chebyshev's inequality implies (see [ABP])

$$\mathbb{P}\{x \in \mathbb{R}^n : | \, |x|^2 - nM_2^2(\mathbb{P})| > \varepsilon n M_2^2(\mathbb{P})\} \le \frac{\sigma_{\mathbb{P}}^2}{n^2 \varepsilon^2}.$$

For $\mathbb{P}$ the uniform measure on an isotropic convex body K, the parameter σ_K is conjectured to be bounded by an absolute constant (the Variance Hypothesis).

When $\mathbb{P}$ has density f, $M_{\mathbb{P}}$ is the Hardy–Littlewood maximal function of f at the origin. It is finite when, for instance, the origin is a regular Lebesgue point of f (Lebesgue differentiation theorem holds), or when f is bounded, in which case $M_{\mathbb{P}} \le \|f\|_\infty$ (the supremum norm of f). Also observe that $M_{\mathbb{P}} < \infty$ implies $\mathbb{P}(\{0\}) = 0$.

Remark 1.2. For $M_{\mathbb{P}}$ and $M_2(\mathbb{P})$ finite the parameter $M_2(\mathbb{P}) M_{\mathbb{P}}^{1/n}$ plays an important role. In the particular case of $\mathbb{P}$ being the uniform measure of an isotropic convex body K, this constant is L_K ($= M_2(\mathbb{P})$ and $M_{\mathbb{P}} = 1$). If $\mathbb{P}$ has density f that is an even log-concave function, the constant $M_2(\mathbb{P}) M_{\mathbb{P}}^{1/n}$ is the isotropy constant of the function since $M_{\mathbb{P}} = f(0)$ (see [B]).

The following fact due to Hensley [H], whose proof follows from [B, Lemma 6], will be extensively used along the paper:

LEMMA 1.3. *There exists an absolute constant $c > 0$ such that for any probability $\mathbb{P}$ on $\mathbb{R}^n$, $M_2(\mathbb{P})M_{\mathbb{P}}^{1/n} \geq c$.*

We finish this section with some

1.1. *Technical preliminaries.* Let $\mathbb{P}$ be a Borel probability on $\mathbb{R}^n$ with $M_{\mathbb{P}} < \infty$ and $M_2 = M_2(\mathbb{P}) < \infty$. Define

$$\gamma_{\mathbb{P}}^k(s) = \frac{1}{(2\pi)^{k/2} M_2^k}\, e^{-|s|^2/2M_2^2}, \quad s \in \mathbb{R}^k, \qquad \Gamma_{\mathbb{P}}^k(t) = \int\limits_{|s|\leq t} \gamma_{\mathbb{P}}^k(s)\, ds, \quad t \geq 0.$$

In the next three lemmas we state some useful inequalities. Given $g, h\colon [0,\infty) \to \mathbb{R}$, write $g \sim h$ if $c_1 h(x) \leq g(x) \leq c_2 h(x)$ for all $x \geq 0$.

LEMMA 1.4. *The following estimates are well known*:

(i)
$$\Gamma(x+1) = x^x e^{-x}\sqrt{2\pi x}\left(1 + \frac{1}{12x} + O\left(\frac{1}{x^2}\right)\right),$$

(ii)
$$|S^{n-1}| = n\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad \omega_n \leq \frac{c^n}{n^{n/2}}, \qquad \omega_n^{1/n} \sim \frac{\sqrt{2\pi e}}{\sqrt{n}},$$
$$\frac{|S^{n-k-1}|}{|S^{n-1}|} \leq C\frac{n^{k/2}}{(2\pi)^{k/2}} \qquad \textit{for } k = o(n).$$

LEMMA 1.5.

(i)
$$\frac{t^k\omega_k}{(\sqrt{2\pi}\, M_2)^k}\, e^{-t^2/2M_2^2} \leq \Gamma_{\mathbb{P}}^k(t) \leq \frac{t^k\omega_k}{(\sqrt{2\pi}\, M_2)^k}, \qquad \forall t \geq 0,$$

(ii)
$$\Gamma_{\mathbb{P}}^k(t) \geq 1 - 2^{k/2} e^{-t^2/4M_2^2}, \qquad \forall t \geq 0,$$

(iii)
$$\Gamma_{\mathbb{P}}^k(t+\delta) \leq (1+\delta/t)^k \Gamma_{\mathbb{P}}^k(t), \qquad \forall \delta, t > 0.$$

Proof. (i) is straightforward; as for (ii),

$$1 - \Gamma_{\mathbb{P}}^k(t) = \int\limits_{|s|\geq t} \gamma_{\mathbb{P}}^k(s)\, ds \leq \frac{e^{-t^2/4M_2^2}}{(\sqrt{2\pi}\, M_2)^k} \int\limits_{|s|\geq t} e^{-|s|^2/4M_2^2}\, ds \leq 2^{k/2} e^{-t^2/4M_2^2};$$

and (iii) follows from

$$\frac{\Gamma_{\mathbb{P}}^k(t+\delta)}{\Gamma_{\mathbb{P}}^k(t)} = 1 + \frac{\int_{t<|s|\leq t+\delta} \gamma_{\mathbb{P}}^k(s)\, ds}{\int_{|s|\leq t} \gamma_{\mathbb{P}}^k(s)\, ds}$$
$$\leq 1 + \omega_k\, \frac{((t+\delta)^k - t^k)e^{-t^2/2M_2^2}}{\int_{|s|\leq t} \gamma_{\mathbb{P}}^k(s)\, ds} \leq 1 + \frac{(t+\delta)^k - t^k}{t^k}. \ \blacksquare$$

LEMMA 1.6. *Let $n \geq k+3$. There exists an absolute $C > 0$ such that*

(i)
$$\left|\left(1 - \frac{2u}{n}\right)^{(n-k-2)/2} - e^{-u}\right| \leq C\,\frac{k}{n-k} \qquad \forall u \in [0, n/2],$$

(ii) $$\left|\left(\frac{|S^{n-k-1}|}{|S^{n-1}|}\,\frac{(2\pi)^{k/2}}{n^{k/2}}\right)-1\right|\le C\,\frac{k^2}{n},$$

(iii) $$\left|e^u\left(1-\frac{2u}{n}\right)^{(n-k-2)/2}-1\right|\le 8\left(\frac{ku}{n}+\frac{u^2}{n}\right)\qquad \forall u\in[0,\sqrt{n}/4],$$

provided that $ku/n+u^2/n\le 1/8$.

Proof. The proof of (i) is the same as in [BK]. As for (ii), it is a consequence of the formula $|S^{n-1}|=2\pi^{n/2}/\Gamma(n/2)$ and the asymptotic formula for the Gamma function in Lemma 1.4. To prove (iii), write $y=u+\frac{n-k-2}{2}\log\left(1-\frac{2u}{n}\right)$. We use the inequality $|e^y-1|\le 2|y|$ for $|y|\le 1$ and Taylor's formula with Lagrange's error term $\log(1-x)=-x-x^2/2(1-\xi)^2$ for $0<\xi<x\le 1$ with $x=2u/n$ to obtain

$$\begin{aligned}\left|e^u\left(1-\frac{2u}{n}\right)^{(n-k-2)/2}-1\right| &\le 2\left|u+\frac{n-k-2}{2}\log\left(1-\frac{2u}{n}\right)\right|\\ &\le 2\left|u-\frac{n-k-2}{2}\left(\frac{2u}{n}+\frac{4u^2}{2n^2(1-\xi)^2}\right)\right|\le 2\left(\frac{k+2}{n}\,u+\frac{(n-k-2)u^2}{(n-2u)^2}\right)\\ &\le 2\left(\frac{3ku}{n}+\frac{nu^2}{(n-\sqrt{n}/2)^2}\right)\le 8\left(\frac{ku}{n}+\frac{u^2}{n}\right).\ \blacksquare\end{aligned}$$

2. Average of k-dimensional marginals. Let $\mathbb{P}$ be a Borel probability on $\mathbb{R}^n$. For every $k\in\mathbb{N}$ with $1\le k\le n$ we define the following average of k-marginals:

$$A_k(\mathbb{P})(B)=\int_{O(n)}\mathbb{P}(U(B+\mathbb{R}^{n-k}))\,dU,\qquad B\subset\mathbb{R}^k.$$

Then $A_k(\mathbb{P})$ is a Borel probability on $\mathbb{R}^k$ invariant under the action of the orthogonal group in $\mathbb{R}^k$. Clearly, $A_k(A_n(\mathbb{P}))=A_k(\mathbb{P})$.

The following proposition was proved in [BV], [BK] and [So] for $k=1$.

PROPOSITION 2.1. *Let $\mathbb{P}$ be a Borel probability on $\mathbb{R}^n$. Then, for all $1\le k<n$ and any Borel set $B\subset\mathbb{R}^k$, we have*

$$A_k(\mathbb{P})(B)=\mathbb{P}(\{0\})\delta_0(B)+\int_B\frac{|S^{n-k-1}|}{|S^{n-1}|}\int_{\{|x|\ge|s|\}}\left(1-\frac{|s|^2}{|x|^2}\right)^{(n-k-2)/2}\frac{d\mathbb{P}(x)}{|x|^k}\,ds$$

where δ_0 is the Dirac measure at 0. The density function of the absolutely continuous part is denoted by $s\in\mathbb{R}^k\mapsto\varphi_{\mathbb{P}}^k(s)$.

Proof. Since $A_k(A_n(\mathbb{P}))=A_k(\mathbb{P})$ and the inner integrand is radial it is enough to prove the formula for probabilities $\mathbb{P}$ that are invariant under orthogonal transformations. First we consider the case $\mathbb{P}=\sigma_{n-1}$. It is enough to prove the equality for dilates of the Euclidean ball, that is, to show that

$$A_k(\sigma_{n-1})(rD_k) = \int\limits_{rD_k} \frac{|S^{n-k-1}|}{|S^{n-1}|} \int\limits_{\{|x|\geq|s|\}} \left(1 - \frac{|s|^2}{|x|^2}\right)^{(n-k-2)/2} \frac{d\sigma_{n-1}(x)}{|x|^k}\, ds$$

$$= \frac{|S^{n-k-1}|}{|S^{n-1}|} \int\limits_{rD_k} (1-|s|^2)^{(n-k-2)/2} \chi_{D_k}(s)\, ds.$$

If $r \geq 1$, then

$$A_k(\sigma_{n-1})(rD_k) = 1 = \frac{|S^{n-k-1}|}{|S^{n-1}|} \int\limits_{D_k} (1-|s|^2)^{(n-k-2)/2}\, ds.$$

If $r < 1$, after passing to polar coordinates, the right hand side equals

$$\frac{|S^{n-k-1}|\,|S^{k-1}|}{|S^{n-1}|} \int\limits_0^r (1-t^2)^{(n-k-2)/2} t^{k-1}\, dt.$$

On the other hand,

$$\begin{aligned} &A_k(\sigma_{n-1})(rD_k) \\ &\quad = \sigma_{n-1}(rD_k \times \mathbb{R}^{n-k}) \\ &\quad = \frac{\omega_k |S^{n-k-1}|}{\omega_n} \left(\frac{r^k}{n} (1-r^2)^{(n-k)/2} + \int\limits_{\sqrt{1-r^2}}^1 t^{n-k-1} (1-t^2)^{k/2}\, dt \right). \end{aligned}$$

Now, the derivatives of the two expressions are equal and we have the result. Observe that, by rescaling, the formula also holds for the Haar probabilities on $\lambda S^{n-1}, \lambda > 0$.

In the general case, we use the fact that any probability $\mathbb{P}$ invariant under orthogonal transformations is, up to $\mathbb{P}(\{0\})$, the product measure of a positive measure on $(0,\infty)$ and the Haar measure on S^{n-1}, and so it can be approximated by convex combinations of Haar probabilities on $\lambda S^{n-1},\ \lambda > 0$. For $\lambda = 0$, the associated probability is δ_0. ∎

Remark 2.2. If $\mathbb{P}$ is a probability with density f, that is, $\mathbb{P}(C) = \int_C f(x)\, dx$, then $A_k(\mathbb{P})(B) = \int_B \varphi^k_{\mathbb{P}}(s)\, ds$ where

$$\varphi^k_{\mathbb{P}}(s) = \int\limits_{O(n)} \int\limits_{\mathbb{R}^{n-k}} f(s_1\xi_1 + \cdots + s_n\xi_n)\, ds_{k+1} \ldots ds_n\, dU$$

and $s = (s_1, \ldots, s_k)$.

In the particular case of $\mathbb{P}(C) = |K \cap C|$ for a Borel set $K \subset \mathbb{R}^n$ of volume 1 we have $A_k(\mathbb{P})(B) = \int_B \varphi^k_K(s)\, ds$ where

$$\varphi^k_K(s) = \int\limits_{O(n)} |(s_1\xi_1 + \cdots + s_k\xi_k + \{\xi_1, \ldots, \xi_k\}^{\perp}) \cap K|_{n-k}\, dU.$$

This integral, an average of sections by $n-k$-dimensional subspaces at distance $|s|$ from the origin, is the density function of a certain average of

k-dimensional marginals of K (further applications of this formula appear in [BBR]).

The following proposition gives a more geometrical interpretation of that function.

Proposition 2.3. *Let $K \subset \mathbb{R}^n$ be a Borel set of volume 1. Then for any $1 \le k < n$ and $s \in \mathbb{R}^k$ we have*

$$\varphi_K^k(s) = \int\limits_{S^{n-1}} \Big(\int\limits_{G(\theta^\perp, n-k)} |(|s|\theta + E) \cap K|_{n-k} \, d\nu(E) \Big) \, d\sigma_{n-1}(\theta)$$

where $G(\theta^\perp, n-k)$ is the Grassmann manifold of $n-k$-dimensional subspaces of the hyperplane $\theta^\perp$, and $d\nu$ its Haar measure.

That is, consider the sphere $|s|S^{n-1}$; for any $\theta \in S^{n-1}$ we first average over all the $(n-k)$-dimensional sections of K at distance $|s|$ from the origin in direction θ, that is, inside $|s|\theta + \theta^\perp$; and then we average over the sphere.

Proof. Since $\varphi_K^k(s)$ is radial,

$$\varphi_K^k(s) = \int\limits_{O(n)} |(|s|\xi_1 + \{\xi_1, \dots, \xi_k\}^\perp) \cap K|_{n-k} \, dU.$$

Next we consider the following consequence of the conditional expectation theorem as it appears in [Ko, Lemma 1]: for any (say) continuous function F on $O(n)$,

$$\int\limits_{O(n)} F(U) \, dU = \int\limits_{G(n,k)} \int\limits_{\xi_{k+1}, \dots, \xi_n \in E^\perp} dU_{n-k} \int\limits_{\xi_1, \dots, \xi_k \in E} F(U) \, dU_k \, d\nu(E)$$

where dU_{n-k} and dU_k are the Haar measures on $O(n-k)$ and $O(k)$. We apply this formula for $k = 1$ and any continuous function and we have in particular

$$\int\limits_{O(n)} F(\xi_1, \dots, \xi_n) \, dU = \int\limits_{S^{n-1}} \Big(\int\limits_{O(\xi_1^\perp)} F(\xi_1, \xi_2, \dots, \xi_n) \, dU_1 \Big) \, d\sigma_{n-1}(\theta)$$

where $O(\xi_1^\perp)$ is the orthogonal group in the hyperplane $\xi_1^\perp$, and dU_1 its Haar measure (this formula can also be proved for any (say) continuous function F directly, by using the uniqueness of the Haar measure on $O(n)$). Applying again Koldobsky's formula in the whole space $\xi_1^\perp$ and $n-k$ for the function $F(\xi_1, \xi_2, \dots, \xi_n) = |(|s|\xi_1 + \{\xi_1, \dots, \xi_k\}^\perp) \cap K|_{n-k}$ we eventually get the result. ∎

Remark 2.4. Let E be a k-dimensional subspace of $\mathbb{R}^n$. We show some relations between the function $F_K(t, E) := |\{x \in K : |P_E(x)| \le t\}|$ (formula (1.1)) and the average marginal density $\varphi_K^k(s)$.

Fix an orthonormal basis $\{\xi_1, \dots, \xi_k\} \subset \mathbb{R}^n$ of E. By Fubini's theorem we have

$$F_K(t,E) = \int\limits_{|s|\le t} \Big| \Big(\sum_{i=1}^{k} s_i \xi_i + E^{\perp} \Big) \cap K \Big|_{n-k} ds_1 \dots ds_k.$$

We now integrate when $U = (\xi_1 \ \dots \ \xi_k)$ runs over the orthogonal group $O(E)$, which allows us to express $F_K(t,E)$ as a convenient average of marginal densities:

$$\begin{aligned}
F_K(t,E) &= \int\limits_{O(E)} \Big(\int\limits_{|s|\le t} \Big| \Big(\sum_{i=1}^{k} s_i \xi_i + E^{\perp} \Big) \cap K \Big|_{n-k} ds_1 \dots ds_k \Big) \, dU \\
&= \int\limits_{|s|\le t} \Big(\int\limits_{O(E)} \Big| \Big(\sum_{i=1}^{k} s_i \xi_i + E^{\perp} \Big) \cap K \Big|_{n-k} dU \Big) \, ds_1 \dots ds_k \\
&= \int\limits_{|s|\le t} \Big(\int\limits_{O(E)} |(|s|\xi_1 + E^{\perp}) \cap K|_{n-k} \, dU \Big) \, ds_1 \dots ds_k \\
&\qquad \text{(by the invariance under the orthogonal group)} \\
&= \int\limits_{|s|\le t} \Big(\int\limits_{S_E} |(|s|\theta + E^{\perp}) \cap K|_{n-k} \, d\sigma_E(\theta) \Big) \, ds_1 \dots ds_k \\
&\qquad \text{(by using Lemma 1 in [Ko])} \\
&= |S^{k-1}| \int_0^t r^{k-1} f_K(r,E) \, dr \\
&\qquad \text{(by passing to polar coordinates in } E)
\end{aligned}$$

where $S_E = S^{n-1} \cap E$, σ_E is its Haar probability and

$$f_K(r,E) = \int\limits_{S_E} |(r\theta + E^{\perp}) \cap K|_{n-k} \, d\sigma_E(\theta), \qquad r \ge 0.$$

Finally, observe that we also obtain formula (1.2),

$$F_K^k(t) = \int\limits_{G_{n,k}} F_K(t,E) \, d\nu(E) = \int\limits_{\{|s|\le t\}} \varphi_K^k(s) \, ds.$$

Our last lemma provides bounds for $f_K(r,E)$ and $F_K(t,E)$ that will be useful in the next section.

LEMMA 2.5. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body and $E \in G_{n,k}$. Set $\mathcal{L}_k = \sup\{L_M : M \subset \mathbb{R}^k$, isotropic$\}$. There exists an absolute constant*

$c_1 > 0$ *such that*

$$\text{(i)} \qquad f_K(r,E) \le e^k f_K(r',E) \le \frac{(c_1\mathcal{L}_k)^k}{L_K^k} \qquad \forall r \ge r' \ge 0,$$

$$\text{(ii)} \qquad F_K(t,E) \ge 1 - c_2 \exp\left(-\frac{c_1 t}{L_K\sqrt{k}}\right) \qquad \forall t \ge 0.$$

Proof. (i) A result by Fradelizi [F] states that

$$|(r\theta + E^\perp) \cap K|_{n-k} \le e^k |(r'\theta + E^\perp) \cap K|_{n-k}, \qquad \forall r \ge r' \ge 0,$$

and the first inequality follows. The second inequality is a consequence of the previous one (for $r' = 0$) and a result by Ball, Milman and Pajor [B], [MP] which states that $|E^\perp \cap K|_{n-k} \le (c_2\mathcal{L}_k)^k / L_K^k$.

(ii) It is a consequence of a more general result: if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map such that $\dim T(\mathbb{R}^n) = k$, then

$$|\{x \in K : |T(x)| \le t\}| \ge 1 - c_2 \exp\left(-\frac{c_1 t}{L_K \|T\|_{\mathrm{HS}}}\right) \qquad \forall t \ge 0$$

where $\|T\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm. Indeed, Borell's inequality (see [MS]) states

$$\left|\left\{x \in K : \frac{|T(x)|}{\left(\int_K |Tx|^2\,dx\right)^{1/2}} > t\right\}\right| \le c_2 \exp(-c_1 t) \qquad \forall t \ge 0.$$

We can suppose that $T = \sum_{j=1}^k u_j \otimes e_j$, where $\{u_j\}_{j=1}^k$ are k vectors in $\mathbb{R}^n$ and $\{e_j\}_{j=1}^k$ is an orthonormal basis in the subspace $T(\mathbb{R}^n)$. Then

$$\int_K |Tx|^2\,dx = \int_K \sum_{j=1}^k |\langle u_j, x\rangle|^2\,dx = \sum_{j=1}^k |u_j|^2 \int_K \left|\left\langle \frac{u_j}{|u_j|}, x\right\rangle\right|^2 dx = L_K^2 \|T\|_{\mathrm{HS}}^2.$$

In our case simply take $T = P_E$.

3. Estimating the quotient.

3. Estimating the quotient. Our aim is to estimate the quantity $|F_K(t,E)/\Gamma_K^k(t) - 1|$ for a random k-dimensional subspace $E \subset \mathbb{R}^n$. Some of the steps hold true for more general probabilities $\mathbb{P}$ and we will state them in full generality. The following hypothesis will be imposed on $\mathbb{P}$ throughout the section.

Concentration Hypothesis ([So]):

$$\text{(3.3)} \qquad \mathbb{P}\{x \in \mathbb{R}^n : |\,|x| - \sqrt{n}\,M_2| > t\sqrt{n}\,M_2\} \le A \exp(-Bn^\alpha t^\beta)$$

for all $0 \le t \le 1$ and for some constants $\alpha, \beta, A, B > 0$.

3.1. *Gaussian approximation of the average density and distribution.* We first consider Gaussian approximation of the average density $\varphi_{\mathbb{P}}^k(s)$.

THEOREM 3.1. *Let $\mathbb{P}$ be a probability on $\mathbb{R}^n$ satisfying* (3.3) *and suppose that $M_2, M_{\mathbb{P}} < \infty$. Define $h(n) = n^{\min\{\alpha,\alpha/\beta,1/2\}}$ and let $\widetilde{h}(n)$ be such that $\widetilde{h}(n) < c(B,\beta)h(n)$ with $c = c(B,\beta) = \min\{1/8,(B/2)^{\min\{1,1/\beta\}}\}$. If $k \le c\widetilde{h}(n)/\log(1+M_2M_{\mathbb{P}}^{1/n})$, then*

$$\sup_{|s|\le\sqrt{\widetilde{h}(n)}\,M_2}\left|\frac{\varphi_{\mathbb{P}}^k(s)}{\gamma_{\mathbb{P}}^k(s)}-1\right| \le c_1\,\frac{\widetilde{h}(n)}{h(n)}$$

for some constant $c_1 = c_1(\alpha, A, \beta, B) > 0$.

Proof. Recall that, by Proposition 2.1, $\varphi_{\mathbb{P}}^k(s) = \int_{\{|x|\ge|s|\}} g_{|s|}(|x|)\,d\mathbb{P}(x)$, where

$$g_t(r) = \frac{|S^{n-k-1}|}{|S^{n-1}|}\,\frac{1}{r^k}\left(1-\frac{t^2}{r^2}\right)^{(n-k-2)/2}, \qquad r \ge t > 0.$$

Consider the image probability of $\mathbb{P}$ under the map $x \mapsto |x|$, that is, the probability on $[0,\infty)$ also denoted by $\mathbb{P}$ with distribution function $\mathbb{P}\{x \in \mathbb{R}^n : |x| \le r\}$. With this notation,

$$\varphi_{\mathbb{P}}^k(s) = \int\limits_{[|s|,\infty)} g_{|s|}(r)\,d\mathbb{P}(r).$$

In order to estimate the asymptotic behaviour of $\varphi_{\mathbb{P}}^k(s)$ as $n\to\infty$ we write

$$\varphi_{\mathbb{P}}^k(s) = g_{|s|}\big(\sqrt{n}\,M_2\big)\mathbb{P}\{|x| \ge |s|\} + \int\limits_{[|s|,\infty)} (g_{|s|}(r) - g_{|s|}(\sqrt{n}\,M_2))d\mathbb{P}(r). \tag{3.4}$$

Write $g_{|s|}(r) - g_{|s|}\big(\sqrt{n}\,M_2\big) = \int_{\sqrt{n}\,M_2}^r g'_{|s|}(u)du$. By using Fubini's theorem in (3.4), it is easy to see that

$$\begin{aligned}
&\varphi_{\mathbb{P}}^k(s) - g_{|s|}(\sqrt{n}\,M_2)\\
&\quad= -\,g_{|s|}(2\sqrt{n}\,M_2)\mathbb{P}\{2\sqrt{n}\,M_2 \le |x|\} - \int\limits_{|s|}^{\sqrt{n}\,M_2} g'_{|s|}(r)\mathbb{P}\{|x|\le r\}\,dr\\
&\qquad+ \int\limits_{\sqrt{n}\,M_2}^{2\sqrt{n}\,M_2} g'_{|s|}(r)\mathbb{P}\{|x|>r\}\,dr + \int\limits_{[2\sqrt{n}\,M_2,\infty)} g_{|s|}(r)\,d\mathbb{P}(r).
\end{aligned}$$

The summands above are estimated with the help of the following three technical lemmas which extend the ideas in [So] to a general k for $|s|$ far from the origin. The behaviour at the origin (discussed in Lemma 3.4) is estimated via the parameter $M_{\mathbb{P}}$.

LEMMA 3.2. *If $|s| \le \sqrt{n/2}\,M_2$, then*

$$\int\limits_{[2\sqrt{n}\,M_2,\infty)} \frac{g_{|s|}(r)}{g_{|s|}(\sqrt{n}\,M_2)}\,d\,\mathbb{P}(r) \le \frac{A}{2^k}\exp\left(\frac{|s|^2}{M_2^2} - Bn^\alpha\right)$$

and

$$\frac{g_{|s|}(2\sqrt{n}\,M_2)}{g_{|s|}(\sqrt{n}\,M_2)}\,\mathbb{P}\{2\sqrt{n}\,M_2 \le |x|\} \le \frac{A}{2^k}\exp\biggl(\frac{|s|^2}{M_2^2} - Bn^{\alpha}\biggr).$$

Proof. Use the elementary inequalities $(1-x)^{-1} \le e^{2x}$ for $0 \le x \le 1/2$ and $1-x \le e^{-x}$ for $x \ge 0$. ∎

LEMMA 3.3. *If* $(|s|^2/M_2^2)^{\max\{\beta,1\}} < Bn^{\alpha}/2$, *then*

$$\int_{\sqrt{n}\,M_2}^{2\sqrt{n}\,M_2} \frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}\,M_2)}\mathbb{P}\{|x|>r\}\,dr \le \max\biggl\{\frac{|s|^2}{M_2^2},k\biggr\}\frac{A\ c(\beta)}{(Bn^{\alpha})^{1/\beta}}.$$

Proof. By straightforward computations and the inequalities $(1-x)^{-1} \le e^{2x}$ for $0 \le x \le 1/2$ and $1-x \le e^{-x}$ for $x \ge 0$,

$$\frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}\,M_2)} \le \frac{|(n-2)|s|^2 - kr^2|}{r^3}\ e^{-(n-k-4)|s|^2/2r^2}\ e^{(n-k-2)|s|^2/2nM_2^2}.$$

For $r \in [\sqrt{n}\,M_2, 2\sqrt{n}\,M_2]$,

$$\frac{|(n-2)|s|^2 - kr^2|}{r^3} \le \frac{c}{r}\max\biggl\{\frac{|s|^2}{M_2^2},k\biggr\}.$$

On the other hand,

$$\begin{aligned}\frac{(n-k-2)|s|^2}{2nM_2^2} - \frac{(n-k-4)|s|^2}{2r^2} &\le 1 + \frac{(n-k-4)|s|^2}{2nM_2^2}\biggl(1-\frac{nM_2^2}{r^2}\biggr)\\ &\le 1 + \frac{|s|^2}{2M_2^2}\biggl(1-\frac{nM_2^2}{r^2}\biggr).\end{aligned}$$

Upon using such bounds, the change of variables $r = (1+u)\sqrt{n}\,M_2$ and the inequality $1 - 1/(1+u)^2 \le 2u$ for $u \ge 0$ yield

$$\int_{\sqrt{n}\,M_2}^{2\sqrt{n}\,M_2} \frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}\,M_2)}\,\mathbb{P}\{|x|>r\}\,dr \le c\int_0^1 e^{|s|^2u/M_2^2}\mathbb{P}\{|x| > (1+u)\sqrt{n}\,M_2\}\,du.$$

Now use the concentration hypothesis (3.3). The proof finishes by estimating the remaining integral with the aid of the following Claim (with $K = |s|^2/M_2^2$ and $L = Bn^{\alpha}$); see [So, Lemma 9].

CLAIM. *Let* $K, L > 0$ *be such that* $K^{\max\{\beta,1\}} < L/2$. *Then*

$$\int_0^1 \exp(Ku - Lu^{\beta})\,du \le \frac{c(\beta)}{L^{1/\beta}}.$$ ∎

LEMMA 3.4. *There exists $c > 0$ such that if $|s| \le \sqrt{n/2}\, M_2$, $k < n/2$ and $(8k\log(cM_2M_{\mathbb{P}}^{1/n}))^{\max\{1,\beta\}} < Bn^\alpha/2$, then*

$$\int_{|s|}^{\sqrt{n}\,M_2} \frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}\,M_2)}\,\mathbb{P}\{|x| \le r\}\,dr \le \max\left\{\frac{|s|^2}{M_2^2}, k\right\}\frac{Ac(\beta)}{(Bn^\alpha)^{1/\beta}} + \frac{1}{2^n}.$$

Proof. Define $\lambda := (cM_2M_{\mathbb{P}}^{1/n})^{-2}$, with $c > 0$ to be chosen later, and split the integral into two parts

$$\int_{\max\{|s|,\lambda\sqrt{n}\,M_2\}}^{\sqrt{n}\,M_2} + \int_{|s|}^{\max\{|s|,\lambda\sqrt{n}\,M_2\}} = I_1 + I_2.$$

By Hensley's Lemma 1.3 we choose c so that $\lambda < 1$. It is easy to see that

$$\frac{|g'_{|s|}(r)|}{g_{|s|}(\sqrt{n}\,M_2)} \le 2\,\frac{|(n-2)|s|^2 - kr^2|}{r^{k+3}}\,(\sqrt{n}\,M_2)^k.$$

The change of variables $r = \sqrt{n}\,M_2u$ and the inequality

$$\frac{|(n-2)|s|^2 - knM_2^2u^2|}{nM_2^2} \le \max\left\{\frac{|s|^2}{M_2^2}, k\right\}, \qquad 0 \le u \le 1,$$

yield

$$I_1 \le 2\max\left\{\frac{|s|^2}{M_2^2}, k\right\}\int_{\max\{|s|,\lambda\sqrt{n}\,M_2\}/\sqrt{n}\,M_2}^{1} \frac{\mathbb{P}\{|x| \le \sqrt{n}\,M_2u\}}{u^{k+3}}\,du.$$

Set $a = \max\{|s|, \lambda\sqrt{n}\,M_2\}/\sqrt{n}\,M_2$. We have $0 \le a \le 1$. By the change of variables $u = 1 - v$ and the concentration hypothesis (3.3),

$$I_1 \le 2A\max\left\{\frac{|s|^2}{M_2^2}, k\right\}\int_0^{1-a} \exp(-(k+3)\log(1-v) - Bn^\alpha v^\beta)\,dv.$$

Finally, use the inequalities

$$-\log(1-v) \le \frac{-\log a}{1-a}\,v \le \log\left(\frac{3}{a}\right)v \le 2\log(cL_{\mathbb{P}})\ v, \quad v \in [0, 1-a],$$

and the Claim above.

For the second integral I_2 we can suppose $|s| \le \lambda\sqrt{n}\,M_2$. Proceeding as before, we have

$$I_2 \le 2\int_{|s|}^{\lambda\sqrt{n}\,M_2} \frac{|(n-2)|s|^2 - kr^2|}{r^{k+3}}\,(\sqrt{n}\,M_2)^k\mathbb{P}\{|x| \le r\}\,dr.$$

By the inequality $|(n-2)|s|^2 - kr^2| \le nr^2$, the definition of $M_{\mathbb{P}}$ and $k < n/2$

we have

$$I_2 \le 2n(\sqrt{n}\, M_2)^k M_{\mathbb{P}}\, \omega_n \int\limits_{|s|}^{\lambda\sqrt{n}\, M_2} r^{n-k-1}\, dr \le 4(L_{\mathbb{P}} \omega_n^{1/n} \sqrt{n}\, \lambda^{1/2})^n,$$

since $\lambda^{n-k} < \lambda^{n/2}$. Finally, the sequence $\omega_n^{1/n}\sqrt{n}$ is bounded by an absolute constant and we can choose $c > 0$ in the definition of λ so that $I_2 \le 2^{-n}$. ■

End of proof of Theorem 3.1. Notice that the hypotheses of the lemmas are satisfied and therefore

$$\left|\frac{\varphi_{\mathbb{P}}^k(s)}{g_{|s|}(\sqrt{n}\, M_2)} - 1\right| \le \frac{A}{2^{k-1}} \exp\left(\frac{|s|^2}{M_2^2} - Bn^\alpha\right) + \frac{c(A,B,\beta)}{n^{\alpha/\beta}} \max\left\{\frac{|s|^2}{M_2^2}, k\right\} + \frac{1}{2^n}.$$

Finally, use the inequality

$$\left|\frac{\varphi_{\mathbb{P}}^k(s)}{\gamma_{\mathbb{P}}^k(s)} - 1\right| \le \frac{g_{|s|}(\sqrt{n}\, M_2)}{\gamma_{\mathbb{P}}^k(s)} \left|\frac{\varphi_{\mathbb{P}}^k(s)}{g_{|s|}(\sqrt{n}\, M_2)} - 1\right| + \left|\frac{g_{|s|}(\sqrt{n}\, M_2)}{\gamma_{\mathbb{P}}^k(s)} - 1\right|.$$

By Lemma 1.6(ii), (iii) for $u = |s|^2/2M_2^2$, we have $g_{|s|}(\sqrt{n}\, M_2)/\gamma_{\mathbb{P}}^k(s) \le c_1$ and

$$\left|\frac{\varphi_{\mathbb{P}}^k(s)}{\gamma_{\mathbb{P}}^k(s)} - 1\right| \le c_1\left(\exp(\widetilde{h}(n) - Bn^\alpha) + \frac{\widetilde{h}(n)}{n^{\alpha/\beta}} + \frac{1}{2^n}\right) + \left[\frac{8}{n}\left(\frac{k|s|^2}{2M_2^2} + \frac{|s|^4}{4M_2^4}\right)\right] \le c_1\left(\frac{\widetilde{h}(n)}{n^{\alpha/\beta}} + \frac{\widetilde{h}(n)^2}{n}\right),$$

which finishes the proof. ■

For $\mathbb{P}$ the uniform measure on an isotropic convex body K we obtain as a corollary

Theorem 3.5. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body satisfying the concentration hypothesis* (3.3). *For some c, c_1 depending on the constants in* (3.3),

(1) *If $k \le c b \widetilde{h}(n)/\log(1 + L_K)$, then*

$$\sup_{|s| \in I} \left|\frac{\varphi_K^k(s)}{\gamma_K^k(s)} - 1\right| \le c_1 \frac{\widetilde{h}(n)}{h(n)}, \qquad I = \left[0, L_K\sqrt{\widetilde{h}(n)}\right].$$

(2) *If $k \le c\widetilde{h}(n)/\log^2 n$, then*

$$\sup_{t \ge 0} \left|\frac{F_K^k(t)}{\Gamma_K^k(t)} - 1\right| \le c_1 \frac{\widetilde{h}(n)}{h(n)}.$$

Proof. Statement (1) is a consequence Theorem 3.1, since in our case $M_{\mathbb{P}} = 1$, $M_2 = L_K$.

Part (2) follows from (1). Indeed, by Lemmas 2.5(ii) and 1.5(ii),

$$\begin{aligned}|F_K(t,E) - \Gamma_K^k(t)| &= |(1 - F_K(t,E)) - (1 - \Gamma_K^k(t))| \\ &\le e^{-c_2 t/L_K\sqrt{k}} + 2^{k/2} e^{-t^2/4L_K^2}.\end{aligned}$$

Therefore, in the range $t \ge C\sqrt{k}\, L_K \log n$ (for suitable $C > 0$) we trivially have $|F_K(t,E) - \Gamma_K^k(t)| \le 2/n$ for every k-dimensional subspace E. For that range of t, Lemma 1.5(ii) gives $\Gamma_K^k(t) \ge c_0 > 0$ and so

$$\left|\frac{F_K(t,E)}{\Gamma_K^k(t)} - 1\right| \le \frac{c_1}{n}.$$

Finally, observe that $t \le C\sqrt{k}\, L_K \log n$ implies $t \le L_K\sqrt{\widetilde{h}(n)}$, and so by integrating (1) and formula (1.2) we have the result. ∎

Example. It is proved in [So] that the uniform probability on the unit ball of ℓ_p^n, $p \ge 1$, satisfies the concentration hypothesis (3.3) for $\alpha = \frac{1}{2}\min\{p,2\}$ and $\beta = \min\{p,2\}$. So, $h(n) = \sqrt{n}$ and by taking $\widetilde{h}(n) = o(h(n))$, Theorem 3.5(1) implies that $\sup_{|s|\in I} |\varphi_K^k(s)/\gamma_K^k(s) - 1| \to 0$ as $n \to \infty$ for $I = [0, o(n^{1/4})]$ and $k = o(n^{1/2})$ (since in this case L_K is uniformly bounded by a constant depending only on p).

If we study the behaviour at $t = 0$ of Theorem 3.1(2), we obtain the following strong form of reverse Hölder inequality in the spirit of [V].

Corollary 3.6. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body satisfying* (3.3). *If $k = o(h(n)/\log^2 n)$, then*

$$\Big(\int_K |x|^2\,dx\Big)^{1/2}\Big(\int_K \frac{dx}{|x|^k}\Big)^{1/k} \to 1 \qquad \text{as } n \to \infty.$$

Proof. By Remark 2.4 and L'Hopital's rule,

$$\lim_{t\to 0^+} \frac{F_K(t,E)}{\Gamma_K^k(t)} = \lim_{t\to 0^+} \frac{f_K(t,E)}{(\sqrt{2\pi}\,L_K)^{-k} e^{-t^2/2L_K^2}} = (\sqrt{2\pi}\,L_K)^k |E^\perp \cap K|_{n-k}.$$

Therefore,

$$\lim_{t\to 0^+} \frac{F_K^k(t)}{\Gamma_K^k(t)} = (\sqrt{2\pi}\,L_K)^k \int_{G_{n,n-k}} |E \cap K|_{n-k}\,d\nu.$$

But this is equal to

$$(\sqrt{2\pi}\,L_K)^k \frac{\omega_{n-k}}{\omega_n} \widetilde{W}_k(K) = (\sqrt{2\pi}\,L_K)^k \frac{|S^{n-k-1}|}{|S^{n-1}|} \int_K \frac{dx}{|x|^k}$$

by the dual Kubota formula, where $\widetilde{W}_k(K)$ denotes the kth dual mixed volume of K (see [BBR]). Since $L_K^2 = \frac{1}{n}\int_K |x|^2\,dx$, we have

$$\lim_{t\to 0^+} \frac{F_K^k(t)}{\Gamma_K^k(t)} = \frac{(2\pi)^{k/2}}{n^{k/2}} \frac{|S^{n-k-1}|}{|S^{n-1}|}\Big(\int_K |x|^2\,dx\Big)^{k/2}\Big(\int_K \frac{dx}{|x|^k}\Big).$$

By Lemma 1.6(ii) and Theorem 3.5(2), the result follows. ∎

3.2. *Gaussian behaviour of a typical subspace.* The main tool of this subsection is the concentration of measure phenomenon in the space $G_{n,k}$ equipped with its Haar probability and the distance $\|P_{E_1}-P_{E_2}\|_{\mathrm{HS}}$, $E_1, E_2 \in G_{n,k}$, where P_E is the orthogonal projection onto E. Recall that the modulus of continuity of a continuous $f\colon G_{n,k}\to\mathbb{R}$ is

$$\omega(a) = \sup_{\|P_{E_1}-P_{E_2}\|_{\mathrm{HS}}\le a} |f(E_1)-f(E_2)|, \qquad a>0.$$

Theorem 3.7 (Concentration of measure). *Denote by ν the Haar probability on $G_{n,k}$. Let $f\colon G_{n,k}\to\mathbb{R}$ be continuous. There exist absolute constants $c_1, c_2>0$ such that for every $a>0$,*

$$\nu\{E\in G_{n,k} : |f(E)-\mathbb{E}(f(E))| > \omega(a)\} \le c_1\exp(-c_2 n a^2).$$

Proof. For $G_{n,k}$ equipped with the distance

$$d(E_1,E_2)$$
$$= \min\Big\{\Big(\sum_{j=1}^k |u_j-v_j|^2\Big)^{1/2} : (u_j),(v_j) \text{ orthonormal bases of } E_1, E_2\Big\}$$

for $E_1, E_2\in G_{n,k}$, the inequality above is stated in [MS].

To finish the proof we show $\|P_{E_1}-P_{E_2}\|_{\mathrm{HS}} \le \sqrt{2}\,d(E_1,E_2)$. Indeed, for any orthonormal bases $(u_j),(v_j)$ of E_1,E_2 we write $P_{E_1}=\sum_{j=1}^k u_j\otimes u_j$ and $P_{E_2}=\sum_{i=1}^k v_i\otimes v_i$ and by definition

$$\|P_{E_1}-P_{E_2}\|_{\mathrm{HS}}^2 = 2k - 2\sum_{i,j=1}^k \langle u_j,v_i\rangle^2 \le 2\sum_{j=1}^k (1-\langle u_j,v_j\rangle^2) \le 2\sum_{j=1}^k |u_j-v_j|^2$$

since $1-\langle u_j,v_j\rangle^2 \le 2(1-\langle u_j,v_j\rangle) = |u_j-v_j|^2$. ∎

We will compute the modulus of continuity of $E\mapsto F_K(t,E)/\Gamma_K^k(t)$:

Lemma 3.8. *Let $0<\varepsilon<1$ and $K\subset\mathbb{R}^n$ be an isotropic convex body. Let $0<t<c_1\sqrt{k}\,L_K\log n$. Then for every $E_1,E_2\in G_{n,k}$ and some universal constant $c>0$ we have*

$$\Big|\frac{F_K(t,E_1)}{\Gamma_K^k(t)} - \frac{F_K(t,E_2)}{\Gamma_K^k(t)}\Big| \le \varepsilon$$

provided that $\|P_{E_1} - P_{E_2}\|_{\mathrm{HS}} \le a$ *where*

$$a = \begin{cases} \dfrac{c^k}{\mathcal{L}_k^k}\Big(\dfrac{\varepsilon t}{L_K}\Big)^{5/4} & \text{if } t \le 2\sqrt{k}\,L_K, \\ \dfrac{c^k \varepsilon^2}{\mathcal{L}_k^k (\log n)^{k-1}} & \text{otherwise.} \end{cases}$$

Proof. Let $0 < \delta$ $(< t)$ to be fixed later. By the triangle inequality,

$$F_K(t, E_2) - F_K(t, E_1) \le F_K(t+\delta, E_1) - F_K(t, E_1) + |\{x \in K : |(P_{E_1} - P_{E_2})(x)| \ge \delta\}|.$$

Let us estimate each summand. By Remark 2.4,

$$F_K(t+\delta, E_1) - F_K(t, E_1) = |S^{k-1}| \int_t^{t+\delta} r^{k-1} f_K(r, E_1)\, dr \le |S^{k-1}| c^k \frac{\mathcal{L}_k^k}{L_K^k} \frac{1}{k} \left((t+\delta)^k - t^k\right)$$

(by Lemma 2.5(i)). By the mean value theorem, $(t+\delta)^k - t^k \le k(t+\delta)^{k-1}\delta \le k2^{k-1}t^{k-1}\delta$, so

$$F_K(t+\delta, E_1) - F_K(t, E_1) \le |S^{k-1}| c^k \frac{\mathcal{L}_k^k}{L_K^k}\, t^{k-1}\delta.$$

Now we compute the second summand. Repeating the arguments in Lemma 2.5(ii) with $T = P_{E_1} - P_{E_2}$, we have

$$|\{x \in K : |(P_{E_1} - P_{E_2})(x)| \ge \delta\}| \le 2\exp\Big(-\frac{c_1\delta}{L_K\|P_{E_1} - P_{E_2}\|_{\mathrm{HS}}}\Big).$$

Put the estimates together, exchange E_1 and E_2 and conclude that

$$\left|\frac{F_K(t, E_1)}{\Gamma_K^k(t)} - \frac{F_K(t, E_2)}{\Gamma_K^k(t)}\right| \le \frac{|S^{k-1}| c^k \mathcal{L}_k^k}{L_K^k \Gamma_K^k(t)} t^{k-1}\delta + \frac{2}{\Gamma_K^k(t)} \exp\Big(-\frac{c_1\delta}{L_K a}\Big). \tag{3.5}$$

If $t \ge 2\sqrt{k}\,L_K$, Lemma 1.5(ii) gives $\Gamma_K^k(t) \ge c'$. Take

$$\delta = \frac{L_K\varepsilon}{|S^{k-1}| c_0^k \mathcal{L}_k^k (c\sqrt{k}\log n)^{k-1}} \ (< t).$$

Substituting in formula (3.5) together with $|S^{k-1}| \le c_0^k/k^{k/2}$ (Lemma 1.4(ii)), $\mathcal{L}_k \le c_1 k^{1/4}$ ([Kl2]) and $L_K \ge c_2$, we have

$$\left|\frac{F_K(t, E_1)}{\Gamma_K^k(t)} - \frac{F_K(t, E_2)}{\Gamma_K^k(t)}\right| \le \frac{\varepsilon}{2} + \exp\Big(\frac{-c_3\varepsilon}{c_0^k \mathcal{L}_k^k (\log n)^{k-1} a}\Big) \quad \text{if } \|P_{E_1} - P_{E_2}\|_{\mathrm{HS}} \le a.$$

Set $a = c_3\varepsilon^2 / c_0^k \mathcal{L}_k^k (\log n)^{k-1}$ so the second summand reads $\exp(-1/\varepsilon) \le \varepsilon/2$.

If $t \leq 2\sqrt{k}\, L_K$, Lemma 1.5(i) implies $\Gamma_K^k(t) \geq e^{-2k} t^k \omega_k / (\sqrt{2\pi}\, L_K)^k$. We substitute this estimate in formula (3.5) to obtain

$$\left| \frac{F_K(t,E_1)}{\Gamma_K^k(t)} - \frac{F_K(t,E_2)}{\Gamma_K^k(t)} \right| \leq c^k \mathcal{L}_k^k \, \frac{\delta}{t} + \frac{c^k L_K^k}{\omega_k t^k} \exp\left(- \frac{c\delta}{L_K a} \right).$$

We take $\delta = c'\varepsilon t / c^k \mathcal{L}_k^k$ so that the first summand is less than $\varepsilon/2$. With this choice of δ, if we also write $u = t/2L_K \in [0, \sqrt{k}]$, the second summand becomes

$$\frac{c_1^k}{\omega_k u^k} \exp\left(- \frac{\varepsilon u}{\mathcal{L}_k^k c_2^k a} \right).$$

Finally, set $a = c\varepsilon^{5/4} u^{5/4} / c_2^k k^{9/8} \mathcal{L}_k^k$ for some appropriately chosen $c > 1$ and substitute in the previous formula to get

$$\frac{c_1^k}{\omega_k u^k} \exp\left(- \frac{k^{9/8} c}{(\varepsilon u)^{1/4}} \right) =: h(u).$$

The maximum value of h is attained at u_0 so that $h'(u_0) = 0$, that is,

$$u_0 = \frac{c^4 \sqrt{k}}{\varepsilon} \quad \text{and} \quad h(u_0) = \frac{c_1^k \varepsilon^k}{\omega_k c^k k^{k/2}}\, e^{-k} \leq \frac{\varepsilon}{2}. \ \blacksquare$$

Next, we apply Theorem 3.7. Recall that $c_1 \widetilde{h}(n)/h(n)$ is the error term in Theorem 3.5.

Lemma 3.9. *Let* $0 < \varepsilon < 1$, $0 < t < c_1 \sqrt{k}\, L_K \log n$, $K \subset \mathbb{R}^n$ *an isotropic convex body satisfying the concentration hypothesis* (3.3), *and* $k \leq c\widetilde{h}(n)/\log^2 n$. *Then*

$$\nu\left\{ E \in G_{n,k} : \left| \frac{F_K(t,E)}{\Gamma_K^k(t)} - 1 \right| > \varepsilon + c_1 \frac{\widetilde{h}(n)}{h(n)} \right\} \leq c_1 \exp(-c_2 a^2 n)$$

where

$$a = \begin{cases} \dfrac{c^k}{\mathcal{L}_k^k} \left(\dfrac{\varepsilon t}{L_K} \right)^{5/4} & \text{if } t \leq 2\sqrt{k}\, L_K, \\ \dfrac{c^k \varepsilon^2}{\mathcal{L}_k^k (\log n)^{k-1}} & \text{otherwise.} \end{cases}$$

Proof. Theorem 3.5 states that

$$\left| \frac{F_K^k(t)}{\Gamma_K^k(t)} - 1 \right| \leq c_1 \frac{\widetilde{h}(n)}{h(n)}.$$

Hence,

$$\nu\left\{E : \left|\frac{F_K(t,E)}{\mathit{\Gamma}_K^k(t)} - 1\right| > \varepsilon + c_1\frac{\widetilde{h}(n)}{h(n)}\right\}$$
$$\leq \nu\left\{E : \left|\frac{F_K(t,E)}{\mathit{\Gamma}_K^k(t)} - \frac{F_K^k(t)}{\mathit{\Gamma}_K^k(t)}\right| > \varepsilon\right\} \leq c_1\exp(-c_2na^2),$$

since Lemma 3.8 reads $\omega(a) \leq \varepsilon$. ∎

In our last result of this section we pass from Lemma 3.9, valid for any fixed t, to a statement that holds for all t simultaneously.

LEMMA 3.10. *Let* $0 < \varepsilon < 1/2$, $t_0 > 0$, $K \subset \mathbb{R}^n$ *an isotropic convex body satisfying* (3.3) *and* $k \leq c\widetilde{h}(n)/\log^2 n$. *Suppose* $c_1\widetilde{h}(n)/h(n) \leq 1/2$. *Then*

$$\nu\left\{E \in G_{n,k} : \left|\frac{F_K(t,E)}{\mathit{\Gamma}_K^k(t)} - 1\right| \leq 2\varepsilon + 2c_1\frac{\widetilde{h}(n)}{h(n)},\ \forall t \geq t_0\right\} \geq 1 - N\exp(-cA^2n)$$

where

$$N \leq \left(\frac{c\sqrt{k}\,n^{1/4}\log n}{t_0}\right)^{c_1k/\varepsilon} \quad \textit{and} \quad A \geq \frac{c_2^k}{k^{k/4}}\,\varepsilon^2\min\left\{\frac{t_0^{5/4}}{n^{5/16}}, \frac{1}{(\log n)^{k-1}}\right\}.$$

Proof. By the arguments in the proof of Theorem 3.5(2), we only need to compute the probability for $t \in [t_0, T]$ with $T = C\sqrt{k}\,L_K\log n$.

Pick $0 < t_0 < t_1 \leq \cdots \leq t_N = T$ in the following way:

$$t_i = t_0\prod_{j=1}^{i}\left(1 + \frac{\varepsilon}{8kj}\right) \sim t_0 i^{c_1\varepsilon/k}, \qquad i = 1, \ldots, N.$$

Write $\eta = 2\varepsilon + 2c_1\widetilde{h}(n)/h(n)$, $0 < \eta < 2$. By Lemma 3.9,

$$\nu\left\{E : \left|\frac{F_K(t_i,E)}{\mathit{\Gamma}_K^k(t_i)} - 1\right| > \frac{\eta}{2} \text{ for some } i\right\} \leq c_1\sum_{i=0}^{N}\exp(-c_2na_i^2)$$

where

$$a_i = \begin{cases} \dfrac{c^k}{\mathcal{L}_k^k}\left(\dfrac{\varepsilon t_i}{L_K}\right)^{5/4} & \text{if } t \leq 2\sqrt{k}\,L_K, \\[2ex] \dfrac{c^k\varepsilon^2}{\mathcal{L}_k^k(\log n)^{k-1}} & \text{otherwise.} \end{cases}$$

If $t \in [t_i, t_{i+1}]$, the fact that

$$\left|\frac{F_K(t,E)}{\mathit{\Gamma}_K^k(t)} - 1\right| > \eta$$

implies that either

$$F_K(t_{i+1},E) > (1+\eta)\mathit{\Gamma}_K^k(t_i) \quad \text{or} \quad F_K(t_i,E) < (1-\eta)\mathit{\Gamma}_K^k(t_{i+1}).$$

Taking into account the choice of t_i, Lemma 1.5(iii) (with $t = t_i$, $\delta = t_{i+1} - t_i$) reads

$$\frac{\Gamma_K^k(t_{i+1})}{\Gamma_K^k(t_i)} \le \left(\frac{t_{i+1}}{t_i}\right)^k \le \left(1 + \frac{\varepsilon}{8k(j+1)}\right)^k \le e^{\varepsilon/8} \le 1 + \frac{\eta}{4}$$

and so, by the elementary inequalities $(1+\eta)(1+\eta/4)^{-1} \ge 1 + \eta/2$ and $(1-\eta)(1+\eta/4) < 1 - \eta/2$ we find that either

$$F_K(t_{i+1}, E) > (1+\eta/2)\Gamma_K^k(t_{i+1}) \quad \text{or} \quad F_K(t_i, E) < (1-\eta/2)\Gamma_K^k(t_i).$$

Thus,

$$\nu\left\{E \in G_{n,k} : \left|\frac{F_K(t,E)}{\Gamma_K^k(t)} - 1\right| > \eta \text{ for some } t \in [t_0, T]\right\}$$
$$\le G_{n,k} : \left|\frac{F_K(t_i,E)}{\Gamma_K^k(t_i)} - 1\right| > \frac{\eta}{2} \text{ for some } i\Big\} \le c_1 N \exp(-c_2 n A^2)$$

where $A = \min_{1 \le i \le N} a_i$. By definition,

$$c\sqrt{k}\, L_K \log n = T = t_N \sim t_0 N^{c_1\varepsilon/k}.$$

That is,

$$N \sim \left(\frac{c\sqrt{k}\, L_K \log n}{t_0}\right)^{c_1 k/\varepsilon}, \qquad A = \frac{c_2^k}{k^{k/4}}\, \varepsilon^2 \min\left\{\left(\frac{t_0}{L_K}\right)^{5/4}, \frac{1}{(\log n)^{k-1}}\right\}.$$

Eventually, we use $L_K \le Cn^{1/4}$, and the result follows. ∎

THEOREM 3.11. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body satisfying condition* (3.3), *and $t_0 > 0$. Then for every $0 < \varepsilon < 1$ and $1 \le k \le c_1\varepsilon \log n/(\log\log n)^2$ we have*

$$\nu\left\{E \in G_{n,k} : \sup_{t \ge t_0}\left|\frac{F_K(t,E)}{\Gamma_K^k(t)} - 1\right| \le \varepsilon\right\} \ge 1 - \exp(-c_2 n^{0.37})$$

where c_1 depends only on the constants appearing in (3.3) *and c_2 depends only on t_0.*

Proof. By hypothesis,

$$\frac{k}{\varepsilon} \le \frac{c_1 \log n}{(\log\log n)^2} \quad \text{and} \quad \varepsilon \ge \frac{(\log\log n)^2}{c_1 \log n}.$$

We can clearly choose $\widetilde{h}(n)$ to fulfill the hypothesis of Lemma 3.10 and moreover c_1 can be adjusted in order that

$$c_1 \frac{\widetilde{h}(n)}{h(n)} \le \frac{(\log\log n)^2}{c_1 \log n} \ (\le \varepsilon).$$

Now, direct computations ($\log N \le c\frac{k}{\varepsilon}\log n$ and $-\log A^2 \le \frac{5}{8}\log n + c_2\frac{\log n}{\log\log n}$) show that

$$\log N - cnA^2 \le -c_2 n^{0.37},$$

and the result follows. ∎

REMARK 3.12. The expression $\ge 1-\exp(-c_2 n^{0.37})$ only points out that the probability tends "very fast" to 1. The exponent 0.37 is simply a choice of a number close to $1-5/8 = 0.375$. (Actually, by changing the exponent $5/4$ to, say, 1.001 in Lemma 3.8 we could reach 0.49...).

REMARK 3.13. The method of proof seems to have the limitation given by Lemma 1.5 ($f_K(r,E) \le e^k|E^\perp \cap K|$) so that from this fact one has $A \ge c^k$. This means that, in order to make $\exp(-A^2 n)$ tend to 0 "fast", $k \ll \log n$ is necessary. It is in this sense that our result is sharp for the method up to $\log\log n$ factors.

Using the results in [ABBP] one can show that for *random* subspaces $E \in G_{n,k}$ one has an improvement of Lemma 1.5, $f_K(r,E) \le c^k L_K^{-k}$ (thus getting rid of $\mathcal{L}_k$). Hence, it is possible to improve Lemma 3.8 for these subspaces and still be able to use a concentration of measure argument to improve Lemmas 3.9 and 3.10. This will result in an improvement by a $\log\log n$ factor.

4. Asymptotic results on the average density and distribution

4.1. *Gaussian approximation of the average density and distribution.* In this section we show that, for a range of k and a class of probabilities $\mathbb{P}$, the average density is uniformly close to the Gaussian density. Furthermore, if $\mathbb{P}$ has exponential tails on half spaces (see definition below), we can also approximate the average distribution. Recall that $F_{\mathbb{P}}(t,E) = \mathbb{P}\{x \in \mathbb{R}^n : |P_E(x)| \le t\}$.

DEFINITION 4.1. Let $c>0$. Denote by $\mathcal{P}_{c,n}$ the set of Borel probabilities such that $\sigma_{\mathbb{P}}, M_2, M_{\mathbb{P}}^{1/n} \le c$.

THEOREM 4.2. *Let $k \le c\sqrt{\log n}/(\log\log n)^{1/2+\delta}$ for some $\delta>0$. Then there exists $c_1>0$ (depending only on c and δ) such that for all $\mathbb{P}\in\mathcal{P}_{c,n}$,*

$$\sup_{s\in\mathbb{R}^k} |\varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s)| \le \frac{c_1^k k^{k/2}}{n^{1/(k+3)}}. \tag{1}$$

Furthermore, if $\mathbb{P}$ satisfies $\mathbb{P}\{x\in\mathbb{R}^n : |\langle\theta,x\rangle| > t\} \le c_2\exp(-c_3 t/M_2)$ for some $c_2, c_3 > 0$ and all $t>0$ and $\theta\in S^{n-1}$, then

$$\sup_{t\ge 0} |F_K^k(t) - \Gamma_{\mathbb{P}}^k(t)| \le \frac{c_4^k k^{k/2}}{n^{1/(k+3)}}(\log n)^k \tag{2}$$

for some $c_4>0$ depending only on the constants.

Proof. Observe, by straightforward computation, that the bound on k ensures that the error terms in (1) and (2) tend to 0 as $n \to \infty$.

The proof of (1) will be done in three steps. Step 3 takes care of very large values of $|s|$, Step 2 of values of $|s|$ near (and including) the origin, and Step 1 of the remaining case. Fix $c_0 > 0$ small enough that will be chosen below. It is used to separate these three steps.

STEP 1. *Let $k = o(n)$. There exists a constant $C > 0$ such that for $0 < |s| \le c_0\sqrt{n}\,/M_{\mathbb{P}}^{1/n}$ and every Borel probability $\mathbb{P}$ we have*

$$|\varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s)| \le Ck^{k/2}\biggl(\frac{\sigma_{\mathbb{P}} M_2}{\sqrt{n}\,|s|^{k+1}} + \frac{1}{nM_{\mathbb{P}}^{k/n}}\biggr).$$

Proof of Step 1. By formula (3.4),

$$\begin{aligned}\varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s) = (g_{|s|}(\sqrt{n}\,M_2) - \gamma_{\mathbb{P}}^k(s)) + g_{|s|}(\sqrt{n}\,M_2)\mathbb{P}\{|x| < |s|\}\\ + \int\limits_{[|s|,\infty)} (g_{|s|}(r) - g_{|s|}(\sqrt{n}\,M_2))\,d\mathbb{P}(r).\end{aligned}$$

We compute the second and third summands with the aid of the following lemmas:

LEMMA 4.3. *Let $k = o(n)$. There exists an absolute constant $C > 0$ such that*

(i) $$\sup_{r\ge t} g_t(r) \le \frac{1}{t^k}\,\frac{Ck^{k/2}}{(2\pi e)^{k/2}},$$

(ii) $$\sup_{r\ge t} |g_t'(r)| \le \frac{1}{\sqrt{n}\,t^{k+1}}\,\frac{Ck^{(k+3)/2}}{(2\pi e)^{k/2}}.$$

Proof. By Lemma 1.4, $|S^{n-k-1}|/|S^{n-1}| \le Cn^{k/2}/(2\pi)^{k/2}$. Proceed as in [BK]. ∎

LEMMA 4.4. *Let $k = o(n)$. There exists $C > 0$ such that for all $s \in \mathbb{R}^k$ with $0 < |s| \le \sqrt{n}\,M_2$,*

(i) $$g_{|s|}(\sqrt{n}\,M_2)\mathbb{P}\{|x| < |s|\} \le \frac{Ck^{k/2}}{(2\pi e)^{k/2}}\,M_{\mathbb{P}}\omega_n|s|^{n-k},$$

(ii) $$\int\limits_{\{r\ge|s|\}} (g_{|s|}(r) - g_{|s|}(\sqrt{n}\,M_2))\,d\mathbb{P}(r) \le \frac{Ck^{k/2}\,\sigma_{\mathbb{P}} M_2}{\sqrt{n}\,|s|^{k+1}}.$$

Proof. (i) By Lemma 4.3,

$$g_{|s|}(\sqrt{n}\,M_2) \le \frac{C}{|s|^k}\,\frac{k^{k/2}}{(2\pi e)^{k/2}}$$

and, by definition of $M_{\mathbb{P}}$, $\mathbb{P}\{|x| \le |s|\} \le M_{\mathbb{P}}\omega_n|s|^n$.

(ii) By the mean value theorem,

$$\int_{\{|x|\geq|s|\}} |g_{|s|}(|x|) - g_{|s|}(\sqrt{n}\, M_2)|\, d\mathbb{P}(x) \leq \sup_{r\geq|s|} |g'_{|s|}(r)| \int_{\mathbb{R}^n} |\,|x| - \sqrt{n}\, M_2|\, d\mathbb{P}(x).$$

Now use Lemma 4.3 and the inequality $\int_{\mathbb{R}^n} |\,|x| - \sqrt{n}\, M_2|\, d\mathbb{P}(x) \leq \sigma_{\mathbb{P}} M_2$ (see [BK]). ∎

Observe that, by suitably choosing c_0 we have: (a) $|s| \leq c_0\sqrt{n}\,/M_{\mathbb{P}}^{1/n}$ implies $|s| \leq \sqrt{n}\, M_2$, by Hensley's Lemma 1.3, and (b) the second error term in the above lemma absorbs the first one.

It remains to estimate the first summand, $|\gamma_{\mathbb{P}}^k(s) - g_{|s|}(\sqrt{n}\, M_2)|$ where

$$g_{|s|}(\sqrt{n}\, M_2) = \frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{1}{n^{k/2} M_2^k} \left(1 - \frac{|s|^2}{n M_2^2}\right)^{(n-k-2)/2}.$$

Write $|s|^2 = 2M_2^2 u$. Then $0 < |s| \leq \sqrt{n}\, M_2$ is equivalent to $0 < u \leq n/2$, and so for such values of u we need to estimate

$$\frac{1}{(2\pi)^{k/2} M_2^k} \left| \frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{(2\pi)^{k/2}}{n^{k/2}} \left(1 - \frac{2u}{n}\right)^{(n-k-2)/2} - e^{-u}\right|.$$

By Lemma 1.3 we have $1/(2\pi)^{k/2} M_2^k \leq C^k M_{\mathbb{P}}^{k/n}$. Finally, add the value

$$\pm \frac{|S^{n-k-1}|}{|S^{n-1}|} \frac{(2\pi)^{k/2}}{n^{k/2}}\, e^{-u}$$

and use Lemma 1.6 to conclude the proof of Step 1.

Step 2. *Let $\mathbb{P} \in \mathcal{P}_{c,n}$ and $k = o(n)$. Then*

$$|\varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s)| \leq \frac{c_1^k k^{k/2}}{n^{1/(k+3)}}$$

for all $|s| \leq c_0\sqrt{n}\,/M_{\mathbb{P}}^{1/n}$ (c_1 depending only on c).

Proof of Step 2. By Lemma 1.3 we also have $M_2, M_{\mathbb{P}}^{1/n} \geq c_2 > 0$.

Let (s_n) be a sequence such that $\sqrt{n}\,|s_n|^{k+1} = n^{1/(k+3)}$, or equivalently, $|s_n| = n^{-1/2(k+3)}$. For $|s| \geq |s_n|$ we have

$$|\varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s)| \leq C k^{k/2} \left(\frac{\sigma_{\mathbb{P}} M_2}{\sqrt{n}\,|s|^{k+1}} + \frac{1}{n M_{\mathbb{P}}^{k/n}} \right) \leq c_1^k k^{k/2} n^{-1/(k+3)}.$$

If $0 \leq |s| \leq |s_n|$, write

$$|\varphi_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s)| \leq |\varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(s_n)| + |\varphi_{\mathbb{P}}^k(s_n) - \gamma_{\mathbb{P}}^k(s_n)| + |\gamma_{\mathbb{P}}^k(s_n) - \gamma_{\mathbb{P}}^k(s)|.$$

The second summand was estimated above. As for the third one, the inequality $|e^{-x} - e^{-y}| \leq |x|$ for $x \geq y > 0$ implies

$$|\gamma_{\mathbb{P}}^k(s) - \gamma_{\mathbb{P}}^k(s_n)| \leq \frac{|s_n|^2}{2(2\pi)^{k/2} M_2^{k+2}} \leq \frac{c_1^k}{n^{1/(k+3)}}.$$

For the first summand, we use the following lemma:

LEMMA 4.5. *Let $n \geq 2k$. There exist $c_0, c_1 > 0$ such that for all $s \in \mathbb{R}^k$ with $|s| \leq c_0\sqrt{n}\,/M_{\mathbb{P}}^{1/n}$,*

$$|\varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(0)| \leq c_1^k(n^{k/2}M_{\mathbb{P}}\omega_n|s|^{n-k} + M_{\mathbb{P}}^{(k+2)/n}|s|^2).$$

This finishes the proof of Step 2 since the estimate of the remaining first summand readily follows from

$$|\varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(s_n)| \leq |\varphi_{\mathbb{P}}^k(s_n) - \varphi_{\mathbb{P}}^k(0)| + |\varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(0)|.$$

Proof of Lemma 4.5. By definition, $|\varphi_{\mathbb{P}}^k(s) - \varphi_{\mathbb{P}}^k(0)|$ equals

$$\frac{|S^{n-k-1}|}{|S^{n-1}|}\int_{\{|x|\leq|s|\}}\frac{d\mathbb{P}(x)}{|x|^k} + \frac{|S^{n-k-1}|}{|S^{n-1}|}\int_{\{|x|\geq|s|\}}\frac{1}{|x|^k}\bigg(\bigg(1-\frac{|s|^2}{|x|^2}\bigg)^{(n-k-2)/2} - 1\bigg)\,d\mathbb{P}(x).$$

We estimate the first summand. By Fubini's theorem,

$$\int_{\{|x|\leq|s|\}}\frac{d\mathbb{P}(x)}{|x|^k} = \int_0^\infty \mathbb{P}\{|x|\leq|s|,\, 1/|x|^k > t\}\,dt = \int_0^{1/|s|^k} + \int_{1/|s|^k}^\infty .$$

The first integral is equal to $\int_0^{1/|s|^k}\mathbb{P}\{|x|\leq|s|\}\,dt$ and by definition of $M_{\mathbb{P}}$, this is bounded by $|s|^{n-k}M_{\mathbb{P}}\omega_n$.

The second integral is equal to

$$\int_{1/|s|^k}^\infty \mathbb{P}\{1/|x|^k > t\}\,dt \leq \int_{1/|s|^k}^\infty M_{\mathbb{P}}\omega_n t^{-n/k}\,dt = M_{\mathbb{P}}\omega_n\,\frac{k}{n-k}\,|s|^{n-k}.$$

Therefore, by Lemma 1.4,

$$\frac{|S^{n-k-1}|}{|S^{n-1}|}\int_{\{|x|\leq|s|\}}\frac{d\mathbb{P}(x)}{|x|^k} \leq \frac{cn^{k/2}}{(2\pi)^{k/2}}\,M_{\mathbb{P}}\omega_n\frac{n}{n-k}|s|^{n-k} \leq c^k n^{k/2}M_{\mathbb{P}}\omega_n|s|^{n-k}.$$

Next we compute the second summand. Use in the integrand the elementary inequality $|a^p - b^p| \leq p|a-b|$ for $a, b \in [0,1]$ with $p = (n-k-2)/2$ to conclude that the second summand is bounded by

$$\frac{|S^{n-k-1}|}{|S^{n-1}|}\,\frac{n-k-2}{2}\int_{\{|x|\geq|s|\}}\frac{|s|^2}{|x|^{k+2}}\,d\mathbb{P}(x) = \frac{(n-k-2)|S^{n-k-1}|}{2|S^{n-k-3}|}\,|s|^2\varphi_{\mathbb{P}}^{k+2}(0).$$

By Proposition 4.7 below we have, for $|s| \leq c(\omega_n M_{\mathbb{P}})^{-1/n} \sim c_0\sqrt{n}/M_{\mathbb{P}}^{1/n}$,

$$\varphi_{\mathbb{P}}^{k+2}(0) \leq c_1 M_{\mathbb{P}}^{(k+2)/n}\,\frac{\omega_{n-k-2}}{\omega_n^{1-(k+2)/n}} \leq c_1^k M_{\mathbb{P}}^{(k+2)/n}.$$

Finally,

$$(n-k-2)\,\frac{|S^{n-k-1}|}{|S^{n-k-3}|}=\pi(n-k)\,\frac{\Gamma\left(\frac{n-k-2}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)}\le c_2$$

by Lemma 1.4, and putting the estimates together, we see that the second summand is bounded by $c_1^k|s|^2 M_{\mathbb{P}}^{(k+2)/n}$, which finishes the proof of the lemma. ∎

STEP 3. *For every probability* $\mathbb{P}$ *with* $M_{\mathbb{P}}^{1/n}\le c$, $|s|\ge c_0\sqrt{n}\,/M_{\mathbb{P}}^{1/n}$ *and* $k\le n/\mathrm{log}n$,

$$|\varphi_{\mathbb{P}}^k(s)-\gamma_{\mathbb{P}}^k(s)|\le\frac{c_1^k k^{k/2}}{n^{k/2}}$$

where $c_1>0$ *depends only on* c.

Proof of Step 3. By Lemma 1.3, $M_2\ge c_2>0$ (depending on c) and trivially

$$\gamma_{\mathbb{P}}^k(s)\le c_1^k\exp(-c_3 n)\le c_1^k/n^{k/2}.$$

On the other hand, by Lemma 4.3,

$$\varphi_{\mathbb{P}}^k(s)\le\max_{|s|\le|x|}g_{|s|}(x)\mathbb{P}\{x:|x|\ge|s|\}\le\frac{k^{k/2}}{(2\pi e)^{k/2}|s|^k}\le\frac{c_1^k k^{k/2}}{n^{k/2}}.$$

This finishes the proof of (1).

Now we prove (2). Let $t\le C\sqrt{k}\,M_2\log n$ (for suitable $C>0$). By applying $\int_{|s|\le t}ds$ to the result in (1) and using the identity (1.2) and Lemma 1.4 we have

$$|F_K^k(t)-\Gamma_{\mathbb{P}}^k(t)|\le\frac{c_1^k}{n^{1/(k+3)}}\,t^k\le\frac{c_2^k k^{k/2}}{n^{1/(k+3)}}\,(\log n)^k.$$

In the range $t\ge C\sqrt{k}\,M_2\log n$, we proceed as in Theorem 3.5(2). Observe that if we write $P_E(x)=\sum_{i=1}^k\langle x,u_i\rangle u_i$ for some orthonormal basis (u_i) of E then

$$1-F_{\mathbb{P}}(t,E)=\mathbb{P}\Big\{\sum_{i=1}^k|\langle x,u_i\rangle|^2>t^2\Big\}\le\mathbb{P}\{\sqrt{k}\max_{1\le i\le k}|\langle x,u_i\rangle|>t\},$$

and so, by hypothesis, $1-F_{\mathbb{P}}(t,E)\le c_2 k\exp(-c_3 t/\sqrt{k}\,M_2)$. By this estimate and Lemma 1.5(ii),

$$\begin{aligned}|F_{\mathbb{P}}(t,E)-\Gamma_{\mathbb{P}}^k(t)|&=|(1-F_{\mathbb{P}}(t,E))-(1-\Gamma_{\mathbb{P}}^k(t))|\\&\le c_2 k e^{-c_3 t/M_2\sqrt{k}}+2^{k/2}e^{-t^2/4M_2^2},\end{aligned}$$

and we conclude, as in Theorem 3.5(2), that $|F_{\mathbb{P}}(t,E)-\Gamma_{\mathbb{P}}^k(t)|\le 2/n$ for every k-dimensional subspace E. ∎

REMARK 4.6. The hypotheses on $M_{\mathbb{P}}$, M_2 and $\sigma_{\mathbb{P}}$ are necessary due to the behaviour at $s=0$. Indeed, consider the probability given by $\mathbb{P}=$

$\frac{1}{2}\sigma_{n-1} + \frac{1}{2}\widetilde{\sigma}_{n-1}$ where $\widetilde{\sigma}_{n-1}$ is the Haar probability on $2S^{n-1}$. Straightforward computations show that $M_2 \sim cn^{-1/2}$, $M_{\mathbb{P}}^{1/n} \sim cn^{1/2}$, $\sigma_{\mathbb{P}} \sim c\sqrt{n}$ and $|\varphi_{\mathbb{P}}^k(0) - \gamma_{\mathbb{P}}^k(0)| \sim cn^{k/2}$, and so this difference tends to $+\infty$ as $n \to \infty$.

Examples. We now give some examples with $\sigma_{\mathbb{P}}, M_2, M_{\mathbb{P}}^{1/n}$ uniformly bounded.

1. Let $\mathbb{P}$ be the uniform measure on K, the unit ball in the space ℓ_p^n, $p > 0$. Clearly $M_{\mathbb{P}} = 1$. The parameters M_2 $(= L_K)$ and σ_K are uniformly bounded in n, as shown in [ABP] for $p \geq 1$; by similar arguments this also holds for $0 < p < 1$.

2. Let $\mathbb{P}$ be a Borel probability on $\mathbb{R}$ with finite fourth moment. Consider the product measure $\widetilde{\mathbb{P}} = \mathbb{P} \otimes \cdots \otimes \mathbb{P}$ on $\mathbb{R}^n$ and suppose $M_{\widetilde{\mathbb{P}}} = 1$. A simple computation shown that $M_2(\widetilde{\mathbb{P}}) = M_2(\mathbb{P})$ and $\sigma_{\widetilde{\mathbb{P}}} = \sigma_{\mathbb{P}}$.

3. Consider the density function on $\mathbb{R}^n$ given by $f(|x|)$ where $f\colon \mathbb{R} \to [0,\infty)$ is an even log-concave function. Then $M_{\mathbb{P}} = f(0)$ and, by Lemma 2.6 in [Kl1], $\sigma_{\mathbb{P}}, M_2$ are bounded by an absolute constant. (This can also be deduced from the results in [Bo].)

4. Let $f(x) = \exp(-a^p|x|^p)$, $0 < p < 1$, be a density function on $\mathbb{R}^n$. Then $M_{\mathbb{P}} = 1$ and $\sigma_{\mathbb{P}}, M_2$ are bounded by constants depending only on p.

4.2. *Upper bounds for a fast growth of k.* A Gaussian behaviour for large k is not expected: Consider the case $K = \omega_n^{-1/n} D_n$. We have

$$\varphi_K^k(s) = \begin{cases} \omega_{n-k}\omega_n^{(k-n)/n}(1 - |s|^2\omega_n^{2/n})^{(n-k)/2} & \text{for } |s| \leq \omega_n^{-1/n}, \\ 0 & \text{otherwise.} \end{cases}$$

- If $k = n - \ell$ with ℓ fixed, then the equivalence
$$\omega_{n-k}\, \omega_n^{k/n-1} \sim n^{\ell/2}(2\pi e)^{-\ell/2}$$
implies $\varphi_K^{n-\ell}(s)n^{-\ell/2} \to \omega_\ell(2\pi e)^{-\ell/2}$.
- If $k = (1-\lambda)n$ with $0 < \lambda < 1$, we have
$$\omega_{n-k}\, \omega_n^{k/n-1} \sim \lambda(2\pi)^{\lambda/2}/\lambda^{\lambda n/2} n^{(1-\lambda)/2},$$
which implies $\varphi_K^{(1-\lambda)n}(s)\lambda^{\lambda n/2}n^{(1-\lambda)/2} \to \lambda(2\pi)^{\lambda/2}e^{-\pi e\lambda|s|^2}$.

For general probabilities we find the following upper bounds of $\varphi_{\mathbb{P}}^k(s)$.

Proposition 4.7. *Let $\mathbb{P}$ be a probability measure on $\mathbb{R}^n$ with $M_{\mathbb{P}} < \infty$. Then there exist numerical constants $c, C > 0$ so that*:

(i) *If $1 \leq k \leq n-2$, then*
$$\varphi_{\mathbb{P}}^k(s) \leq CM_{\mathbb{P}}^{k/n}\,\frac{\omega_{n-k}}{\omega_n^{1-k/n}}\left(1 - \frac{k}{n}\,\frac{|s|^{n-k}}{(\omega_n M_{\mathbb{P}})^{k/n-1}}\right)$$
whenever $|s| \leq (k/n)^{1/(n-k)}(\omega_n M_{\mathbb{P}})^{-1/n}$.

(ii) *If $k = n-1$ and $\mathbb{P}$ has bounded density f, then*

$$\varphi_{\mathbb{P}}^{n-1}(s) \le C\|f\|_\infty$$

whenever $|s| \le c\sqrt{n}\,\|f\|_\infty^{-1/n}$.

Proof. (i) *Case* $1 \le k \le n-2$. Recall

$$\begin{aligned}\varphi_{\mathbb{P}}^{k}(s) &= \frac{|S^{n-k-1}|}{|S^{n-1}|}\int\limits_{\{|s|\le|x|\}}\left(1-\frac{|s|^2}{|x|^2}\right)^{(n-k-2)/2}\frac{d\mathbb{P}(x)}{|x|^k}\\ &\le \frac{|S^{n-k-1}|}{|S^{n-1}|}\int\limits_{\{|s|\le|x|\}}\frac{d\mathbb{P}(x)}{|x|^k}.\end{aligned}$$

Let $A \ge |s|$ to be chosen later. Then

$$\begin{aligned}\varphi_{\mathbb{P}}^{k}(s) &\le \frac{|S^{n-k-1}|}{|S^{n-1}|}\Bigg(\int\limits_{|s|\le|x|\le A}\frac{d\mathbb{P}(x)}{|x|^k}+\int\limits_{A\le|x|}\frac{d\mathbb{P}(x)}{|x|^k}\Bigg)\\ &\le \frac{|S^{n-k-1}|}{|S^{n-1}|}\Bigg(\int\limits_{|s|\le|x|\le A}\frac{d\mathbb{P}(x)}{|x|^k}+\frac{1}{A^k}\Bigg).\end{aligned}$$

Fix $I > 1$ and let N_s be a natural number such that $A/I^{N_s+1} \le |s| < A/I^{N_s}$. Since $\int_{tD_n} d\mathbb{P}(x) \le M_{\mathbb{P}} t^n \omega_n$ for all $t > 0$, we have

$$\begin{aligned}\int\limits_{|s|\le|x|\le A}\frac{d\mathbb{P}(x)}{|x|^k} &\le \sum_{m=0}^{N_s}\int\limits_{A/I^{m+1}\le|x|<A/I^m}\frac{d\mathbb{P}(x)}{|x|^k}\\ &\le \sum_{m=0}^{N_s}\left(\frac{I^{m+1}}{A}\right)^k\int\limits_{|x|\le A/I^m}d\mathbb{P}(x)\\ &\le \frac{I^k}{A^k}\sum_{m=0}^{N_s}I^{mk}\left(\frac{A}{I^m}\right)^n\omega_n M_{\mathbb{P}} = I^k A^{n-k}\omega_n M_{\mathbb{P}}\sum_{m=0}^{N_s}\left(\frac{1}{I^{n-k}}\right)^m\\ &\le I^k A^{n-k}\omega_n M_{\mathbb{P}}\left(1-\left(\frac{1}{I^{n-k}}\right)^{N_s+1}\right)\left(1-\frac{1}{I^{n-k}}\right)^{-1}\\ &\le I^k\left(1-\frac{1}{I^{n-k}}\right)^{-1}A^{n-k}\omega_n M_{\mathbb{P}}\left(1-\left(\frac{|s|}{IA}\right)^{n-k}\right).\end{aligned}$$

We choose $I = (n/k)^{1/(n-k)}$ to get

$$\int\limits_{|s|\le|x|\le A}\frac{d\mathbb{P}(x)}{|x|^k} \le \frac{k}{n-k}\left(\frac{n}{k}\right)^{n/(n-k)}A^{n-k}\omega_n M_{\mathbb{P}}\left(1-\left(\frac{|s|}{A}\right)^{n-k}\frac{k}{n}\right).$$

We now optimise by taking $A = (k/n)^{1/(n-k)}(\omega_n M_{\mathbb{P}})^{-1/n}$ whenever $|s| \le (k/n)^{1/(n-k)}(\omega_n M_{\mathbb{P}})^{-1/n}$, and we arrive at the result taking also into account that $|S^{m-1}| = m\omega_m$.

(ii) *Case* $k = n-1$. We have

$$\begin{aligned}\varphi_{\mathbb{P}}^{n-1}(s) &= \frac{2}{|S^{n-1}|}\int\limits_{\{|s|\le|x|\}} \frac{f(x)\,dx}{|x|^{n-2}\sqrt{|x|^2-|s|^2}}\\ &\le \frac{2}{|S^{n-1}|}\Big(\int\limits_{|s|\le|x|\le|s|+A} + \int\limits_{|s|+A\le|x|}\Big)\\ &\le \frac{2}{|S^{n-1}|}\bigg(|S^{n-1}|\,\|f\|_\infty\sqrt{(|s|+A)^2-|s|^2} + \frac{1}{(|s|+A)^{n-2}\sqrt{2|s|A+A^2}}\bigg).\end{aligned}$$

Assume $|s| \le A$. Then

$$\varphi_{\mathbb{P}}^{n-1}(s) \le \frac{2}{|S^{n-1}|}\bigg(|S^{n-1}|\,\|f\|_\infty\sqrt{3}\,A + \frac{1}{A^{n-1}}\bigg).$$

We optimise by taking

$$A = \bigg(\frac{n-1}{\sqrt{3}\,|S^{n-1}|\,\|f\|_\infty}\bigg)^{1/n}$$

and then

$$\begin{aligned}\varphi_{\mathbb{P}}^{n-1}(s) &\le \frac{2}{|S^{n-1}|}(|S^{n-1}|\,\|f\|_\infty\sqrt{3})^{1-1/n}\bigg((n-1)^{1/n} + \Big(\frac{1}{n-1}\Big)^{1-1/n}\bigg)\\ &\le 2\sqrt{3}\,\|f\|_\infty\end{aligned}$$

whenever $|s| \le C\sqrt{n}\,\|f\|_\infty^{-1/n}$ for some absolute constant $C > 0$. ∎

Remark 4.8. Our result (i) gives (assume $M_{\mathbb{P}} = 1$ for simplicity) an upper bound in the range $|s| \le (k/n)^{1/(n-k)}\omega_n^{-1/n}$ $(\le c\sqrt{n}\,)$. By looking at the trivial estimate given by

$$\varphi_{\mathbb{P}}^{k}(s) \le \Big(1-\frac{k}{n}\Big)\omega_{n-k}\omega_n^{-1}\frac{1}{|s|^k}$$

we conclude that in the range $|s| \ge C\omega_n^{-1/n}$ $(\sim C\sqrt{n})$,

$$\varphi_{\mathbb{P}}^{k}(s) \le \frac{1-k/n}{C^k}\,\frac{\omega_{n-k}}{\omega_n^{1-k/n}}.$$

The computations at the beginning of this section show that for $k = (1-\lambda)n$ or $k = n-\ell$, $2 \le \ell$, the function $\varphi_{\mathbb{P}}^{k}(s)$ is bounded in the range $|s| \ge C\sqrt{n}$ by $c_1 e^{-cn}$. Therefore, in both cases the distribution of $\varphi_{\mathbb{P}}^{k}(s)$ is concentrated on $|s| \le c\sqrt{n}$ (with constants depending only on λ or ℓ respectively).

COROLLARY 4.9. *Under the hypothesis of Proposition* 4.7 *we have in particular*:

(i) *if* $k = (1-\lambda)n$, *then* $\varphi_{\mathbb{P}}^{(1-\lambda)n}(s)\lambda^{\lambda n/2}n^{(1-\lambda)/2} \leq C(\lambda)M_{\mathbb{P}}^{1-\lambda}$ *for all* $s \in \mathbb{R}^k$ *and for some constant* $C(\lambda) > 0$ *depending on* λ,

(ii) *if* $k = n - \ell$ *with* $\ell \geq 2$ *fixed*, *then* $\varphi_{\mathbb{P}}^{n-\ell}(s)n^{-\ell/2} \leq C(\ell)M_{\mathbb{P}}^{1-\ell/n}$ *for all* $s \in \mathbb{R}^k$ *and for some constant* $C(\ell) > 0$ *depending on* ℓ.

By comparing with the case of the Euclidean ball, we see that the bounds are sharp for all s in case (ii) and also in the range $|s| \leq 1$ (say) for all values of k.

REMARK 4.10. We can improve the numerical constants for central sections of star-shaped bodies: Let $1 \leq k \leq n-2$ and let $K \subseteq \mathbb{R}^n$ be a star-shaped body of volume $|K| = 1$. Let rD_n be the Euclidean ball of volume 1 (of radius $r = \omega_n^{-1/n}$). Then

$$\varphi_K^k(s) \leq \varphi_K^k(0) \leq \varphi_{rD_n}^k(0) = \omega_{n-k}\omega_n^{(k-n)/n} \qquad \forall s \in \mathbb{R}^k.$$

Indeed,

$$\begin{aligned}\varphi_K^k(s) &= \frac{|S^{n-k-1}|}{|S^{n-1}|} \int\limits_{K\cap\{|x|\geq|s|\}} \left(1 - \frac{|s|^2}{|x|^2}\right)^{(n-k-2)/2} \frac{dx}{|x|^k} \\ &\leq \frac{|S^{n-k-1}|}{|S^{n-1}|} \int\limits_K \frac{dx}{|x|^k} = \varphi_K^k(0) = \frac{\omega_{n-k}}{\omega_n}\widetilde{W}_k(K)\end{aligned}$$

where $\widetilde{W}_k(K)$ denotes the kth dual mixed volume of K (see [BBR]). Now, by the dual Minkowski inequality, $\widetilde{W}_k(K) \leq \omega_n^{k/n}$ since $|K| = 1$, and the result follows.

Addendum. Since this paper was submitted for publication, several papers concerning the results in this paper have appeared, especially [Kl3] (where, in particular, Klartag proves that every isotropic convex body satisfies the Concentration Hypothesis with $\alpha = 0.33, \beta = 3.33$), [FGP], [Kl4], [EK]. The authors have also obtained new results in a joint work [ABBP], and using the methods developed there (the computation of the Lipschitz constant for the parallel section function), the dependence on t_0 in Theorem 3.11 can be deleted.

We thank the referee for encouraging us to improve an earlier version of the main Theorem 3.11.

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Received September 28, 2006
Revised version June 13, 2008 (6013)