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Non-normal elements in Banach *-algebras

by

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Abstract. Let A be a Banach *-algebra with an identity, continuous involution, center Z and set of self-adjoint elements Σ. Let $h \in \Sigma$. The set of $v \in \Sigma$ such that $(h + iv)^n$ is normal for no positive integer n is dense in Σ if and only if $h \notin Z$. The case where A has no identity is also treated.

1. Introduction. Normal operators are important in the theory of bounded linear operators on a Hilbert space. For the more general situation of a Banach *-algebra A it is then natural to consider its *normal elements*, those $x \in A$ for which $xx^* = x^*x$. Suppose that A has an identity, a continuous involution and is not commutative. It was shown in [10, Cor. 3.5] that the set of $x \in A$ such that x^n is normal for no positive integer n is dense in A. This investigation started with a desire to be more specific in important cases.

Throughout this paper A is a Banach *-algebra with a continuous involution $x \mapsto x^*$, center Z and set of self-adjoint elements Σ. Suppose first that A has an identity. Let $h \in \Sigma$. We show that the set of $v \in \Sigma$ such that $(h + iv)^n$ is normal for no positive integer n is dense in Σ if and only if $h \notin Z$. This has the following consequence.

Proposition 1.1. *Let A be the algebra of all bounded linear operators on a Hilbert space. Let $T \in \Sigma$. Then if T is not a scalar multiple of the identity the set of $V \in \Sigma$ such that $(T + iV)^n$ is normal for no positive integer n is dense in Σ.*

The argument used here makes use of the identity. A separate argument is needed to show the following result.

Proposition 1.2. *Let A be the algebra of all compact linear operators on an infinite-dimensional Hilbert space. Let $T \neq 0$, $T \in \Sigma$. Then the set of $V \in \Sigma$ such that $(T + iV)^n$ is normal for no positive integer n is dense in Σ.*

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2. On non-normal elements. As usual we set $[x, y] = xy - yx$. We say that x is *normal modulo* Z if $[x, x^*] \in Z$. Let $p(t) = \sum_{j=0}^{n} b_j t^j$ be a polynomial in the real variable t with coefficients in A. If $p(t) \in Z$ for an infinite subset of the reals then each $b_j \in Z$.

THEOREM 2.1. *Suppose that A has an identity e and that n is a positive integer. Let $h \in \Sigma$. Let Δ (resp. Δ_1) be the set of $k \in \Sigma$ such that $(h+ik)^n$ is not normal (resp. not normal modulo Z). If $h \notin Z$ then Δ dense in Σ. If $h \notin Z$ and A is semi-prime then Δ_1 is dense in Σ.*

Proof. Let E be the closed linear subspace of A which will be successively (0) and Z in the arguments to follow.

Suppose that Δ or Δ_1 is not dense in Σ. Then there is a non-void open set Ω in Σ such that, for each $k \in \Omega$, $(h + ik)^n$ is normal modulo E. Let $a \in \Omega$ and $v \in \Sigma$. For some $\varepsilon > 0$ we have

$$[(h + i(a + tv))^n, (h - i(a + tv))^n] \in E$$

for all t, $0 \le t \le \varepsilon$. Hence this relation holds for all real t. It follows that $(h + iv)^n$ is normal modulo E for all $v \in \Sigma$.

From this it follows that $(v + ith)^n$ is normal modulo E for all $v \in \Sigma$ and real values of t. As shown in [11, Lemma 3.1] we have

$$\Big[v^n, \sum_{j=0}^{n-1} v^j h v^{n-1-j}\Big] \in E$$

for all $v \in \Sigma$. Suppose that $t \neq 0$. Set

$$w_1 = t^{-1}\{(e + tv)^n - e\}, \qquad w_2 = \sum_{j=0}^{n-1} (e + tv)^j h (e + tv)^{n-1-j}.$$

Then $[w_1, w_2] \in E$. Let $t \to 0$ to see that $[v, h] \in E$ for all $v \in \Sigma$. Hence $[h, x] \in E$ for all $x \in A$.

Now consider the above analysis for Δ and $E = (0)$. We then have $h \in Z$. Next suppose that A is semi-prime and we are treating Δ_1 with $E = Z$. Then $[h, x] \in Z$ for all $x \in A$. It follows from [12, Th. 3.1] that $h \in Z$ in this case also. ∎

THEOREM 2.2. *Suppose that A has an identity. Let $h \in \Sigma$, $h \notin Z$. Then the set of $v \in \Sigma$ such that $(h + iv)^n$ is normal for no positive integer n is dense in Σ. If A is semi-prime the set of $v \in \Sigma$ such that $(h + iv)^n$ is normal modulo Z for no positive integer n is dense in Σ.*

Proof. The set of $v \in \Sigma$ such that $(h + iv)^n$ is not normal modulo E is an open set which is dense by Theorem 2.1. The conclusion follows from the Baire Category Theorem. ∎

We let Γ denote the set of $h \in \Sigma$ such that $h^n \in Z$ for no positive integer n.

LEMMA 2.3. *Suppose that A is semi-prime and is not commutative. Then Γ is dense in Σ.*

Proof. Let $W_m = \{h \in \Sigma\colon h^m \notin Z\}$. Then $\Gamma = \bigcap_m W_m$. By the Baire Category Theorem it is sufficient to show that each W_m is dense in Σ. Suppose that, for some positive integer r, W_r is not dense. Then there is a non-empty open subset Ω of Σ in the complement of W_r.

Let $a \in \Omega$ and let $h \in \Sigma$. For all real t, the element $(a+th)^r$ lies in Z so $h^r \in Z$. By [10, Lemma 3.1] we see that $x^r \in Z$ for all $x \in A$. By standard ring theory [5, Th. 3.22], A is commutative. This contradiction shows that Γ is dense in Σ. ■

THEOREM 2.4. *Suppose that A is semi-prime and is not commutative. For each h in the dense subset Γ of Σ the set of $v \in \Sigma$ such that $(h+iv)^n$ is normal modulo Z for no positive integer n is dense in Σ.*

Proof. By the arguments used for Theorems 2.1 and 2.2 it is sufficient to show that, for each positive integer n, the set of $v \in \Sigma$ such that $(h+iv)^n$ is not normal modulo Z is dense in Σ.

Suppose otherwise. As shown in the proof of Theorem 2.1 the element $(h+iv)^n$ is normal modulo Z for each $v \in \Sigma$. As shown in [11, Lemma 3.1] this implies that $[h^n,[h^n,v]] = 0$ for all $v \in \Sigma$. But then $[h^n,[h^n,x]] = 0$ for all $x \in A$. By a result of Herstein [6, p. 5] we see that $h^n \in Z$. But this contradicts $h \in \Gamma$. ■

COROLLARY 2.5. *Let A be a C^*-algebra with $Z = (0)$. Then for any $h \neq 0$, $h \in \Sigma$, the set of $v \in \Sigma$ such that $(h+iv)^n$ is normal for no positive integer n is dense in Σ.*

Proof. Let $h \in \Sigma$ with $h^n = 0$ for a positive integer n. Then $\|h^n\| = \|h\|^n$ so that $h = 0$. Thus if $h \neq 0$ then $h \in \Gamma$. We apply Theorem 2.4. ■

Proposition 1.2 is a special case of Corollary 2.5.

For each $x \in A$ let $r(x) = \lim \|x^n\|^{1/n}$. In [4, p. 420] the involution in A is said to be *regular* if $r(h) = 0$, $h \in \Sigma$, imply that $h = 0$. In that case it is readily seen that A is semisimple and so $x \mapsto x^*$ is continuous in virtue of the uniqueness of norm theorem [3, p. 130].

If A has a regular involution and $h^n = 0$ for $h \in \Sigma$ then $r(h) = 0$ so that $h = 0$.

THEOREM 2.6. *Let A be a semisimple topologically simple infinite-dimensional annihilator Banach *-algebra. Suppose that $x^*x = 0$ implies that $x = 0$. Then the conclusion of Corollary 2.5 holds.*

Proof. As A is semisimple the involution is continuous. By [9, Th. 4.10.16], A has a faithful *-representation $x \mapsto T(x)$ as bounded linear operators on a Hilbert space. This *-representation is continuous ([3, p. 196, Th. 3]). For each $h \in \Sigma$ we have $r(T(h)) \leq r(h)$. Thus whenever $r(h) = 0$ we have $T(h) = 0$ and also $h = 0$ so that the involution is regular.

Now (0) is the only primitive ideal of A so that A is a primitive Banach algebra. By [9, Cor. 2.4.5] either $Z = (0)$ or Z is the set of all scalar multiples of a non-zero idempotent e. We rule out the latter possibility. First of all e cannot be an identity element for A, since otherwise A would be finite-dimensional by [2, Prop. 6.3]. If $e \neq 0$ and e is not the identity we would have $A = Ae \oplus A(1-e)$ contrary to the fact that A is topologically simple. Therefore $Z = (0)$. Again we apply Theorem 2.4. ∎

Proposition 1.2 is a special case of Theorem 2.6; the latter also applies to some H*-algebras of [1] such as a full matrix algebra [9, Theorem 4.10.32].

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