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Restriction theorems for the Fourier transform to homogeneous polynomial surfaces in $\mathbb{R}^3$

by

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Abstract. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$, let $\Sigma = \{(x, \varphi(x)) : |x| \leq 1\}$ and let σ be the Borel measure on Σ defined by $\sigma(A) = \int_B \chi_A(x, \varphi(x))\, dx$ where B is the unit open ball in $\mathbb{R}^2$ and dx denotes the Lebesgue measure on $\mathbb{R}^2$. We show that the composition of the Fourier transform in $\mathbb{R}^3$ followed by restriction to Σ defines a bounded operator from $L^p(\mathbb{R}^3)$ to $L^q(\Sigma, d\sigma)$ for certain p, q. For $m \geq 6$ the results are sharp except for some border points.

1. Introduction. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth enough function, let B be the open unit ball in $\mathbb{R}^n$ and let $\Sigma = \{(x, \varphi(x)) : x \in B\}$. For $f \in S(\mathbb{R}^{n+1})$, let $\mathcal{R}f : \Sigma \to \mathbb{C}$ be defined by $(\mathcal{R}f)(x, \varphi(x)) = \widehat{f}(x, \varphi(x))$, $x \in B$, where $\widehat{f}$ denotes the usual Fourier transform of f defined by $\widehat{f}(\xi) = \int f(u) e^{-i\langle u,\xi\rangle}\, du$. Let σ be the Borel measure on Σ defined by $\sigma(A) = \int_B \chi_A(x, \varphi(x))\, dx$ and let E be the *type set* for the operator $\mathcal{R}$, i.e. the set of pairs $(1/p, 1/q) \in [0, 1] \times [0, 1]$ such that $\|\widehat{f}\|_{L^q(\Sigma)} \leq c\|f\|_{L^p(\mathbb{R}^{n+1})}$ for some $c > 0$ and all $f \in S(\mathbb{R}^{n+1})$, where the spaces $L^p(\mathbb{R}^{n+1})$ and $L^q(\Sigma)$ are taken with respect to the Lebesgue measure in $\mathbb{R}^{n+1}$ and the measure σ respectively.

The $L^p(\mathbb{R}^{n+1})$-$L^q(\Sigma)$ boundedness properties of the restriction operator $\mathcal{R}$ have been widely studied. It is well known that for Σ as above, if $(1/p, 1/q) \in E$ then

$$\frac{1}{q} \geq -\frac{n+2}{n}\,\frac{1}{p} + \frac{n+2}{n}.$$

In [10], it is proved, for the case where φ is a nondegenerate quadratic form in $\mathbb{R}^{n+1}$, that $(1/p, 1/2) \in E$ if $(n+4)/(2n+4) \leq 1/p \leq 1$, and the method given there provides a general tool to obtain, from suitable estimates for $\widehat{\sigma}$, $L^p(\mathbb{R}^{n+1})$-$L^2(\Sigma)$ estimates for $\mathcal{R}$. Moreover, a general theorem, due to Stein,

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holds for smooth enough hypersurfaces with never vanishing Gaussian curvature (see e.g. [8, p. 386]). There it is shown that, in this case, $(1/p, 1/q) \in E$ if

$$\frac{1}{q} \geq -\frac{n+2}{n}\frac{1}{p} + \frac{n+2}{n} \quad \text{and} \quad \frac{n+4}{2n+4} \leq \frac{1}{p} \leq 1.$$

For the case $n = 1$ a restriction theorem (also under the assumption of nonvanishing curvature) is given in [3] where it is proved that, in this case, $(1/p, 1/q) \in E$ if $3/4 < 1/p \leq 1$ and $1/q \geq -3/p+3$, and this result is sharp, i.e. the conditions on p and q are also necessary. Also in [1], [5], [4] and [6] restriction theorems for curves of finite type are obtained. Concerning the homogeneous case, the type set E is studied in [2] for $\varphi(x) = (\sum_{j=1}^{n} |x_j|^r)^\alpha$. The main tools used in [2] are a dyadic decomposition of Σ combined with Strichartz's method applied to the these dyadic pieces and interpolation techniques.

In this paper we consider the case $n = 2$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ a homogeneous polynomial function. We study the type set E following in part the approach in [2].

Let us describe our results. Let E° denote the relative interior of E in $[0,1] \times [0,1]$.

If $\det \varphi''(x) \equiv 0$ we characterize E° (see Theorem 3.3).

If $\det \varphi''(x)$ is not identically zero and if it vanishes somewhere on $\mathbb{R}^2 - \{0\}$, since φ is a homogeneous polynomial function, the set of the points x where $\det \varphi''(x)$ vanishes is a finite union of lines $L_1, \ldots, L_k$ through the origin. For a point $x_j \in L_j - \{0\}$, $j = 1, \ldots, k$, we consider the vanishing order α_j of $\det \varphi''(x)$ at x_j along a transversal direction to L_j. A simple computation using the homogeneity of $\det \varphi'$ shows that α_j is independent of the point x_j and of the transversal direction chosen. Let

$$\widetilde{m} = \max\{m, \alpha_1 + 2, \ldots, \alpha_k + 2\}. \tag{1.1}$$

In this case, for $\widetilde{m} \geq 6$ we characterize E°, and for $\widetilde{m} < 6$ we characterize $E^\circ \cap ((3/4, 1] \times [0,1])$ and we prove that $(3/4, 1/q) \in E$ for $(\widetilde{m}+2)/8 < 1/q \leq 1$ (see Theorem 3.4). These results still hold if $\det \varphi''(x)$ never vanishes on $\mathbb{R}^2 - \{0\}$ provided that we define $\widetilde{m} = m$ in this case.

Finally, for every case, we give (see Theorems 4.2 and 4.4) a sharp $L^p(\mathbb{R}^3)$-$L^2(\Sigma)$ estimate for the restriction operator $\mathcal{R}$.

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2. Preliminaries

REMARK 2.1. Let us introduce some additional notation and state some general facts concerning restriction operators. If $V \subset \mathbb{R}^2$ is a measurable set and if $\psi : V \to \mathbb{R}$ is a continuous function, let $\Sigma_{V,\psi}$, $\sigma_{V,\psi}$, $\mathcal{R}_{V,\psi}$ be the surface, the measure and the restriction operator defined as Σ, σ and $\mathcal{R}$ at the beginning of the introduction, with $n = 2$, but taking now V and ψ instead of B and φ respectively. Finally, let $E_{V,\psi}$ be the type set for $\mathcal{R}_{V,\psi}$. Let us recall some well known facts about the operators $\mathcal{R}_{V,\psi}$.

(a) The Riesz–Thorin theorem implies that $E_{V,\psi}$ is a convex set. Moreover, for $f \in S(\mathbb{R}^3)$, we have

$$\|\mathcal{R}_{V,\psi}\|_{L^1(\mathbb{R}^3),L^\infty(\Sigma_{V,\psi})} \le 1 \quad \text{and} \quad \|\mathcal{R}_{V,\psi}\|_{L^1(\mathbb{R}^3),L^1(\Sigma_{V,\psi})} \le |V|$$

where $|V|$ denotes the Lebesgue measure of V. So, by the Riesz-Thorin theorem, if $|V| < \infty$ the closed segment with endpoints $(1,0)$ and $(1,1)$ is contained in $E_{V,\psi}$. In particular we get the estimate

$$\|\mathcal{R}_{V,\psi}\|_{L^1(\mathbb{R}^3),L^2(\Sigma_{V,\psi})} \le |V|^{1/2}.$$

(b) If $T \in \mathrm{GL}(\mathbb{R}^2)$ a computation shows that $E_{T(V),\psi\circ T^{-1}} = E_{V,\psi}$. Also, $E_{V,a\psi} = E_{V,\psi}$ for $a \in \mathbb{R} - \{0\}$.

(c) Let us recall the well known homogeneity argument (see e.g. [10], [11]). If $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a continuous and homogeneous function of degree m then $E_{tV,\varphi} = E_{V,\varphi}$ for all $t > 0$. Indeed, a computation gives, for $f \in S(\mathbb{R}^3)$ and $t > 0$,

$$\|\widehat{f}\|_{L^q(\Sigma_{tV,\varphi})} = t^{2/q-(m+2)}\|(f_{t^{-1}})^\wedge\|_{L^q(\Sigma_{V,\varphi})} \tag{2.1}$$

where $f_t(v_1, v_2, v_3) = f(tv_1, tv_2, t^m v_3)$. From (2.1) it easily follows that

$$\|\mathcal{R}_{tV,\varphi}\|_{L^p(\mathbb{R}^3),L^q(\Sigma_{tV,\varphi})} = t^{2/q+(m+2)/p-(m+2)}\|\mathcal{R}_{V,\varphi}\|_{L^p(\mathbb{R}^3),L^q(\Sigma_{V,\varphi})} \tag{2.2}$$

for all $t > 0$ and so $E_{tV,\varphi} = E_{V,\varphi}$.

(d) Let φ be as in (c), let $W = \bigcup_{k\in\mathbb{N}\cup\{0\}} 2^{-k}V$ and suppose that $(1/p, 1/q) \in E_{V,\varphi}$ and

$$\frac{1}{q} > -\left(\frac{m}{2}+1\right)\frac{1}{p} + \frac{m}{2} + 1.$$

Then $(1/p, 1/q) \in E_{W,\varphi}$. Indeed, since

$$\|\mathcal{R}_{W,\varphi}\|^q_{L^p(\mathbb{R}^3),L^q(\Sigma_{W,\varphi})} \le \sum_{k\in\mathbb{N}\cup\{0\}} \|\mathcal{R}_{2^{-k}V,\varphi}\|^q_{L^p(\mathbb{R}^3),L^q(\Sigma_{2^{-k}V,\varphi})}$$

the statement follows from (2.2).

(e) Another consequence of the homogeneity argument is the following. For φ and W as in (d), since

$$\|\mathcal{R}_{W,\varphi}\|_{L^p(\mathbb{R}^3),L^q(\Sigma_{W,\varphi})} \ge \|\mathcal{R}_{2^{-k}V,\varphi}\|_{L^p(\mathbb{R}^3),L^q(\Sigma_{2^{-k}V,\varphi})}$$

for all $k \in \mathbb{N}$, from (2.2) it follows that

$$\frac{1}{q} \ge -\left(\frac{m}{2}+1\right)\frac{1}{p} + \frac{m}{2} + 1$$

is a necessary condition in order to have $(1/p, 1/q) \in E_{W,\varphi}$. ∎

The following Lemmas 2.2 and 2.4 allow us to compute the vanishing order of $\det\varphi''(x)$ along the x_1 axis for an arbitrary homogeneous polynomial function $\varphi:\mathbb{R}^2\to\mathbb{R}$. Let α be the order of $x_2=0$ as zero of the function $x_2\mapsto\det\varphi''(1,x_2)$, with the convention that $\alpha=0$ if $\det\varphi''(1,0)\neq 0$, and $\alpha=\infty$ if $\det\varphi''(1,x_2)$ vanishes identically (i.e., by the homogeneity of φ, if $\det\varphi''(x)$ vanishes identically on $\mathbb{R}^2$).

LEMMA 2.2. *Let $\varphi:\mathbb{R}^2\to\mathbb{R}$ be a homogeneous polynomial function of degree $m\geq 2$ of the form*

$$\varphi(x_1,x_2)=a_0x_1^m+\sum_{1\leq l\leq m}a_lx_1^{m-l}x_2^l \tag{2.3}$$

for some $a_0,\dots,a_m\in\mathbb{R}$ with $a_0\neq 0$ and let α be defined as above.

(i) *Suppose that*

$$\frac{a_s}{a_0}=\binom{m}{s}m^{-s}\left(\frac{a_1}{a_0}\right)^s\quad \textit{for } s=1,\dots,r$$

with $r<m$ and that

$$\frac{a_{r+1}}{a_0}\neq\binom{m}{r+1}m^{-r-1}\left(\frac{a_1}{a_0}\right)^{r+1}.$$

Then $\alpha=r-1$.

(ii) *If*

$$\frac{a_s}{a_0}=\binom{m}{s}m^{-s}\left(\frac{a_1}{a_0}\right)^s\quad \textit{for } s=1,\dots,m,$$

then $\alpha=\infty$.

Proof. To prove (i), without loss of generality we can assume that $a_0=1$. Let r be as in (i), so $1\leq r\leq m-1$. We have $\det\varphi''(x_1,x_2)=AB-C^2$ where

$$A=\sum_{0\leq l\leq m-2}(m-l)(m-l-1)a_lx_1^{m-l-2}x_2^l,$$
$$B=\sum_{0\leq j\leq m-2}(j+2)(j+1)a_{j+2}x_1^{m-j-2}x_2^j,$$
$$C=\sum_{l=0}^{m-2}(l+1)(m-l-1)a_{l+1}x_1^{m-l-2}x_2^l.$$

A computation of $AB-C^2$ gives

$$\begin{aligned}\det\varphi''(x_1,x_2)\\ =\sum_{0\leq i\leq 2m-4}\sum_{l+j=i}&[(m-l)(m-l-1)(j+2)(j+1)a_la_{j+2}\\ &-(l+1)(m-l-1)(m-j-1)(j+1)a_{l+1}a_{j+1}]x_1^{2m-4-i}x_2^i.\end{aligned}\tag{2.4}$$

For $0 \le i \le 2m-4$, let $c_i x_1^{2m-4-i}$ be the coefficient of x_2^i in $\det \varphi''(x_1, x_2)$. For $s = 1, \dots, r-1$ we have

$$c_s = \sum_{l+j=s} (m-l)(m-l-1)(j+2)(j+1) a_l a_{j+2} - \sum_{l+j=s} (l+1)(m-l-1)(m-j-1)(j+1) a_{l+1} a_{j+1}.$$

Thus (recalling that $a_0 = 1$) the hypothesis of (i) gives

$$\begin{aligned} c_s &= m(m-1)(s+2)(s+1)a_{s+2} \\ &\quad + \sum_{l+j=s, l\neq 0} (m-l)(m-l-1)(j+2)(j+1)\binom{m}{l}\binom{m}{j+2} m^{-s-2} a_1^{s+2} \\ &\quad - \sum_{l+j=s} (l+1)(m-l-1)(m-j-1)(j+1)\binom{m}{l+1}\binom{m}{j+1} m^{-s-2} a_1^{s+2}. \end{aligned}$$

Since

$$(l+1)(m-l-1)(m-j-1)(j+1)\binom{m}{l+1}\binom{m}{j+1} = (m-l)(m-l-1)(j+2)(j+1)\binom{m}{l}\binom{m}{j+2}$$

we get

$$c_s = m(m-1)(s+1)(s+2)\left[a_{s+2} - \binom{m}{s+2} m^{-s-2} a_1^{s+2}\right]$$

and so $c_0 = \ldots = c_{r-2} = 0$ and $c_{r-1} \neq 0$, hence $\alpha = r-1$.

To see (ii) observe that if

$$\frac{a_s}{a_0} = \binom{m}{s} m^{-s} \left(\frac{a_1}{a_0}\right)^s \quad \text{for } s = 1, \dots, m$$

then $\varphi(x_1, x_2) = a_0(x_1 + b x_2)^m$ for some $b \in \mathbb{R}$ and that in this case $\det \varphi''(x)$ is identically zero. ∎

REMARK 2.3. Let φ, α be as in Lemma 2.2. Observe that this lemma implies that $\alpha < m-2$ except in the cases where φ is either of the form $\varphi(x_1, x_2) = a_0(x_1 + b x_2)^m$ or $\varphi(x_1, x_2) = a_0(x_1 + b x_2)^m + b' x_2^m$ for some $a_0, b, b' \in \mathbb{R}$ with $a_0 \neq 0$, $b' \neq 0$, and that in these exceptional cases we have $\alpha = \infty$ and $\alpha = m-2$ respectively.

LEMMA 2.4. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$ given by*

$$\varphi(x_1, x_2) = \sum_{k \le l \le m} a_l x_1^{m-l} x_2^l \tag{2.5}$$

for some $a_k, \dots, a_m \in \mathbb{R}$ *with* $1 \le k \le m$, *and* $a_k \ne 0$. *Let* α *be as in Lemma* 2.2. *Then* $\alpha = 2k-2$ *if* $k < m$, *and* $\alpha = \infty$ *if* $k = m$.

Proof. If $k = m$, then $\varphi(x_1, x_2) = a_m x_2^m$, so $\det \varphi''(1, x_2)$ vanishes identically and then $\alpha = \infty$.

If $k < m$, then $\varphi(x_1, x_2) = a_k x_1^{m-k} x_2^k + x_2^{k+1} \psi(x_1, x_2)$ for some polynomial function ψ satisfying $\psi(1, 0) \ne 0$, and so

$$\det \varphi''(x_1, x_2) = -k(m-1)(m-k) a_k^2 x_1^{2m-2k-2} x_2^{2k-2} + x_2^{2k-1} \Theta(x_1, x_2)$$

where $\Theta(x_1, x_2)$ is a polynomial function. Since $k(m-1)(m-k) \ne 0$ and $a_k \ne 0$, we get $\alpha = 2k-2$. ∎

REMARK 2.5. For an arbitrary homogeneous polynomial φ of degree $m \ge 2$, from Lemmas 2.2 and 2.4 it follows that $\det \varphi''(x_1, x_2) \equiv 0$ if and only if $\varphi(x_1, x_2) = (ax_1 + bx_2)^m$ for some $a, b \in \mathbb{R}$ and all $(x_1, x_2) \in \mathbb{R}^2$. ∎

REMARK 2.6. We will need the following Strichartz theorem (see [10]) whose proof relies on Stein's complex interpolation theorem which gives $L^p(\mathbb{R}^3)$-$L^2(\Sigma_{V,\psi})$ estimates for the operator $\mathcal{R}_{V,\psi}$. Since we will need information about the size of the constants we give a sketch of its proof.

Let V be a measurable subset of $\mathbb{R}^2$ such that $|V| > 0$ and let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose that $|(\sigma_{V,\psi})^\wedge(\xi)| \le A(1+|\xi_3|)^{-\tau}$ for some $\tau > 0$ and for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. Then

$$\|\mathcal{R}_{V,\psi}\|_{L^p(\mathbb{R}^3), L^2(\Sigma_{V,\psi})} \le c_\tau A^{1/(2(1+\tau))}$$

for $p = (2+2\tau)/(2+\tau)$ where c_τ is a positive constant depending only on τ. Indeed, as in [8, p. 381] we define the analytic family of distributions I_z given, for $\mathrm{Re}(z) > 0$, by

$$I_z(t) = \begin{cases} \dfrac{e^{z^2}}{\Gamma(z)}\, t^{z-1} \varsigma(t) & \text{for } t > 0, \\ 0 & \text{for } t \le 0, \end{cases}$$

where $\varsigma \in C_c^\infty$ and $\varsigma(t) = 1$ for $|t| \le 1$. Also we define $J_z = \delta \otimes \delta \otimes I_z$. For $-\tau \le \mathrm{Re}(z) \le 1$ and $f \in S(\mathbb{R}^3)$, let $T_z f = (J_z * \sigma_{V,\psi})^\wedge * f$. A computation shows that if $\mathrm{Re}(z) = 1$, then $\|J_z * \sigma_{V,\psi}\|_{L^\infty(\mathbb{R}^3)} \le c < \infty$, so

$$\|T_z f\|_{L^2(\mathbb{R}^3)} = \|(T_z f)^\wedge\|_{L^2(\mathbb{R}^3)} \le c\|\widehat{f}\|_{L^2(\mathbb{R}^3)} = c\|f\|_{L^2(\mathbb{R}^3)}.$$

Also since $|(J_z)^\wedge(\xi)| \le c_\tau(1+|\xi_3|)^\tau$ for $\mathrm{Re}(z) = -\tau$, Young's inequality gives

$$\|T_z f\|_{L^\infty(\mathbb{R}^3)} = \|((\sigma_{V,\psi})^\wedge (J_z)^\wedge) * f\|_{L^\infty(\mathbb{R}^3)} \le c_\tau A \|f\|_{L^1(\mathbb{R}^3)}$$

and so Stein's complex interpolation theorem (as stated e.g. in [9, Ch. V]) entails that the operator $T_0 f = (\sigma_{V,\psi})^\wedge * f$ satisfies

$$\|T_0\|_{L^p(\mathbb{R}^3), L^{p'}(\mathbb{R}^3)} \le c'_\tau A^{1/(1+\tau)}$$

for $p = (2+2\tau)/(2+\tau)$. This implies (see e.g. [8, p. 253]) that

$$\|\mathcal{R}_{V,\psi}\|_{L^p(\mathbb{R}^3),L^2(\Sigma_{V,\psi})} \le c''_\tau A^{1/(2(1+\tau))}. \ \blacksquare$$

LEMMA 2.7. *Let* $\varphi(x_1,x_2) = \sum_{k\le l\le m} a_l x_1^{m-l} x_2^l$ *where* $0 \le k \le m$ *and* $a_k,\ldots,a_m \in \mathbb{R}$ *with* $a_k \ne 0$, *and let* Σ *and* E *be defined as in the introduction. If* $(1/p,1/q) \in E$ *then*

$$\frac{1}{q} \ge -(k+1)\frac{1}{p} + k + 1.$$

Proof. If $k=0$ the lemma follows from Remark 2.1(e). Suppose $k \ne 0$. For $0 < \varepsilon < 1$, let f_ε be the characteristic function of the set $[0,1] \times [0,\varepsilon^{-1/k}] \times [0,\varepsilon^{-1}]$. Then for $x = (x_1,x_2)$,

$$\begin{aligned}\widehat{f_\varepsilon}(x,\varphi(x)) &= \int_0^1 \int_0^{\varepsilon^{-1/k}} \int_0^{\varepsilon^{-1}} e^{-i(x_1\xi_1 + x_2\xi_2 + \varphi(x_1,x_2)\xi_3)}\, d\xi_1\, d\xi_2\, d\xi_3 \\ &= \varepsilon^{-(1+1/k)} \int_Q e^{-i(x_1u_1 + \varepsilon^{-1/k}x_2u_2 + \varepsilon^{-1}\varphi(x_1,x_2)u_3)}\, du_1\, du_2\, du_3\end{aligned}$$

where $Q = [0,1]\times[0,1]\times[0,1]$. Let $c = \min\big(1, \big(\frac{1}{3}\sum_{k\le l\le m}|a_l|\big)^{-1/k}\big)$ and $D_\varepsilon = [0,1/3] \times [0,(c/3)\varepsilon^{1/k}]$. So $\Sigma_{D_\varepsilon,\varphi} \subset \Sigma$ and $\|\widehat{f_\varepsilon}\|_{L^q(\Sigma)} \ge \|\widehat{f_\varepsilon}\|_{L^q(\Sigma_{D_\varepsilon,\varphi})}$. Now

$$\begin{aligned}&\varepsilon^{q+q/k}\|\widehat{f_\varepsilon}\|^q_{L^q(\Sigma_{D_\varepsilon,\varphi})} \\ &\quad= \int_{D_\varepsilon} \Big| \int_Q e^{-i(x_1u_1 + \varepsilon^{-1/k}x_2u_2 + \varepsilon^{-1}\varphi(x_1,x_2)u_3)}\, du_1\, du_2\, du_3 \Big|^q dx_1\, dx_2 \\ &\quad\ge \int_{D_\varepsilon} \Big| \int_Q \cos(x_1u_1 + \varepsilon^{-1/k}x_2u_2 + \varepsilon^{-1}\varphi(x_1,x_2)u_3)\, du_1\, du_2\, du_3 \Big|^q dx_1\, dx_2\end{aligned}$$

For $(x_1,x_2,\varphi(x_1,x_2)) \in \Sigma_{D_\varepsilon,\varphi}$ we have

$$\begin{aligned}\varepsilon^{-1}|\varphi(x_1,x_2)| &\le \varepsilon^{-1}(|a_k {x_1}^{m-k} x_2^k| + \ldots + |a_m {x_2}^m|) \\ &\le c^k|a_k| + |a_{k+1}|c^{k+1}\varepsilon^{1/k} + \ldots + |a_m|c^m\varepsilon^{(m-k)/k} \le 1/3\end{aligned}$$

and so for $(u_1,u_2,u_3) \in Q$ we get $|x_1u_1 + \varepsilon^{-1/k}x_2u_2 + \varepsilon^{-1}\varphi(x_1,x_2)u_3| \le 1 < \pi/3$. Thus $\varepsilon^{q+q/k}\|\widehat{f_\varepsilon}\|^q_{L^q(\Sigma_{D_\varepsilon,\varphi})} \ge c'\varepsilon^{1/k}$ with c' independent of ε and f. Now, $\|f_\varepsilon\|_{L^p(\mathbb{R}^3)} = \varepsilon^{-(1+1/k)/p}$. So from the inequality $\|\widehat{f_\varepsilon}\|_{L^q(\Sigma)} \le c\|f_\varepsilon\|_{L^p(\mathbb{R}^3)}$, applied with ε small enough, we get

$$-\Big(1+\frac{1}{k}\Big) + \frac{1}{k}\frac{1}{q} \ge -\Big(1+\frac{1}{k}\Big)\frac{1}{p}$$

and the lemma follows. ∎

Remark 2.8. It is known that if $(\int_0^1 |\widehat{h}(x_2, x_2^2)|^q\,dx_2)^{1/q} \le c\|h\|_{L^p(\mathbb{R}^2)}$ for some $c > 0$ and all $h \in S(\mathbb{R}^2)$ then $1/p > 3/4$ (see [4, Theorem 2]). This result implies the following.

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a quadratic homogeneous polynomial function such that $\det \varphi''(x) \equiv 0$, and let Σ and E be defined as in the introduction. If $1 \le p, q \le \infty$ and there exists $c > 0$ such that

$$\|\widehat{f}\|_{L^q(\Sigma)} \le c\|f\|_{L^p(\mathbb{R}^3)} \tag{2.6}$$

for all $f \in S(\mathbb{R}^3)$, then $1/p > 3/4$. Indeed, from Remark 2.5 we have $\varphi(x_1, x_2) = (ax_1 + bx_2)^2$ for some $a, b \in \mathbb{R}$ and all $(x_1, x_2) \in \mathbb{R}^2$. So, from Remark 2.1(b) the problem reduces (after composing with a suitable rotation followed by a dilation) to the case $\varphi(x_1, x_2) = x_2^2$. Let $g \in S(\mathbb{R})$ be such that $\widehat{g} > 0$ on $[0, 1]$. For $h \in S(\mathbb{R}^2)$ we take $f(x_1, x_2, x_3) = g(x_1)h(x_2, x_3)$ in (2.6) to obtain

$$\|\widehat{g}\|_{L^q(0,1)} \Big(\int_0^1 |\widehat{h}(x_2, x_2^2)|^q\,dx_2\Big)^{1/q} \le c\|g\|_{L^p(\mathbb{R})}\|h\|_{L^p(\mathbb{R}^2)}$$

and so $(\int_0^1 |\widehat{h}(x_2, x_2^2)|^q\,dx_2)^{1/q} \le c\|h\|_{L^p(\mathbb{R}^2)}$ for some $c > 0$ and all $h \in S(\mathbb{R}^2)$. ∎

3. $L^p(\mathbb{R}^3)$-$L^q(\Sigma)$ estimates for $\mathcal{R}$. For $\delta > 0$ we set

$$V_\delta = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \le 1,\ |x_2| \le \delta|x_1|\}. \tag{3.1}$$

Lemma 3.1. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function such that* $\det \varphi''(x)$ *does not vanish identically, let α be defined as in the preliminaries and let $m^* = \max(m, \alpha + 2)$. Let V_δ be defined by* (3.1)*. Then for δ positive and small enough*:

(i) *if $m^* < 6$,*

$$\frac{3}{4} \le \frac{1}{p} \le 1 \quad \textit{and} \quad -\Big(\frac{m^*}{2} + 1\Big)\frac{1}{p} + \frac{m^*}{2} + 1 < \frac{1}{q} \le 1,$$

then $(1/p, 1/q) \in E_{V_\delta,\varphi}$,

(ii) *if $m^* \ge 6$ and*

$$-\Big(\frac{m^*}{2} + 1\Big)\frac{1}{p} + \frac{m^*}{2} + 1 < \frac{1}{q} \le 1,$$

then $(1/p, 1/q) \in E_{V_\delta,\varphi}$.

Proof. From Remark 2.1(a), to prove the lemma it suffices to show that, for δ positive and small enough, the following assertions hold:

(i′) If $m - 2 \le \alpha < 4$ and $(\alpha + 4)/8 < 1/q \le 1$ then $(3/4, 1/q) \in E_{V_\delta,\varphi}$.

(ii′) If $m-2 \leq \alpha$, $\alpha \geq 4$ and $(\alpha+2)/(\alpha+4) < 1/p \leq 1$ then $(1/p, 1) \in E_{V_\delta,\varphi}$.

(iii′) If $0 \leq \alpha < m-2$, $m \geq 6$, and $m/(m+2) < 1/p \leq 1$ then $(1/p, 1) \in E_{V_\delta,\varphi}$.

(iv′) If $0 \leq \alpha < m-2$, $m < 6$ and $(m+2)/8 < 1/q \leq 1$ then $(3/4, 1/q) \in E_{V_\delta,\varphi}$.

Let $\delta_0 > 0$ be such that $\det \varphi''(x) \neq 0$ for all $x = (x_1, x_2) \in V_{\delta_0}$ with $x_2 \neq 0$. Our assumptions imply that there exist positive constants c_1, c_2 such that, if $(x_1, x_2) \in V_{\delta_0}$ and $1/2 \leq |x_1| \leq 1$, then

$$c_1|x_2|^\alpha \leq |\det \varphi''(x_1, x_2)| \leq c_2|x_2|^\alpha. \tag{3.2}$$

For $k \in \mathbb{N} \cup \{0\}$, let

$$Q_k = \{(x_1, x_2) \in \mathbb{R}^2 : 1/2 \leq |x_1| \leq 1,\ 2^{-k-1} \leq |x_2| \leq 2^{-k}\}, \tag{3.3}$$

let $\varphi_k : Q_0 \to \mathbb{R}$ be defined by $\varphi_k(x_1, x_2) = \varphi(x_1, 2^{-k}x_2)$ and let $\sigma_{Q_k,\varphi}$ be defined as at the beginning of the preliminaries. A change of variable gives

$$(\sigma_{Q_k,\varphi})^\wedge(\xi) = 2^{-k}(\sigma_{Q_0,\varphi_k})^\wedge(\xi_1, 2^{-k}\xi_2, \xi_3). \tag{3.4}$$

Pick $k(\delta_0) \in \mathbb{N}$ such that $Q_{k(\delta_0)} \subset V_{\delta_0}$. Since

$$|\det \varphi_k''(x_1, x_2)| = 2^{-2k}|\det \varphi''(x_1, 2^{-k}x_2)|,$$

from (3.2) it follows that there exists $c_1 > 0$ such that $|\det \varphi_k''(x_1, x_2)| \geq c_1 2^{-k(\alpha+2)}$ for all $k \geq k(\delta_0)$, $(x_1, x_2) \in Q_0$. Then Proposition 6 on p. 344 of [8] implies that there exists a positive constant c_3 such that

$$|(\sigma_{Q_0,\varphi_k})^\wedge(\xi_1, 2^{-k}\xi_2, \xi_3)| \leq c_3 2^{k(\alpha+2)/2}|(\xi_1, 2^{-k}\xi_2, \xi_3)|^{-1} \leq c_3 2^{k(\alpha+2)/2}|\xi_3|^{-1}$$

for all $k \geq k(\delta_0)$ and $\xi \in \mathbb{R}^3$. For these k, from (3.4) we obtain $|(\sigma_{Q_k,\varphi})^\wedge(\xi)| \leq c_3 2^{k\alpha/2}|\xi_3|^{-1}$. So, Remark 2.6 implies that

$$\|\widehat{f}\|_{L^2(\Sigma_{Q_k,\varphi})} \leq c_4 2^{k\alpha/8}\|f\|_{L^{4/3}(\mathbb{R}^3)}, \qquad f \in S(\mathbb{R}^3), \tag{3.5}$$

with c_4 independent of k and f. From (3.5), Hölder's inequality gives, for $1 \leq q < 2$ and $f \in S(\mathbb{R}^3)$,

$$\begin{aligned} \|\widehat{f}\|_{L^q(\Sigma_{Q_k,\varphi})} &\leq \sigma(\Sigma_{Q_k,\varphi})^{(2-q)/(2q)}\|\widehat{f}\|_{L^2(\Sigma_{Q_k,\varphi})} \\ &\leq c_4 2^{k(\alpha/8-(2-q)/(2q))}\|f\|_{L^{4/3}(\mathbb{R}^3)}. \end{aligned} \tag{3.6}$$

Suppose that $\alpha < 4$. If $(\alpha+4)/8 < 1/q \leq 1$ then $\alpha/8 - (2-q)/(2q) < 0$ and so for some $c > 0$ and all $f \in S(\mathbb{R}^3)$,

$$\sum_{k \geq k(\delta_0)} \|\widehat{f}\|_{L^q(\Sigma_{Q_k,\varphi})} \leq c\|f\|_{L^{4/3}(\mathbb{R}^3)}. \tag{3.7}$$

For $\delta > 0$ and $j \in \mathbb{N} \cup \{0\}$, let

$$A_{j,\delta} = \{(x_1, x_2) \in \mathbb{R}^2 : 2^{-j-1} \leq |x_1| \leq 2^{-j},\ |x_2| \leq \delta|x_1|\}, \tag{3.8}$$

thus $V_\delta = \bigcup_{j\ge 0} A_{j,\delta}$. For δ small enough, (3.7) gives

$$\|\widehat{f}\|_{L^q(\Sigma_{A_{0,\delta},\varphi})} \le c\|f\|_{L^{4/3}(\mathbb{R}^3)}.$$

If $m-2 \le \alpha < 4$ the condition $1/q > (\alpha+4)/8$ implies

$$\frac{1}{q} > -\left(\frac{m}{2}+1\right)\frac{3}{4} + \frac{m}{2} + 1$$

and so (i′) follows from Remark 2.1(d).

For $t \in [0,1]$ let p_t be defined by $1/p_t = 3t/4 + 1 - t$. For $k \ge k(\delta_0)$ and any α, from (3.6) we get $\|\widehat{f}\|_{L^1(\Sigma_{Q_k,\varphi})} \le c_4 2^{k(\alpha/8-1/2)}\|f\|_{L^{4/3}(\mathbb{R}^3)}$. Also $\|\widehat{f}\|_{L^1(\Sigma_{Q_k,\varphi})} \le c_4 2^{-k}\|f\|_{L^1(\mathbb{R}^3)}$, so an application of the Riesz–Thorin theorem gives

$$\|\widehat{f}\|_{L^1(\Sigma_{Q_k,\varphi})} \le c2^{k(((\alpha+4)/8)t-1)}\|f\|_{L^{p_t}(\mathbb{R}^3)}$$

for all $f \in S(\mathbb{R}^3)$. So, for δ small enough,

$$\|\widehat{f}\|_{L^1(\Sigma_{A_{0,\delta},\varphi})} \le \sum_{k\ge k(\delta_0)} \|\widehat{f}\|_{L^1(\Sigma_{Q_k,\varphi})} \le c\|f\|_{L^{p_t}(\mathbb{R}^3)} \tag{3.9}$$

for all $t \in [0, 8/(\alpha+4))$ if $\alpha \ge 4$, and for all $t \in [0,1]$ if $\alpha < 4$.

Suppose that $m-2 \le \alpha < \infty$ and $\alpha \ge 4$. If $(\alpha+2)/(\alpha+4) < 1/p \le 1$ then $1/p = 3t/4 + 1 - t$ for some $t \in [0, 8/(\alpha+4))$. Also

$$1 > -\left(\frac{m}{2}+1\right)\frac{1}{p} + \frac{m}{2} + 1.$$

So, Remark 2.1(d) and (3.9) imply $(1/p, 1) \in E_{V_\delta,\varphi}$ and then (ii′) holds.

Consider now the case $0 \le \alpha < m-2$, $m \ge 6$ and suppose $m/(m+2) < 1/p \le 1$. A computation shows that $1/p = 1/p_t$ for some $t \in [0, 8/(m+2))$ and so $t < 8/(\alpha+4)$. If $\alpha \ge 4$ then (3.9) and Remark 2.1(d) imply (iii′) in this case. If $\alpha < 4$, observe that the assumption on p implies $3/4 < 1/p$ and so (3.9) gives $(1/p, 1) \in E_{A_{0,\delta},\varphi}$. Since also

$$1 > -\left(\frac{m}{2}+1\right)\frac{1}{p} + \frac{m}{2} + 1,$$

(iii′) follows in this case from Remark 2.1(d).

Finally, assume that $0 \le \alpha < m-2$, $m < 6$ and $(m+2)/8 < 1/q \le 1$. Then $(\alpha+4)/8 < 1$, thus (3.7) gives $(3/4, 1/q) \in E_{A_{0,\delta},\varphi}$. Also $(m+2)/8 < 1/q$ implies $1/q > -(m/2+1)3/4 + m/2 + 1$ and so Remark 2.1(d) gives (iv′). ∎

Remark 3.2. For $\delta > 0$ let

$$W_\delta = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| \le 1,\ |x_2| \le \delta|x_1|\}$$

and let V_δ be defined by (3.1). Since $W_\delta \subset V_\delta$, it follows that Lemma 3.1 holds for W_δ in place of V_δ. ∎

Theorem 3.3. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \ge 2$ such that* $\det \varphi''(x) \equiv 0$. *Then for $m \ge 3$,*

$$E^\circ = \{(1/p, 1/q) \in [0,1] \times [0,1] : 1/q > -(m+1)/p + m + 1\},$$

and for $m = 2$,

$$E^\circ = \{(1/p, 1/q) \in (3/4,1] \times [0,1] : 1/q > -3/p + 3\}.$$

Proof. From Remark 2.5 we have $\varphi(x_1, x_2) = (ax_1 + bx_2)^m$ for some $a, b \in \mathbb{R}$ and all $(x_1, x_2) \in \mathbb{R}^2$, and so, by Remark 2.1(b), the problem reduces (after composing with a suitable rotation followed by a dilation) to the case $\varphi(x_1, x_2) = x_2^m$. From Remark 2.1(a), Lemma 2.7 and Remark 2.8, it suffices to see that for $m \ge 3$, if $m/(m+1) < 1/p \le 1$ then $(1/p, 1) \in E$, and for $m = 2$, if $3/4 < 1/p \le 1$ and $1/q > -3/p + 3$ then $(1/p, 1/q) \in E$. For $3/4 < 1/p \le 1$ we know that (see e.g. [3]) there exists $c > 0$ such that

$$\Big(\int_{1/2\le|s|\le1} |\widehat{g}(s, s^m)|^p\, ds\Big)^{1/p} \le c_p \|g\|_{L^p(\mathbb{R}^2)} \quad \text{for all } g \in S(\mathbb{R}^2). \tag{3.10}$$

We claim that for such p there exists $c' > 0$ such that

$$\Big(\int_{|x_1|\le1}\int_{1/2\le|x_2|\le1} |\widehat{f}(x_1, x_2, x_2^m)|^p\, dx_1\, dx_2\Big)^{1/p} \le c' \|f\|_{L^p(\mathbb{R}^3)} \tag{3.11}$$

for all $f \in S(\mathbb{R}^3)$. Indeed, for $h : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R}^2 \to \mathbb{C}$, let $h^{\wedge_1}$, $g^{\wedge_2}$ denote their one-and two-dimensional Fourier transforms respectively. Now,

$$\Big(\int_{|x_1|\le1}\int_{1/2\le|x_2|\le1} |\widehat{f}(x_1, x_2, x_2^m)|^p\, dx_1\, dx_2\Big)^{1/p}$$
$$= \Big\|\Big(\int_{1/2\le|x_2|\le1} |((\xi_2,\xi_3) \mapsto f(\cdot, \xi_2, \xi_3)^{\wedge_1}(x_1))^{\wedge_2}(x_2, x_2^m)|^p\, dx_2\Big)^{1/p}\Big\|_X$$

where $X = L^p((-1,1), dx_1)$. From (3.10) we get

$$\Big(\int_{|x_1|\le1}\int_{1/2\le|x_2|\le1} |\widehat{f}(x_1, x_2, x_2^m)|^p\, dx_1\, dx_2\Big)^{1/p} \le c\Big(\int_{\mathbb{R}^2}\Big(\int_{|x_1|\le1} |f(\cdot, \xi_2, \xi_3)^{\wedge_1}(x_1)|^p dx_1\Big)\, d\xi_2\, d\xi_3\Big)^{1/p}. \tag{3.12}$$

Since $p < 2$,

$$\|f(\cdot, \xi_2, \xi_3)^{\wedge_1}\|_{L^p(-1,1)} \le c'' \|f(\cdot, \xi_2, \xi_3)^{\wedge_1}\|_{L^{p'}(-1,1)} \le c''' \|f(\cdot, \xi_2, \xi_3)\|_{L^p(\mathbb{R})}$$

for some positive c'' and c'''. So (3.11) follows.

For $t > 0$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ let $t.x = (x_1, tx_2)$ and $t \circ \xi = (\xi_1, t\xi_2, t^m\xi_3)$. For $g : \mathbb{R}^3 \to \mathbb{R}$, $t > 0$ let $t \circ g : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$(t \circ g)(\xi) = g(t \circ \xi)$. Finally, for $k \in \mathbb{N} \cup \{0\}$ let

$$R_k = \{(x_1, x_2) : |x_1| \le 1,\, 2^{-k-1} \le |x_2| \le 2^{-k}\}.$$

So $R_k = 2^{-k} R_0$ and from (3.11) a standard homogeneity argument gives

$$\begin{aligned} \|\widehat{f}\|_{L^p(\Sigma_{R_k,\varphi})} &\le c 2^{-k/p + k(m+1)} \|2^k \circ f\|_{L^p(\mathbb{R}^3)} \\ &= 2^{-k/p + k(m+1)(1-1/p)} \|f\|_{L^p(\mathbb{R}^3)} \end{aligned} \tag{3.13}$$

and so, by Hölder's inequality,

$$\begin{aligned} \|\widehat{f}\|_{L^1(\Sigma_{R_k,\varphi})} &\le 2^{-k/p + k(m+1)(1-1/p)} |R_k|^{1-1/p} \|f\|_{L^p(\mathbb{R}^3)} \\ &= 2^{k(m-(m+1)/p)} \|f\|_{L^p(\mathbb{R}^3)}. \end{aligned} \tag{3.14}$$

In particular, if $m \ge 3$ and $m/(m+1) < 1/p \le 1$ then $3/4 < 1/p \le 1$ and so from (3.14) we obtain

$$\|\widehat{f}\|_{L^1(\Sigma)} \le \sum_{k \ge 0} \|\widehat{f}\|_{L^1(\Sigma_{R_k,\varphi})} \le c \|f\|_{L^p(\mathbb{R}^3)}$$

and the theorem follows for the case $m \ge 3$.

Consider now the case $m = 2$. Suppose that $3/4 < 1/p \le 1$; then $-1/p + 3(1 - 1/p) < 0$ and so from (3.13),

$$\|\widehat{f}\|_{L^p(\Sigma)} \le \sum_{k \ge 0} \|\widehat{f}\|_{L^p(\Sigma_{R_k,\varphi})} \le c \|f\|_{L^p(\mathbb{R}^3)}.$$

Also, Hölder's inequality gives $\|\widehat{f}\|_{L^q(\Sigma)} \le c\|f\|_{L^p(\mathbb{R}^3)}$ for $1/p \le 1/q \le 1$. So the theorem follows from Remark 2.1(a). ■

THEOREM 3.4. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \ge 2$ such that* $\det \varphi''(x)$ *does not vanish identically, and let $\widetilde{m}$ be defined by* (1.1) *if* $\det \varphi''(x)$ *vanishes somewhere in* $\mathbb{R}^2 - \{0\}$, *and by $\widetilde{m} = m$ if not. Then*

(i) *for $\widetilde{m} \ge 6$,*

$$E^\circ = \left\{(1/p, 1/q) \in [0,1] \times [0,1] : \frac{1}{q} > -\left(\frac{\widetilde{m}}{2} + 1\right)\frac{1}{p} + \frac{\widetilde{m}}{2} + 1\right\},$$

(ii) *for $\widetilde{m} < 6$,*

$$\begin{aligned} &E^\circ \cap ((3/4, 1] \times [0,1]) \\ &\quad = \left\{(1/p, 1/q) \in (3/4, 1] \times [0,1] : \frac{1}{q} > -\left(\frac{\widetilde{m}}{2} + 1\right)\frac{1}{p} + \frac{\widetilde{m}}{2} + 1\right\} \end{aligned}$$

and also $(3/4, 1/q) \in E$ *for* $(\widetilde{m} + 2)/8 < 1/q \le 1$.

Proof. To see that the stated conditions are sufficient, we consider first the case $\det \varphi''(x) \ne 0$ for all $x \in \mathbb{R}^2 - \{0\}$. For $j \in \mathbb{N} \cup \{0\}$ let

$$A_j = \{x \in \mathbb{R}^2 : 2^{-j-1} \le |x| \le 2^j\}.$$

From [8, p. 386] we have $(3/4,1/2)\in E_{A_j,\varphi}$ and so in this case the lemma follows from Remark 2.1(d).

Suppose now that $\det\varphi''(x)=0$ for some $x\in\mathbb{R}^2-\{0\}$. Let $L_1,\dots,L_k$ be as in the introduction. For $\delta>0$ let

$$C_\delta^j=\{x\in B:|\pi_{L_j^\perp}(x)|\le\delta|\pi_{L_j}(x)|\}$$

where π_{L_j} and $\pi_{L_j^\perp}$ denote the orthogonal projections from $\mathbb{R}^2$ onto L_j and $L_j^\perp$ respectively. Choose δ small enough such that $C_\delta^i\cap C_\delta^j=\emptyset$ for $i\ne j$. So $\det\varphi''(x)\ne 0$ for $x\in C_\delta^j-L_j$. Let $T_j\in\mathrm{SO}(2)$ be such that $T_j(L_j)$ is the x_1 axis, let $\psi_j=\varphi\circ T_j^{-1}$, $\widetilde{C}_\delta^j=T_j(C_\delta^j)$ and let $\widetilde{\alpha}_j$ be the vanishing order of $x_2\mapsto\det\psi_j''(1,x_2)$. Since the curvature is invariant under rotations we have $\widetilde{\alpha}_j=\alpha_j$, $j=1,\dots,k$. We also have $E_{C_\delta^j,\varphi}=E_{T_j(C_\delta),\varphi\circ T_j^{-1}}$ (see Remark 2.1(b)). Let $C_\delta=\bigcup_{1\le j\le k}C_\delta^j$, so $E_{C_\delta,\varphi}=\bigcap_{1\le j\le k}E_{C_\delta^j,\varphi}$.

Suppose that $(1/p,1/q)$ belongs to the left side of the equality in either (i) or (ii). If $\widetilde{m}=m$, Remark 2.1(d) applies. If $\widetilde{m}=\alpha_j+2$ for some $j=1,\dots,k$, we apply Lemmas 2.4 and 2.7 to $\varphi\circ T_j^{-1}$ to deduce that $(1/p,1/q)$ belongs to the right side of the equality.

On the other hand, from Lemma 3.1 and Remark 3.2 it follows that, for δ positive and small enough, if $\widetilde{m}<6$,

$$\frac{3}{4}\le\frac{1}{p}\le 1\quad\text{and}\quad -\frac{\widetilde{m}+2}{2p}+\frac{\widetilde{m}+2}{2}<\frac{1}{q}\le 1$$

then $(1/p,1/q)\in E_{C_\delta,\varphi}$, and if $\widetilde{m}\ge 6$,

$$\frac{\widetilde{m}}{\widetilde{m}+2}<\frac{1}{p}\le 1\quad\text{and}\quad -\frac{\widetilde{m}+2}{2p}+\frac{\widetilde{m}+2}{2}<\frac{1}{q}\le 1$$

then $(1/p,1/q)\in E_{C_\delta,\varphi}$. Finally, let $D_\delta=B-C_\delta$. So D_δ is a union of a finite number of angular sectors with vertices at the origin where $\det\varphi''(x)$ never vanishes (except at the origin), and so we can proceed as in the first part of the proof to get $(1/p,1/q)\in E_{D_\delta,\varphi}$ for

$$\frac{3}{4}\le\frac{1}{p}\le 1\quad\text{and}\quad -\frac{\widetilde{m}+2}{2p}+\frac{\widetilde{m}+2}{2}<\frac{1}{q}\le 1.$$

Since $E=E_{C_\delta,\varphi}\cap E_{D_\delta,\varphi}$ the theorem follows. ■

4. Sharp $L^p(\mathbb{R}^3)$-$L^2(\Sigma)$ estimates for $\mathcal{R}$

REMARK 4.1. For our next results we will need to introduce two Littlewood–Paley decompositions on $S(\mathbb{R}^3)$. Let $\Phi\in C_c^\infty(\mathbb{R})$ be an even function satisfying $0\le\Phi\le 1$, $\operatorname{supp}\Phi\subset\{t\in\mathbb{R}:2^{-1}\le|t|\le 2\}$ and such that $\sum_{r\in\mathbb{Z}}\Phi(2^rt)=1$ if $t\ne 0$. For $r\in\mathbb{Z}$ let $\Psi_r:\mathbb{R}\to\mathbb{R}$ be defined by $\Psi_r(t)=2^{-r}\widehat{\Phi}(2^{-r}t)$, and for $f\in S(\mathbb{R}^3)$ let T_rf be given by $T_rf=(\delta\otimes\Psi_r\otimes\delta)*f$.

Thus $f = \sum_{r\in\mathbb{Z}} T_r f$ with convergence in $S'(\mathbb{R}^3)$. Moreover, it can be checked that for $\varepsilon_r = \pm 1$, the one-dimensional multiplier $\sum_{r\in\mathbb{Z}} \varepsilon_r \Phi(2^r t)$ satisfies the hypothesis of Theorem 3 on p. 96 of [7] with constants independent of the choice of ε_r. Hence for $f \in S(\mathbb{R}^3)$, $\|\sum_{r\in\mathbb{Z}} \varepsilon_r T_r f\|_{L^p(\mathbb{R}^3)} \le c\|f\|_{L^p(\mathbb{R}^3)}$ with c independent of f and of the choice of ε_r, so as in [7, p. 105] we have the Littlewood–Paley inequality

$$\Big\| \Big(\sum_{r\in\mathbb{Z}} |T_r f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^3)} \le c\|f\|_{L^p(\mathbb{R}^3)}. \tag{4.1}$$

Similarly, if we start with an even function $\widetilde{\Phi} \in C_c^\infty(\mathbb{R}^2)$ with support contained in the annulus $\{t \in \mathbb{R}^2 : 2^{-1} \le |t| \le 2\}$ such that $0 \le \widetilde{\Phi} \le 1$, $\sum_{r\in\mathbb{Z}} \widetilde{\Phi}(2^r t) = 1$ for $t \in \mathbb{R}^2 - \{0\}$, and if we define $\widetilde{\Psi}_r(t) = 2^{-2r}(\widetilde{\Phi})^\wedge(2^{-r}t)$, $r \in \mathbb{Z}$, and $\widetilde{T}_r f = (\widetilde{\Psi}_r \otimes \delta) * f$, we now see that $f = \sum_{r\in\mathbb{Z}} \widetilde{T}_r f$, $f \in S(\mathbb{R}^3)$, with convergence in $S'(\mathbb{R}^3)$ and that (4.1) is also true for the family $\{\widetilde{T}_r\}_{r\in\mathbb{Z}}$ in place of $\{T_r\}_{r\in\mathbb{Z}}$. ∎

THEOREM 4.2. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \ge 2$ such that* $\det \varphi''(x) \equiv 0$. *Then* $((2m+1)/(2m+2), 1/2) \in E$.

Proof. As in Theorem 3.3 it is enough to prove the theorem for the case $\varphi(x) = x_2^m$. In this case the van der Corput lemma applied with the mth derivative gives

$$\Big| \int_{-1}^{1} e^{-i(y\xi_2 + y^m \xi_3)}\, dy \Big| \le c|\xi_3|^{-1/m},$$

so $|\sigma^\wedge(\xi)| \le c(1 + |\xi_3|)^{-1/m}$. Thus the theorem follows from Remark 2.6. ∎

LEMMA 4.3. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function such that* $\det \varphi''(x)$ *is not identically zero*, *let m^* be as in Lemma* 3.1 *and for $\delta > 0$ let V_δ be defined by* (3.1). *Then for δ positive and small enough*, $((m^*+1)/(m^*+2), 1/2) \in E_{V_\delta,\varphi}$.

Proof. We first consider the case $\alpha < m - 2$. Then $m^* = m$. Let δ_0 and $k(\delta_0)$ be as in the proof of Lemma 3.1 and for $k \in \mathbb{Z} \cup \{0\}$ let Q_k be defined by (3.3). For $\theta \in [0,1]$ let p_θ be defined by $1/p_\theta = 3\theta/4 + 1 - \theta$. For $f \in S(\mathbb{R}^3)$, from (3.5) and $\|\widehat{f}\|_{L^2(\Sigma_{Q_k,\varphi})} \le c2^{-k/2}\|f\|_{L^1(\mathbb{R}^3)}$ (see Remark 2.1(a)), the Riesz–Thorin theorem gives

$$\|\widehat{f}\|_{L^2(\Sigma_{Q_k,\varphi})} \le c2^{k\alpha\theta/8 - k(1-\theta)/2}\|f\|_{L^{p_\theta}(\mathbb{R}^3)}. \tag{4.2}$$

Then for δ positive and small enough we get

$$\|\widehat{f}\|_{L^2(\Sigma_{A_{0,\delta},\varphi})} \le \sum_{k \ge k(\delta_0)} \|\widehat{f}\|_{L^2(\Sigma_{Q_k,\varphi})} \le c\|f\|_{L^{(m+2)/(m+1)}(\mathbb{R}^3)}$$

with $A_{0,\delta}$ defined by (3.8). Thus, from Remark 2.1(c) we find

$$\|\widehat{f}\|_{L^2(\Sigma_{A_{j,\delta},\varphi})} \leq c\|f\|_{L^{(m+2)/(m+1)}(\mathbb{R}^3)} \tag{4.3}$$

for some $c > 0$ and all $j \in \mathbb{Z}$, $f \in S(\mathbb{R}^3)$.

For $r \in \mathbb{Z}$ let $\widetilde{T}_r$, $\widetilde{\Psi}_r$ be as in Remark 4.1. Then for $f \in S(\mathbb{R}^3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ such that $(\xi_1, \xi_2) \neq 0$ we have

$$\widehat{f}(\xi) = \sum_{r\in\mathbb{Z}} \widehat{\widetilde{\Psi}}_r(\xi_1, \xi_2)\widehat{f}(\xi) = \sum_{r\in\mathbb{Z}} (\widetilde{T}_r f)^\wedge(\xi)$$

with convergence in $L^2(\mathbb{R}^3)$. Also, $(\widetilde{T}_r f)^\wedge(\xi) = \widetilde{\Phi}(2^r(\xi_1, \xi_2))\widehat{f}(\xi)$ and so, for each ξ, the set $\{r \in \mathbb{Z} : (\widetilde{T}_r f)^\wedge(\xi) \neq 0\}$ has at most three elements. Then

$$\Big|\sum_{r\in\mathbb{Z}} (\widetilde{T}_r f)^\wedge(x, \varphi(x))\Big|^2 \leq 3 \sum_{r\in\mathbb{Z}} \Big|(\widetilde{T}_r f)^\wedge(x, \varphi(x))\Big|^2.$$

Let $A'_{r,\delta} = \bigcup_{j=r-1}^{r+1} A_{j,\delta}$. From (4.3) we have

$$\int_{A'_{r,\delta}} |(\widetilde{T}_r f)^\wedge(x, \varphi(x))|^2\, dx \leq c\|\widetilde{T}_r f\|^2_{L^{(m+2)/(m+1)}(\mathbb{R}^3)}$$

with c independent of f and r. Also, if $0 < \delta < 1$ then

$$\{x \in V_\delta : (\widetilde{T}_r f)^\wedge(x, \varphi(x)) \neq 0\} \subset A'_{r,\delta}.$$

From these facts we deduce that there exist positive constants c, c' and c'' independent of f such that

$$\begin{aligned}
\|\widehat{f}\|^2_{L^2(\Sigma_{V_\delta,\varphi})} &= \int_{V_\delta} \Big|\sum_{r\in\mathbb{Z}} (\widetilde{T}_r f)^\wedge(x, \varphi(x))\Big|^2 dx \\
&\leq 3 \sum_{r\in\mathbb{Z}} \int_{A'_{r,\delta}} |(\widetilde{T}_r f)^\wedge(x, \varphi(x))|^2\, dx \leq c \sum_{r\in\mathbb{Z}} \|\widetilde{T}_r f\|^2_{L^{(m+2)/(m+1)}(\mathbb{R}^3)} \\
&\leq c' \Big\| \Big(\sum_{r\in\mathbb{Z}} |\widetilde{T}_r f|^2\Big)^{1/2} \Big\|_{L^{(m+2)/(m+1)}(\mathbb{R}^3)} \leq c'' \|f\|^2_{L^{(m+2)/(m+1)}(\mathbb{R}^3)}
\end{aligned}$$

where the last inequality follows from the Littlewood–Paley inequality and the previous one from Minkowski's inequality. Thus the lemma follows for $\alpha < m - 2$.

If $\alpha \geq m-2$ then $m^* = \alpha+2$ and so from (4.2) we obtain $\|\widehat{f}\|_{L^2(\Sigma_{Q_k,\varphi})} \leq c\|f\|_{L^{(m^*+2)/(m^*+1)}(\mathbb{R}^3)}$ for some $c > 0$, for all $k \geq k(\delta_0)$ and $f \in S(\mathbb{R}^3)$.

For $r \in \mathbb{Z}$ let T_r, Ψ_r be as in Remark 4.1. Then for $f \in S(\mathbb{R}^3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ such that $\xi_2 \neq 0$ we have

$$\widehat{f}(\xi) = \sum_{r\in\mathbb{Z}} \widehat{\Psi}_r(\xi_2)\widehat{f}(\xi) = \sum_{r\in\mathbb{Z}} (T_r f)^\wedge(\xi)$$

with convergence in $L^2(\mathbb{R}^3)$. Now we proceed as in the first part of the proof, but with $\widetilde{T}_r$ replaced by T_r, to obtain

$$\|\widehat{f}\|^2_{L^2(\Sigma_{A_{0,\delta},\varphi})} \le c\|f\|^2_{L^{(m^*+2)/(m^*+1)}(\mathbb{R}^3)}.$$

From Remark 2.1(c) we get

$$(4.4)\qquad \|\widehat{f}\|_{L^2(\Sigma_{A_{j,\delta},\varphi})} \le c2^{-j[1+(m+2)(m^*+1)/(m^*+2)-(m+2)]}\|f\|_{L^{(m^*+2)/(m^*+1)}(\mathbb{R}^3)} = c2^{-j(m^*-m)/(m^*+2)}\|f\|_{L^{(m^*+2)/(m^*+1)}(\mathbb{R}^3)}.$$

If $\alpha > m-2$ then $m^* > m$, so we can perform the sum on j to obtain the conclusion.

Finally, for $\alpha = m-2$ the estimates in (4) are uniform on j and then we proceed as in the case $\alpha < m-2$ to get the assertion. ∎

From this lemma we proceed as in the proof of Theorem 3.4 to obtain the following

THEOREM 4.4. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function such that* $\det \varphi''(x)$ *is not identically zero, and let $\widetilde{m}$ be as in Theorem* 3.4. *Then* $((\widetilde{m}+1)/(\widetilde{m}+2), 1/2) \in E$.

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