Restriction theorems for the Fourier transform to homogeneous polynomial surfaces in $\mathbb{R}^3$

by

E. Ferreyra, T. Godoy and M. Urciuolo (Córdoba)

Abstract. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$, let $\Sigma = \{(x, \varphi(x)) : |x| \leq 1\}$ and let $\sigma$ be the Borel measure on $\Sigma$ defined by $\sigma(A) = \int_B \chi_A(x, \varphi(x)) \, dx$ where $B$ is the unit open ball in $\mathbb{R}^2$ and $dx$ denotes the Lebesgue measure on $\mathbb{R}^2$. We show that the composition of the Fourier transform in $\mathbb{R}^3$ followed by restriction defines a bounded operator from $L^p(\mathbb{R}^3)$ to $L^q(\Sigma, d\sigma)$ for certain $p, q$. For $m \leq 6$ the results are sharp except for some border points.

1. Introduction. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a smooth enough function, let $B$ be the open unit ball in $\mathbb{R}^n$ and let $\Sigma = \{(x, \varphi(x)) : x \in B\}$. For $f \in S(\mathbb{R}^{n+1})$, let $\mathcal{R}f : \Sigma \to \mathbb{C}$ be defined by $(\mathcal{R}f)(x, \varphi(x)) = \hat{f}(x, \varphi(x))$, $x \in B$, where $\hat{f}$ denotes the usual Fourier transform of $f$ defined by $\hat{f}(\xi) = \int_B f(u)e^{-i(u, \xi)} \, du$. Let $\sigma$ be the Borel measure on $\Sigma$ defined by $\sigma(A) = \int_B \chi_A(x, \varphi(x)) \, dx$ and let $E$ be the type set for the operator $\mathcal{R}$, i.e. the set of pairs $(1/p, 1/q) \in [0, 1] \times [0, 1]$ such that $\|\hat{f}\|_{L^q(\Sigma)} \leq c \|f\|_{L^p(\mathbb{R}^{n+1})}$ for some $c > 0$ and all $f \in S(\mathbb{R}^{n+1})$, where the spaces $L^p(\mathbb{R}^{n+1})$ and $L^q(\Sigma)$ are taken with respect to the Lebesgue measure in $\mathbb{R}^{n+1}$ and the measure $\sigma$ respectively.

The $L^p(\mathbb{R}^{n+1})$-$L^q(\Sigma)$ boundedness properties of the restriction operator $\mathcal{R}$ have been widely studied. It is well known that for $\Sigma$ as above, if $(1/p, 1/q) \in E$ then

$$\frac{1}{q} \geq -\frac{n+2}{n} + \frac{n+2}{p}.$$ 

In [10], it is proved, for the case where $\varphi$ is a nondegenerate quadratic form in $\mathbb{R}^{n+1}$, that $(1/p, 1/2) \in E$ if $(n+4)/(2n+4) \leq 1/p \leq 1$, and the method given there provides a general tool to obtain, from suitable estimates for $\hat{\sigma}$, $L^p(\mathbb{R}^{n+1})$-$L^2(\Sigma)$ estimates for $\mathcal{R}$. Moreover, a general theorem, due to Stein, 2000 Mathematics Subject Classification: Primary 42B10, 26D10.

Key words and phrases: restriction theorems, Fourier transform.

Research partially supported by Agencia Córdoba Ciencia, CONICET and Secyt-UNC.
holds for smooth enough hypersurfaces with never vanishing Gaussian curvature (see e.g. [8, p. 386]). There it is shown that, in this case, \((1/p, 1/q) \in E\) if
\[
\frac{1}{q} \geq -\frac{n + 2}{n} \frac{1}{p} + \frac{n + 2}{n} \quad \text{and} \quad \frac{n + 4}{2n + 4} \leq \frac{1}{p} \leq 1.
\]
For the case \(n = 1\) a restriction theorem (also under the assumption of nonvanishing curvature) is given in [3] where it is proved that, in this case, \((1/p, 1/q) \in E\) if \(3/4 < 1/p \leq 1\) and \(1/q \geq -3/p + 3\), and this result is sharp, i.e. the conditions on \(p\) and \(q\) are also necessary. Also in [1], [5], [4] and [6] restriction theorems for curves of finite type are obtained. Concerning the homogeneous case, the type set \(E\) is studied in [2] for \(\varphi(x) = (\sum_{j=1}^n |x_j|^\alpha).\)

The main tools used in [2] are a dyadic decomposition of \(E\) combined with Strichartz’s method applied to these dyadic pieces and interpolation techniques.

In this paper we consider the case \(n = 2\) and \(\varphi : \mathbb{R}^2 \to \mathbb{R}\) a homogeneous polynomial function. We study the type set \(E\) following in part the approach in [2].

Let us describe our results. Let \(E^\circ\) denote the relative interior of \(E\) in \([0, 1] \times [0, 1]\).

If \(\det \varphi''{(x)} \equiv 0\) we characterize \(E^\circ\) (see Theorem 3.3).

If \(\det \varphi''{(x)}\) is not identically zero and if it vanishes somewhere on \(\mathbb{R}^2 - \{0\}\), since \(\varphi\) is a homogeneous polynomial function, the set of the points \(x\) where \(\det \varphi''{(x)}\) vanishes is a finite union of lines \(L_1, \ldots, L_k\) through the origin. For a point \(x_j \in L_j - \{0\}, j = 1, \ldots, k\), we consider the vanishing order \(\alpha_j\) of \(\det \varphi''{(x)}\) at \(x_j\) along a transversal direction to \(L_j\). A simple computation using the homogeneity of \(\det \varphi'\) shows that \(\alpha_j\) is independent of the point \(x_j\) and of the transversal direction chosen. Let
\[
(1.1) \quad \tilde{m} = \max\{m, \alpha_1 + 2, \ldots, \alpha_k + 2\}.
\]

In this case, for \(\tilde{m} \geq 6\) we characterize \(E^\circ\), and for \(\tilde{m} < 6\) we characterize \(E^\circ \cap ((3/4, 1] \times [0, 1])\) and we prove that \((3/4, 1/q) \in E\) for \((\tilde{m} + 2)/8 < 1/q \leq 1\) (see Theorem 3.4). These results still hold if \(\det \varphi''{(x)}\) never vanishes on \(\mathbb{R}^2 - \{0\}\) provided that we define \(\tilde{m} = m\) in this case.

Finally, for every case, we give (see Theorems 4.2 and 4.4) a sharp \(L^p(\mathbb{R}^3) - L^2(\Sigma)\) estimate for the restriction operator \(R\).

Acknowledgements. The authors are deeply indebted to Prof. F. Ricci for his invaluable suggestions.

2. Preliminaries

Remark 2.1. Let us introduce some additional notation and state some general facts concerning restriction operators. If \(V \subset \mathbb{R}^2\) is a measurable set
and if \( \psi : V \to \mathbb{R} \) is a continuous function, let \( \Sigma_{V,\psi}, \sigma_{V,\psi}, \mathcal{R}_{V,\psi} \) be the surface, the measure and the restriction operator defined as \( \Sigma \), \( \sigma \) and \( \mathcal{R} \) at the beginning of the introduction, with \( n = 2 \), but taking now \( V \) and \( \psi \) instead of \( B \) and \( \varphi \) respectively. Finally, let \( E_{V,\psi} \) be the type set for \( \mathcal{R}_{V,\psi} \).

Let us recall some well known facts about the operators \( \mathcal{R}_{V,\psi} \).

(a) The Riesz–Thorin theorem implies that \( E_{V,\psi} \) is a convex set. Moreover, for \( f \in S(\mathbb{R}^3) \), we have

\[
\| \mathcal{R}_{V,\psi} \|_{L^1(\mathbb{R}^3), L^\infty(\Sigma_{V,\psi})} \leq 1 \quad \text{and} \quad \| \mathcal{R}_{V,\psi} \|_{L^1(\mathbb{R}^3), L^1(\Sigma_{V,\psi})} \leq |V|
\]

where \( |V| \) denotes the Lebesgue measure of \( V \). So, by the Riesz–Thorin theorem, if \( |V| < \infty \) the closed segment with endpoints \((1,0)\) and \( (1,1) \) is contained in \( E_{V,\psi} \). In particular we get the estimate

\[
\| \mathcal{R}_{V,\psi} \|_{L^1(\mathbb{R}^3), L^2(\Sigma_{V,\psi})} \leq |V|^{1/2}.
\]

(b) If \( T \in \text{GL}(\mathbb{R}^2) \), a computation shows that \( E_{T(V),\psi \circ T^{-1}} = E_{V,\psi} \). Also, \( E_{tV,\psi} = E_{V,\psi} \), for \( a \in \mathbb{R} - \{0\} \).

(c) Let us recall the well known homogeneity argument (see e.g. [10], [11]). If \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) is a continuous and homogeneous function of degree \( m \) then \( E_{tV,\varphi} = E_{V,\varphi} \) for all \( t > 0 \). Indeed, a computation gives, for \( f \in S(\mathbb{R}^3) \) and \( t > 0 \),

\[
(2.1) \quad \| \hat{f} \|_{L^q(\Sigma_{tV,\varphi})} = t^{2/q-(m+2)} \| (f_t^{-1})^\wedge \|_{L^q(\Sigma_{V,\varphi})}
\]

where \( f_t(v_1, v_2, v_3) = f(tv_1, tv_2, t^m v_3) \). From (2.1) it easily follows that

\[
(2.2) \quad \| \mathcal{R}_{tV,\varphi} \|_{L^p(\mathbb{R}^3), L^q(\Sigma_{V,\varphi})} = t^{2/q+(m+2)/p-(m+2)} \| \mathcal{R}_{V,\varphi} \|_{L^p(\mathbb{R}^3), L^q(\Sigma_{V,\varphi})}
\]

for all \( t > 0 \) and so \( E_{tV,\varphi} = E_{V,\varphi} \).

(d) Let \( \varphi \) be as in (c), let \( W = \bigcup_{k \in \mathbb{N} \cup \{0\}} 2^{-k} V \) and suppose that \((1/p, 1/q) \in E_{V,\varphi} \) and

\[
\frac{1}{q} > -\left( \frac{m}{2} + 1 \right) \frac{1}{p} + \frac{m}{2} + 1.
\]

Then \((1/p, 1/q) \in E_{W,\varphi} \). Indeed, since

\[
\| \mathcal{R}_{W,\varphi} \|_{L^p(\mathbb{R}^3), L^q(\Sigma_{W,\varphi})} \leq \sum_{k \in \mathbb{N} \cup \{0\}} \| \mathcal{R}_{2^{-k}V,\varphi} \|_{L^p(\mathbb{R}^3), L^q(\Sigma_{2^{-k}V,\varphi})}
\]

the statement follows from (2.2).

(e) Another consequence of the homogeneity argument is the following.

For \( \varphi \) and \( W \) as in (d), since

\[
\| \mathcal{R}_{W,\varphi} \|_{L^p(\mathbb{R}^3), L^q(\Sigma_{W,\varphi})} \geq \| \mathcal{R}_{2^{-k}V,\varphi} \|_{L^p(\mathbb{R}^3), L^q(\Sigma_{2^{-k}V,\varphi})}
\]

for all \( k \in \mathbb{N} \), from (2.2) it follows that

\[
\frac{1}{q} \geq -\left( \frac{m}{2} + 1 \right) \frac{1}{p} + \frac{m}{2} + 1
\]

is a necessary condition in order to have \((1/p, 1/q) \in E_{W,\varphi} \).
The following Lemmas 2.2 and 2.4 allow us to compute the vanishing order of \( \det \varphi''(x) \) along the \( x_1 \) axis for an arbitrary homogeneous polynomial function \( \varphi : \mathbb{R}^2 \to \mathbb{R} \). Let \( \alpha \) be the order of \( x_2 = 0 \) as zero of the function \( x_2 \mapsto \det \varphi''(1, x_2) \), with the convention that \( \alpha = 0 \) if \( \det \varphi''(1, 0) \neq 0 \), and \( \alpha = \infty \) if \( \det \varphi''(1, x_2) \) vanishes identically (i.e., by the homogeneity of \( \varphi \), if \( \det \varphi''(x) \) vanishes identically on \( \mathbb{R}^2 \)).

**Lemma 2.2.** Let \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be a homogeneous polynomial function of degree \( m \geq 2 \) of the form

\[
\varphi(x_1, x_2) = a_0 x_1^m + \sum_{1 \leq l \leq m} a_l x_1^{m-l} x_2^l
\]

for some \( a_0, \ldots, a_m \in \mathbb{R} \) with \( a_0 \neq 0 \) and let \( \alpha \) be defined as above.

(i) Suppose that

\[
a_s \frac{a_s}{a_0} = \left( \frac{m}{s} \right) m^{-s} \left( \frac{a_1}{a_0} \right)^s \text{ for } s = 1, \ldots, r
\]

with \( r < m \) and that

\[
a_{r+1} \frac{a_{r+1}}{a_0} \neq \left( \frac{m}{r+1} \right) m^{-r-1} \left( \frac{a_1}{a_0} \right)^{r+1}.
\]

Then \( \alpha = r - 1 \).

(ii) If

\[
a_s \frac{a_s}{a_0} = \left( \frac{m}{s} \right) m^{-s} \left( \frac{a_1}{a_0} \right)^s \text{ for } s = 1, \ldots, m,
\]

then \( \alpha = \infty \).

**Proof.** To prove (i), without loss of generality we can assume that \( a_0 = 1 \). Let \( r \) be as in (i), so \( 1 \leq r \leq m - 1 \). We have \( \det \varphi''(x_1, x_2) = AB - C^2 \) where

\[
A = \sum_{0 \leq l \leq m-2} (m-l)(m-l-1)a_l x_1^{m-l-2} x_2^l,
\]

\[
B = \sum_{0 \leq j \leq m-2} (j+2)(j+1)a_{j+2} x_1^{m-j-2} x_2^j,
\]

\[
C = \sum_{l=0}^{m-2} (l+1)(m-l-1)a_{l+1} x_1^{m-l-2} x_2^l.
\]

A computation of \( AB - C^2 \) gives

\[
\det \varphi''(x_1, x_2) = \sum_{0 \leq l \leq 2m-4} \sum_{l+j=i} [(m-l)(m-l-1)(j+2)(j+1)a_la_{j+2}
\]

\[
- (l+1)(m-l-1)(m-j-1)(j+1)a_{l+1}a_{j+1}] x_1^{2m-4-i} x_2^i.
\]
For $0 \leq i \leq 2m - 4$, let $c_i x_1^{2m-4-i}$ be the coefficient of $x_2^i$ in $\det \varphi''(x_1, x_2)$. For $s = 1, \ldots, r - 1$ we have

$$c_s = \sum_{l+j=s} (m-l)(m-l-1)(j+2)(j+1)a_la_{j+2}$$

$$- \sum_{l+j=s} (l+1)(m-l-1)(m-j-1)(j+1)a_{l+1}a_{j+1}.$$ 

Thus (recalling that $a_0 = 1$) the hypothesis of (i) gives

$$c_s = m(m-1)(s+2)(s+1)a_{s+2}$$

$$+ \sum_{l+j=s, l \neq 0} (m-l)(m-l-1)(j+2)(j+1) \left( \begin{array}{c} m \\ l \end{array} \right) \left( \begin{array}{c} m \\ j+2 \end{array} \right) m^{-s-2}a_1^{s+2}$$

$$- \sum_{l+j=s} (l+1)(m-l-1)(m-j-1)(j+1) \left( \begin{array}{c} m \\ l+1 \end{array} \right) \left( \begin{array}{c} m \\ j+1 \end{array} \right) m^{-s-2}a_1^{s+2}.$$ 

Since

$$(l+1)(m-l-1)(m-j-1)(j+1) \left( \begin{array}{c} m \\ l+1 \end{array} \right) \left( \begin{array}{c} m \\ j+1 \end{array} \right)$$

$$= (m-l)(m-l-1)(j+2)(j+1) \left( \begin{array}{c} m \\ l \end{array} \right) \left( \begin{array}{c} m \\ j+2 \end{array} \right)$$

we get

$$c_s = m(m-1)(s+1)(s+2) \left[ a_{s+2} - \left( \begin{array}{c} m \\ s+2 \end{array} \right) m^{-s-2}a_1^{s+2} \right]$$

and so $c_0 = \ldots = c_{r-2} = 0$ and $c_{r-1} \neq 0$, hence $\alpha = r - 1$.

To see (ii) observe that if

$$\frac{a_s}{a_0} = \left( \begin{array}{c} m \\ s \end{array} \right) m^{-s}\left( \frac{a_1}{a_0} \right)^s$$

for $s = 1, \ldots, m$

then $\varphi(x_1, x_2) = a_0(x_1 + bx_2)^m$ for some $b \in \mathbb{R}$ and that in this case $\det \varphi''(x)$ is identically zero.

**Remark 2.3.** Let $\varphi, \alpha$ be as in Lemma 2.2. Observe that this lemma implies that $\alpha < m - 2$ except in the cases where $\varphi$ is either of the form $\varphi(x_1, x_2) = a_0(x_1 + bx_2)^m$ or $\varphi(x_1, x_2) = a_0(x_1 + bx_2)^m + b'x_2^m$ for some $a_0, b, b' \in \mathbb{R}$ with $a_0 \neq 0, b' \neq 0$, and that in these exceptional cases we have $\alpha = \infty$ and $\alpha = m - 2$ respectively.

**Lemma 2.4.** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function of degree $m \geq 2$ given by

$$\varphi(x_1, x_2) = \sum_{k \leq l \leq m} a_{kl}x_1^{m-l}x_2^l$$
for some \( a_k, \ldots, a_m \in \mathbb{R} \) with \( 1 \leq k \leq m \), and \( a_k \neq 0 \). Let \( \alpha \) be as in Lemma 2.2. Then \( \alpha = 2k - 2 \) if \( k < m \), and \( \alpha = \infty \) if \( k = m \).

Proof. If \( k = m \), then \( \varphi(x_1, x_2) = a_m x_2^m \), so \( \det \varphi''(1, x_2) \) vanishes identically and then \( \alpha = \infty \).

If \( k < m \), then \( \varphi(x_1, x_2) = a_k x_1^{m-k} + x_2^{k+1} \psi(x_1, x_2) \) for some polynomial function \( \psi \) satisfying \( \psi(1, 0) \neq 0 \), and so

\[
\det \varphi''(x_1, x_2) = -k(m-1)(m-k)a_k^2 x_1^{2m-2k-2} x_2^{2k-2} + x_2^{2k-1} \Theta(x_1, x_2)
\]

where \( \Theta(x_1, x_2) \) is a polynomial function. Since \( k(m-1)(m-k) \neq 0 \) and \( a_k \neq 0 \), we get \( \alpha = 2k - 2 \).

Remark 2.5. For an arbitrary homogeneous polynomial \( \varphi \) of degree \( m \geq 2 \), from Lemmas 2.2 and 2.4 it follows that \( \det \varphi''(1, x_2) \equiv 0 \) if and only if \( \varphi(x_1, x_2) = (ax_1 + bx_2)^m \) for some \( a, b \in \mathbb{R} \) and all \( (x_1, x_2) \in \mathbb{R}^2 \).

Remark 2.6. We will need the following Strichartz theorem (see [10]) whose proof relies on Stein’s complex interpolation theorem which gives \( L^p(\mathbb{R}^3) \rightarrow L^2(\Sigma_{V, \psi}) \) estimates for the operator \( R_{V, \psi} \). Since we will need information about the size of the constants we give a sketch of its proof.

Let \( V \) be a measurable subset of \( \mathbb{R}^2 \) such that \(|V| > 0\) and let \( \psi : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a continuous function. Suppose that \(|(\sigma_{V, \psi})^\wedge(\xi)| \leq A(1 + |\xi_3|)^{-\tau} \) for some \( \tau > 0 \) and for all \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \). Then

\[
\|R_{V, \psi}\|_{L^p(\mathbb{R}^3), L^2(\Sigma_{V, \psi})} \leq c_\tau A^{1/(2(1+\tau))}
\]

for \( p = (2 + 2\tau)/(2 + \tau) \) where \( c_\tau \) is a positive constant depending only on \( \tau \). Indeed, as in [8, p. 381] we define the analytic family of distributions \( I_z \) given, for \( \text{Re}(z) > 0 \), by

\[
I_z(t) = \begin{cases} 
\frac{e^{z^2}}{\Gamma(z)} t^{z-1} \zeta(t) & \text{for } t > 0, \\
0 & \text{for } t \leq 0,
\end{cases}
\]

where \( \zeta \in C^\infty_c \) and \( \zeta(t) = 1 \) for \( |t| \leq 1 \). Also we define \( J_z = \delta \otimes \delta \otimes I_z \). For \( -\tau \leq \text{Re}(z) \leq 1 \) and \( f \in S(\mathbb{R}^3) \), let \( T_z f = (J_z \ast \sigma_{V, \psi})^\wedge \ast f \). A computation shows that if \( \text{Re}(z) = 1 \), then \( \|J_z \ast \sigma_{V, \psi}\|_{L^\infty(\mathbb{R}^3)} \leq c < \infty \), so

\[
\|T_z f\|_{L^2(\mathbb{R}^3)} = \|(T_z f)^\wedge\|_{L^2(\mathbb{R}^3)} \leq c \|\hat{f}\|_{L^2(\mathbb{R}^3)} = c \|f\|_{L^2(\mathbb{R}^3)}.
\]

Also since \(|(J_z)^\wedge(\xi)| \leq c_\tau (1 + |\xi_3|)^{\tau} \) for \( \text{Re}(z) = -\tau \), Young’s inequality gives

\[
\|T_z f\|_{L^\infty(\mathbb{R}^3)} = \|((\sigma_{V, \psi})^\wedge(J_z)^\wedge) \ast f\|_{L^\infty(\mathbb{R}^3)} \leq c_\tau A \|f\|_{L^1(\mathbb{R}^3)}
\]

and so Stein’s complex interpolation theorem (as stated e.g. in [9, Ch. V]) entails that the operator \( T_0 f = (\sigma_{V, \psi})^\wedge \ast f \) satisfies

\[
\|T_0\|_{L^p(\mathbb{R}^3), L^{p'}(\mathbb{R}^3)} \leq c_\tau A^{1/(1+\tau)}
\]
for $p = (2 + 2\tau)/(2 + \tau)$. This implies (see e.g. [8, p. 253]) that
\[ \|R_{V,\psi}\|_{L^p(\mathbb{R}^3),L^2(\Sigma_{V,\psi})} \leq c_\psi A^{1/(2(1+\tau))}. \]

Lemma 2.7. Let $\varphi(x_1, x_2) = \sum_{k \leq l \leq m} a_k x_1^{m-l} x_2^l$ where $0 \leq k \leq m$ and
\[ a_k, \ldots, a_m \in \mathbb{R} \text{ with } a_k \neq 0, \] and let $\Sigma$ and $E$ be defined as in the introduction. If $(1/p, 1/q) \in E$ then
\[ \frac{1}{q} \geq -(k + 1)\frac{1}{p} + k + 1. \]

Proof. If $k = 0$ the lemma follows from Remark 2.1(e). Suppose $k \neq 0$. For $0 < \epsilon < 1$, let \( f_\epsilon \) be the characteristic function of the set \([0, 1] \times [0, \epsilon^{-1}/k] \times [0, \epsilon^{-1}]\). Then for $x = (x_1, x_2)$,
\[
\hat{f}_\epsilon(x, \varphi(x)) = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} e^{-i(x_1 \xi_1 + x_2 \xi_2 + \varphi(x_1, x_2)\xi_3)} d\xi_1 d\xi_2 d\xi_3 \\
= \epsilon^{-(1+1/k)} \int_{Q} e^{-i(x_1 u_1 + \epsilon^{-1/k} x_2 u_2 + \epsilon^{-1} \varphi(x_1, x_2) u_3)} d u_1 d u_2 d u_3
\]
where $Q = [0, 1] \times [0, 1] \times [0, 1]$. Let $c = \min(1, (\frac{1}{3} \sum_{k \leq l \leq m} |a_l|)^{-1/k})$ and
\[ D_\epsilon = [0, 1/3] \times [0, (c/3)\epsilon^{1/k}] \]. So $\Sigma_{D_\epsilon,\varphi} \subset \Sigma$ and $\|\hat{f}_\epsilon\|_{L^q(\Sigma)} \geq \|\hat{f}_\epsilon\|_{L^q(\Sigma_{D_\epsilon,\varphi})}$. Now
\[
\epsilon^{q+k/\|f_\epsilon\|_{L^q(\Sigma_{D_\epsilon,\varphi})}} \\
= \int_{D_\epsilon} \int_{Q} e^{-i(x_1 u_1 + \epsilon^{-1/k} x_2 u_2 + \epsilon^{-1} \varphi(x_1, x_2) u_3)} d u_1 d u_2 d u_3 \\
\geq \int_{D_\epsilon} \int_{Q} \cos(x_1 u_1 + \epsilon^{-1/k} x_2 u_2 + \epsilon^{-1} \varphi(x_1, x_2) u_3) d u_1 d u_2 d u_3 \\
\]
For $(x_1, x_2, \varphi(x_1, x_2)) \in \Sigma_{D_\epsilon,\varphi}$ we have
\[
\epsilon^{-1} |\varphi(x_1, x_2)| \leq \epsilon^{-1} (|a_k x_1^{m-k} x_2^k| + \ldots + |a_m x_2^m|) \\
\leq c^k |a_k| + |a_{k+1}| c^{k+1} \epsilon^{1/k} + \ldots + |a_m| c^{m} \epsilon^{(m-k)/k} \leq 1/3
\]
and so for $(u_1, u_2, u_3) \in Q$ we get $|x_1 u_1 + \epsilon^{-1/k} x_2 u_2 + \epsilon^{-1} \varphi(x_1, x_2) u_3| \leq 1 < \pi/3$. Thus $\epsilon^{q+k/\|f_\epsilon\|_{L^q(\Sigma_{D_\epsilon,\varphi})}} \geq c' \epsilon^{1/k}$ with $c'$ independent of $\epsilon$ and $f$. Now,
\[ \|f_\epsilon\|_{L^p(\mathbb{R}^3)} = \epsilon^{-(1+1/k)/p}. \]
So from the inequality $\|\hat{f}_\epsilon\|_{L^q(\Sigma)} \leq c\|f_\epsilon\|_{L^p(\mathbb{R}^3)}$, applied with $\epsilon$ small enough, we get
\[ -\left(1 + \frac{1}{k}\right) + \frac{1}{k} \frac{1}{q} \geq -\left(1 + \frac{1}{k}\right) \frac{1}{p} \]
and the lemma follows. ■
Remark 2.8. It is known that if \( \int_0^1 \left| \hat{h}(x_2, x_2^2) \right|^q dx \) \( \leq c \| h \|_{L^p(\mathbb{R}^2)} \) for some \( c > 0 \) and all \( h \in S(\mathbb{R}^2) \) then \( 1/p > 3/4 \) (see [4, Theorem 2]). This result implies the following.

Let \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a quadratic homogeneous polynomial function such that \( \det \varphi''(x) \equiv 0 \), and let \( \Sigma \) and \( E \) be defined as in the introduction. If \( 1 \leq p, q \leq \infty \) and there exists \( c > 0 \) such that
\[
(2.6) \quad \| \mathcal{F} \|_{L^q(\Sigma)} \leq c \| f \|_{L^p(\mathbb{R}^3)}
\]
for all \( f \in S(\mathbb{R}^3) \), then \( 1/p > 3/4 \). Indeed, from Remark 2.5 we have \( \varphi(x_1, x_2) = (ax_1 + bx_2)^2 \) for some \( a, b \in \mathbb{R} \) and all \( (x_1, x_2) \in \mathbb{R}^2 \). So, from Remark 2.1(b) the problem reduces (after composing with a suitable rotation followed by a dilation) to the case \( \varphi(x_1, x_2) = x_2^2 \). Let \( g \in S(\mathbb{R}) \) be such that \( g > 0 \) on \([0, 1]\). For \( h \in S(\mathbb{R}^2) \) we take \( f(x_1, x_2, x_3) = g(x_1)h(x_2, x_3) \) in (2.6) to obtain
\[
\| g \|_{L^q(0,1)} \left( \int_0^1 \left| \hat{h}(x_2, x_2^2) \right|^q dx \right)^{1/q} \leq c \| g \|_{L^p(\mathbb{R})} \| h \|_{L^p(\mathbb{R}^2)}
\]
and so \( \left( \int_0^1 \left| \hat{h}(x_2, x_2^2) \right|^q dx \right)^{1/q} \leq c \| h \|_{L^p(\mathbb{R}^2)} \) for some \( c > 0 \) and all \( h \in S(\mathbb{R}^2) \).

3. \( L^p(\mathbb{R}^3) - L^q(\Sigma) \) estimates for \( \mathcal{R} \). For \( \delta > 0 \) we set
\[
(3.1) \quad V_\delta = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq \delta |x_1|\}.
\]

Lemma 3.1. Let \( \varphi : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a homogeneous polynomial function such that \( \det \varphi''(x) \) does not vanish identically, let \( \alpha \) be defined as in the preliminaries and let \( m^* = \max(m, \alpha + 2) \). Let \( V_\delta \) be defined by (3.1). Then for \( \delta \) positive and small enough:

(i) if \( m^* < 6 \),
\[
\frac{3}{4} \leq \frac{1}{p} \leq 1 \quad \text{and} \quad - \left( \frac{m^*}{2} + 1 \right) \frac{1}{p} + \frac{m^*}{2} + 1 < \frac{1}{q} \leq 1,
\]
then \( (1/p, 1/q) \in EV_{\delta, \varphi} \),

(ii) if \( m^* \geq 6 \) and
\[
- \left( \frac{m^*}{2} + 1 \right) \frac{1}{p} + \frac{m^*}{2} + 1 < \frac{1}{q} \leq 1,
\]
then \( (1/p, 1/q) \in EV_{\delta, \varphi} \).

Proof. From Remark 2.1(a), to prove the lemma it suffices to show that, for \( \delta \) positive and small enough, the following assertions hold:

(i') If \( m - 2 \leq \alpha < 4 \) and \( (\alpha + 4)/8 < 1/q \leq 1 \) then \((3/4, 1/q) \in EV_{\delta, \varphi}\).
(ii') If $m - 2 \leq \alpha$, $\alpha \geq 4$ and $(\alpha + 2)/(\alpha + 4) < 1/p \leq 1$ then $(1/p, 1) \in E_{V_{b, r}}$.

(iii') If $0 \leq \alpha < m - 2$, $m \geq 6$, and $m/(m + 2) < 1/p \leq 1$ then $(1/p, 1) \in E_{V_{b, r}}$.

(iv') If $0 \leq \alpha < m - 2$, $m < 6$ and $(m + 2)/8 < 1/q \leq 1$ then $(3/4, 1/q) \in E_{V_{b, r}}$.

Let $\delta_0 > 0$ be such that $\det \varphi''(x) \neq 0$ for all $x = (x_1, x_2) \in V_{\delta_0}$ with $x_2 \neq 0$. Our assumptions imply that there exist positive constants $c_1, c_2$ such that, if $(x_1, x_2) \in V_{\delta_0}$ and $1/2 \leq |x_1| \leq 1$, then
\[
(3.2) \quad c_1|x_2|^\alpha \leq |\det \varphi''(x_1, x_2)| \leq c_2|x_2|^\alpha.
\]

For $k \in \mathbb{N} \cup \{0\}$, let
\[
(3.3) \quad Q_k = \{(x_1, x_2) \in \mathbb{R}^2 : 1/2 \leq |x_1| \leq 1, 2^{-k-1} \leq |x_2| \leq 2^{-k}\},
\]
let $\varphi_k : Q_0 \to \mathbb{R}$ be defined by $\varphi_k(x_1, x_2) = \varphi(x_1, 2^{-k}x_2)$ and let $\sigma_{Q_k, r}$ be defined as at the beginning of the preliminaries. A change of variable gives
\[
(3.4) \quad (\sigma_{Q_k, r})^\wedge(\xi) = 2^{-k}(\sigma_{Q_0, r})^\wedge(\xi_1, 2^{-k}\xi_2, \xi_3).
\]

Pick $k(\delta_0) \in \mathbb{N}$ such that $Q_{k(\delta_0)} \subset V_{\delta_0}$. Since
\[
|\det \varphi''(x_1, x_2)| = 2^{-2k}|\det \varphi''(x_1, 2^{-k}x_2)|,
\]
from (3.2) it follows that there exists $c_1 > 0$ such that $|\det \varphi''(x_1, x_2)| \geq c_12^{-k(\alpha+2)}$ for all $k \geq k(\delta_0)$, $(x_1, x_2) \in Q_0$. Then Proposition 6 on p. 344 of [8] implies that there exists a positive constant $c_3$ such that
\[
|(\sigma_{Q_0, r})^\wedge(\xi_1, 2^{-k}\xi_2, \xi_3)| \leq c_32^{k(\alpha+2)/2}|(\xi_1, 2^{-k}\xi_2, \xi_3)|^{-1} \leq c_32^{k(\alpha+2)/2}|\xi_3|^{-1}
\]
for all $k \geq k(\delta_0)$ and $\xi \in \mathbb{R}^3$. For these $k$, from (3.4) we obtain $|(\sigma_{Q_k, r})^\wedge(\xi)| \leq c_32^{k/2}|\xi_3|^{-1}$. So, Remark 2.6 implies that
\[
(3.5) \quad \|\hat{f}\|_{L^2(\Sigma_{Q_k, r})} \leq c_42^{k\alpha/8}\|f\|_{L^4(\mathbb{R}^3)}, \quad f \in S(\mathbb{R}^3),
\]
with $c_4$ independent of $k$ and $f$. From (3.5), Hölder’s inequality gives, for $1 \leq q < 2$ and $f \in S(\mathbb{R}^3),
\[
(3.6) \quad \|\hat{f}\|_{L^q(\Sigma_{Q_k, r})} \leq \sigma(\Sigma_{Q_k, r})^{(2-q)/(2q)}\|\hat{f}\|_{L^2(\Sigma_{Q_k, r})} \leq c_42^{k(\alpha/8-(2-q)/(2q))}\|f\|_{L^4(\mathbb{R}^3)}.
\]

Suppose that $\alpha < 4$. If $(\alpha + 4)/8 < 1/q \leq 1$ then $\alpha/8 - (2 - q)/(2q) < 0$ and so for some $c > 0$ and all $f \in S(\mathbb{R}^3),
\[
(3.7) \quad \sum_{k \geq k(\delta_0)} \|\hat{f}\|_{L^q(\Sigma_{Q_k, r})} \leq c\|f\|_{L^4(\mathbb{R}^3)}.
\]

For $\delta > 0$ and $j \in \mathbb{N} \cup \{0\}$, let
\[
(3.8) \quad A_{j, \delta} = \{(x_1, x_2) \in \mathbb{R}^2 : 2^{-j-1} \leq |x_1| \leq 2^{-j}, |x_2| \leq \delta|x_1|\},
\]
thus $V_\delta = \bigcup_{j \geq 0} A_{j, \delta}$. For $\delta$ small enough, (3.7) gives
\[ \| \hat{f} \|_{L^q(S_{\Lambda_0, \delta, \varphi})} \leq c \| f \|_{L^{4/3}(\mathbb{R}^3)}. \]
If $m - 2 \leq \alpha < 4$ the condition $1/q > (\alpha + 4)/8$ implies
\[ \frac{1}{q} > -\left( \frac{m}{2} + 1 \right) \frac{3}{4} + \frac{m}{2} + 1 \]
and so (i') follows from Remark 2.1(d).

For $t \in [0, 1]$ let $p_t$ be defined by $1/p_t = 3t/4 + 1 - t$. For $k \geq k(\delta_0)$ and any $\alpha$, from (3.6) we get $\| \hat{f} \|_{L^1(\Sigma_{Q_k, \varphi})} \leq c_4 2^{k(\alpha/8 - 1/2)} \| f \|_{L^{4/3}(\mathbb{R}^3)}$. Also $\| \hat{f} \|_{L^1(\Sigma_{Q_k, \varphi})} \leq c_4 2^{-k} \| f \|_{L^1(\mathbb{R}^3)}$, so an application of the Riesz–Thorin theorem gives
\[ \| \hat{f} \|_{L^1(\Sigma_{Q_k, \varphi})} \leq c 2^{k((\alpha+4)/8)t-1)} \| f \|_{L^{p_t}(\mathbb{R}^3)} \]
for all $f \in S(\mathbb{R}^3)$. So, for $\delta$ small enough,
\[ (3.9) \quad \| \hat{f} \|_{L^1(\Sigma_{A_0, \delta, \varphi})} \leq \sum_{k \geq k(\delta_0)} \| \hat{f} \|_{L^1(\Sigma_{Q_k, \varphi})} \leq c \| f \|_{L^{p_t}(\mathbb{R}^3)} \]
for all $t \in [0, 8/(\alpha + 4))$ if $\alpha \geq 4$, and for all $t \in [0, 1]$ if $\alpha < 4$.

Suppose that $m - 2 \leq \alpha < \infty$ and $\alpha \geq 4$. If $(\alpha + 2)/(\alpha + 4) < 1/p \leq 1$ then $1/p = 3t/4 + 1 - t$ for some $t \in [0, 8/(\alpha + 4))$. Also
\[ 1 > -\left( \frac{m}{2} + 1 \right) \frac{1}{p} + \frac{m}{2} + 1. \]
So, Remark 2.1(d) and (3.9) imply $(1/p, 1) \in EV_{\delta, \varphi}$ and then (ii') holds.

Consider now the case $0 \leq \alpha < m - 2$, $m \geq 6$ and suppose $m/(m + 2) < 1/p \leq 1$. A computation shows that $1/p = 1/p_t$ for some $t \in [0, 8/(m + 2))$ and so $t < 8/(\alpha + 4)$. If $\alpha \geq 4$ then (3.9) and Remark 2.1(d) imply (iii') in this case. If $\alpha < 4$, observe that the assumption on $p$ implies $3/4 < 1/p$ and so (3.9) gives $(1/p, 1) \in E_{A_0, \delta, \varphi}$. Since also
\[ 1 > -\left( \frac{m}{2} + 1 \right) \frac{1}{p} + \frac{m}{2} + 1, \]
(iii') follows in this case from Remark 2.1(d).

Finally, assume that $0 \leq \alpha < m - 2$, $m < 6$ and $(m + 2)/8 < 1/q \leq 1$. Then $(\alpha + 4)/8 < 1$, thus (3.7) gives $(3/4, 1/q) \in E_{A_0, \delta, \varphi}$. Also $(m + 2)/8 < 1/q$ implies $1/q > -(m/2 + 1)3/4 + m/2 + 1$ and so Remark 2.1(d) gives (iv').

**Remark 3.2.** For $\delta > 0$ let
\[ W_\delta = \{ x = (x_1, x_2) \in \mathbb{R}^2 : |x| \leq 1, |x_2| \leq \delta |x_1| \} \]
and let $V_\delta$ be defined by (3.1). Since $W_\delta \subset V_\delta$, it follows that Lemma 3.1 holds for $W_\delta$ in place of $V_\delta$.■
Theorem 3.3. Let \( \varphi : \mathbb{R}^2 \to \mathbb{R} \) be a homogeneous polynomial function of degree \( m \geq 2 \) such that \( \text{det} \, \varphi''(x) \equiv 0 \). Then for \( m \geq 3 \),
\[
E^\circ = \{(1/p, 1/q) \in [0, 1] \times [0, 1] : 1/q > -(m + 1)/p + m + 1\},
\]
and for \( m = 2 \),
\[
E^\circ = \{(1/p, 1/q) \in (3/4, 1] \times [0, 1] : 1/q > -3/p + 3\}.
\]

Proof. From Remark 2.5 we have \( \varphi(x_1, x_2) = (ax_1 + bx_2)^m \) for some \( a, b \in \mathbb{R} \) and all \( (x_1, x_2) \in \mathbb{R}^2 \), and so, by Remark 2.1(b), the problem reduces (after composing with a suitable rotation followed by a dilation) to the case \( \varphi(x_1, x_2) = x_2^m \). From Remark 2.1(a), Lemma 2.7 and Remark 2.8, it suffices to see that for \( m \geq 3 \), if \( m/(m + 1) < 1/p \leq 1 \) then \( (1/p, 1) \in E \), and for \( m = 2 \), if \( 3/4 < 1/p \leq 1 \) and \( 1/q > -3/p + 3 \) then \( (1/p, 1/q) \in E \). For \( 3/4 < 1/p \leq 1 \) we know that (see e.g. [3]) there exists \( c > 0 \) such that
\[
(3.10) \quad \left( \int_{1/2 \leq |s| \leq 1} |\hat{g}(s, s^m)|^p \, ds \right)^{1/p} \leq c_p \|g\|_{L^p(\mathbb{R}^2)} \quad \text{for all } g \in S(\mathbb{R}^2).
\]

We claim that for such \( p \) there exists \( c' > 0 \) such that
\[
(3.11) \quad \left( \int_{|x_1| \leq 1} \int_{1/2 \leq |x_2| \leq 1} |\hat{f}(x_1, x_2, x_2^m)|^p \, dx_1 \, dx_2 \right)^{1/p} \leq c' \|f\|_{L^p(\mathbb{R}^3)}
\]
for all \( f \in S(\mathbb{R}^3) \). Indeed, for \( h : \mathbb{R} \to \mathbb{C} \) and \( g : \mathbb{R}^2 \to \mathbb{C} \), let \( h^{\wedge 1}, g^{\wedge 2} \) denote their one-and two-dimensional Fourier transforms respectively. Now,
\[
\left( \int_{|x_1| \leq 1} \int_{1/2 \leq |x_2| \leq 1} |\hat{f}(x_1, x_2, x_2^m)|^p \, dx_1 \, dx_2 \right)^{1/p}
\]
\[
= \left\| \left( \int_{1/2 \leq |x_2| \leq 1} |(\xi_2, \xi_3) \mapsto f(\cdot, \xi_2, \xi_3)^{\wedge 1}(x_1))^{\wedge 2}(x_2, x_2^m)|^p \, dx_2 \right)^{1/p} \right\|_X
\]
where \( X = L^p((-1, 1), dx_1) \). From (3.10) we get
\[
(3.12) \quad \left( \int_{|x_1| \leq 1} \int_{1/2 \leq |x_2| \leq 1} |\hat{f}(x_1, x_2, x_2^m)|^p \, dx_1 \, dx_2 \right)^{1/p}
\]
\[
\leq c \left( \int_{\mathbb{R}^2} \left( \int_{|x_1| \leq 1} |f(\cdot, \xi_2, \xi_3)^{\wedge 1}(x_1)|^p \, dx_1 \right) d\xi_2 \, d\xi_3 \right)^{1/p}.
\]
Since \( p < 2 \),
\[
\|f(\cdot, \xi_2, \xi_3)^{\wedge 1}\|_{L^p((-1, 1))} \leq c'' \|f(\cdot, \xi_2, \xi_3)^{\wedge 1}\|_{L^{p'}((-1, 1))} \leq c''' \|f(\cdot, \xi_2, \xi_3)\|_{L^p(\mathbb{R})}
\]
for some positive \( c'' \) and \( c''' \). So (3.11) follows.

For \( t > 0, x = (x_1, x_2) \in \mathbb{R}^2, \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \) let \( t \cdot x = (x_1, tx_2) \) and \( t \circ \xi = (\xi_1, t\xi_2, t^m \xi_3) \). For \( g : \mathbb{R}^3 \to \mathbb{R}, t > 0 \) let \( t \circ g : \mathbb{R}^3 \to \mathbb{R} \) be defined by
\((t \circ g)(\xi) = g(t \circ \xi)\). Finally, for \(k \in \mathbb{N} \cup \{0\}\) let 
\[
R_k = \{(x_1, x_2) : |x_1| \leq 1, 2^{-k-1} \leq |x_2| \leq 2^{-k}\}.
\]
So \(R_k = 2^{-k}R_0\) and from (3.11) a standard homogeneity argument gives 
\[
(3.13) \quad \|\hat{f}\|_{L^p(\Sigma_{R_k,\varphi})} \leq c2^{-k/p+k(m+1)}\|2^k \circ f\|_{L^p(\mathbb{R}^3)} = 2^{-k/p+k(m+1)(1-1/p)}\|f\|_{L^p(\mathbb{R}^3)}
\]
and so, by H"older’s inequality,
\[
(3.14) \quad \|\hat{f}\|_{L^1(\Sigma_{R_k,\varphi})} \leq 2^{-k/p+k(m+1)(1-1/p)}|R_k|^{1-1/p}\|f\|_{L^p(\mathbb{R}^3)} = 2^{k(m-(m+1)/p)}\|f\|_{L^p(\mathbb{R}^3)}.
\]
In particular, if \(m \geq 3\) and \(m/(m+1) < 1/p \leq 1\) then \(3/4 < 1/p \leq 1\) and so from (3.14) we obtain
\[
\|\hat{f}\|_{L^1(\Sigma)} \leq \sum_{k \geq 0} \|\hat{f}\|_{L^1(\Sigma_{R_k,\varphi})} \leq c\|f\|_{L^p(\mathbb{R}^3)}
\]
and the theorem follows for the case \(m \geq 3\).

Consider now the case \(m = 2\). Suppose that \(3/4 < 1/p \leq 1\); then \(-1/p + 3(1 - 1/p) < 0\) and so from (3.13),
\[
\|\hat{f}\|_{L^p(\Sigma)} \leq \sum_{k \geq 0} \|\hat{f}\|_{L^p(\Sigma_{R_k,\varphi})} \leq c\|f\|_{L^p(\mathbb{R}^3)}.
\]
Also, H"older’s inequality gives \(\|\hat{f}\|_{L^q(\Sigma)} \leq c\|f\|_{L^p(\mathbb{R}^3)}\) for \(1/p \leq 1/q \leq 1\). So the theorem follows from Remark 2.1(a). \(\blacksquare\)

**Theorem 3.4.** Let \(\varphi : \mathbb{R}^2 \to \mathbb{R}\) be a homogeneous polynomial function of degree \(m \geq 2\) such that \(\det \varphi''(x)\) does not vanish identically, and let \(\tilde{m}\) be defined by (1.1) if \(\det \varphi''(x)\) vanishes somewhere in \(\mathbb{R}^2 \setminus \{0\}\), and by \(\tilde{m} = m\) if not. Then

(i) for \(\tilde{m} \geq 6\),
\[
E^0 = \left\{(1/p, 1/q) \in [0, 1] \times [0, 1] : \frac{1}{q} > -\left(\frac{\tilde{m}}{2} + 1\right)\frac{1}{p} + \frac{\tilde{m}}{2} + 1\right\},
\]
(ii) for \(\tilde{m} < 6\),
\[
E^0 \cap ((3/4, 1] \times [0, 1])
\]
\[
= \left\{(1/p, 1/q) \in (3/4, 1] \times [0, 1] : \frac{1}{q} > -\left(\frac{\tilde{m}}{2} + 1\right)\frac{1}{p} + \frac{\tilde{m}}{2} + 1\right\}
\]
and also \((3/4, 1/q) \in E\) for \((\tilde{m} + 2)/8 < 1/q \leq 1\).

**Proof.** To see that the stated conditions are sufficient, we consider first the case \(\det \varphi''(x) \neq 0\) for all \(x \in \mathbb{R}^2 \setminus \{0\}\). For \(j \in \mathbb{N} \cup \{0\}\) let
\[
A_j = \{x \in \mathbb{R}^2 : 2^{-j-1} \leq |x| \leq 2^j\}.
\]
From [8, p. 386] we have \((3/4, 1/2) \in E_{A_j, \varphi}\) and so in this case the lemma follows from Remark 2.1(d).

Suppose now that \(\det \varphi''(x) = 0\) for some \(x \in \mathbb{R}^2 \setminus \{0\}\). Let \(L_1, \ldots, L_k\) be as in the introduction. For \(\delta > 0\) let

\[ C_\delta^j = \{x \in B : |\pi_{L_j^\perp}(x)| \leq \delta |\pi_{L_j}(x)|\} \]

where \(\pi_{L_j}\) and \(\pi_{L_j^\perp}\) denote the orthogonal projections from \(\mathbb{R}^2\) onto \(L_j\) and \(L_j^\perp\) respectively. Choose \(\delta\) small enough such that \(C_i^j \cap C_j^j = \emptyset\) for \(i \neq j\). So \(\det \varphi''(x) \neq 0\) for \(x \in C_\delta^j - L_j\). Let \(T_j \in \text{SO}(2)\) be such that \(T_j(L_j)\) is the \(x_1\) axis, let \(\psi_j = \varphi \circ T_j^{-1}\), \(\tilde{C}_\delta^j = T_j(C_\delta^j)\) and let \(\tilde{\alpha}_j\) be the vanishing order of \(x_2 \mapsto \det \varphi''(1, x_2)\). Since the curvature is invariant under rotations we have \(\tilde{\alpha}_j = \alpha_j, j = 1, \ldots, k\). We also have \(E_{C_\delta^j, \varphi} = E_{T_j(C_\delta^j), \varphi \circ T_j^{-1}}\) (see Remark 2.1(b)). Let \(C_\delta = \bigcup_{1 \leq j \leq k} C_\delta^j\), so \(E_{C_\delta, \varphi} = \bigcap_{1 \leq j \leq k} E_{C_\delta^j, \varphi}\).

Suppose that \((1/p, 1/q)\) belongs to the left side of the equality in either (i) or (ii). If \(\tilde{m} = m\), Remark 2.1(d) applies. If \(\tilde{m} = \alpha_j + 2\) for some \(j = 1, \ldots, k\), we apply Lemmas 2.4 and 2.7 to \(\varphi \circ T_j^{-1}\) to deduce that \((1/p, 1/q)\) belongs to the right side of the equality.

On the other hand, from Lemma 3.1 and Remark 3.2 it follows that, for \(\delta\) positive and small enough, if \(\tilde{m} < 6\),

\[
\frac{3}{4} \leq \frac{1}{p} \leq 1 \quad \text{and} \quad -\frac{\tilde{m} + 2}{2p} + \frac{\tilde{m} + 2}{2} < \frac{1}{q} \leq 1
\]

then \((1/p, 1/q) \in E_{C_\delta, \varphi}\), and if \(\tilde{m} \geq 6\),

\[
\frac{\tilde{m}}{\tilde{m} + 2} < \frac{1}{p} \leq 1 \quad \text{and} \quad -\frac{\tilde{m} + 2}{2p} + \frac{\tilde{m} + 2}{2} < \frac{1}{q} \leq 1
\]

then \((1/p, 1/q) \in E_{C_\delta, \varphi}\). Finally, let \(D_\delta = B - C_\delta\). So \(D_\delta\) is a union of a finite number of angular sectors with vertices at the origin where \(\det \varphi''(x)\) never vanishes (except at the origin), and so we can proceed as in the first part of the proof to get \((1/p, 1/q) \in E_{D_\delta, \varphi}\) for

\[
\frac{3}{4} \leq \frac{1}{p} \leq 1 \quad \text{and} \quad -\frac{\tilde{m} + 2}{2p} + \frac{\tilde{m} + 2}{2} < \frac{1}{q} \leq 1.
\]

Since \(E = E_{C_\delta, \varphi} \cap E_{D_\delta, \varphi}\) the theorem follows. \(\blacksquare\)

4. Sharp \(L^p(\mathbb{R}^3)\)-\(L^2(\Sigma)\) estimates for \(\mathcal{R}\)

Remark 4.1. For our next results we will need to introduce two Littlewood–Paley decompositions on \(S(\mathbb{R}^3)\). Let \(\Phi \in C_c^\infty(\mathbb{R})\) be an even function satisfying \(0 \leq \Phi \leq 1\), \(\text{supp} \Phi \subset \{t \in \mathbb{R} : 2^{-1} \leq |t| \leq 2\}\) and such that \(\sum_{r \in \mathbb{Z}} \Phi(2^r t) = 1\) if \(t \neq 0\). For \(r \in \mathbb{Z}\) let \(\Psi_r : \mathbb{R} \to \mathbb{R}\) be defined by \(\Psi_r(t) = 2^{-r} \Phi(2^{-r} t)\), and for \(f \in S(\mathbb{R}^3)\) let \(T_r f\) be given by \(T_r f = (\delta \otimes \Psi_r \otimes \delta) * f\).
Thus $f = \sum_{r \in \mathbb{Z}} T_r f$ with convergence in $S'({\mathbb{R}^3})$. Moreover, it can be checked that for $\varepsilon_r = \pm1$, the one-dimensional multiplier $\sum_{r \in \mathbb{Z}} \varepsilon_r \Phi(2^r t)$ satisfies the hypothesis of Theorem 3 on p. 96 of [7] with constants independent of the choice of $\varepsilon_r$. Hence for $f \in S({\mathbb{R}^3})$, $\| \sum_{r \in \mathbb{Z}} \varepsilon_r T_r f \|_{L^p({\mathbb{R}^3})} \leq c \| f \|_{L^p({\mathbb{R}^3})}$ with $c$ independent of $f$ and of the choice of $\varepsilon_r$, so as in [7, p. 105] we have the Littlewood–Paley inequality

\begin{equation}
(4.1) \quad \left\| \left( \sum_{r \in \mathbb{Z}} |T_r f|^2 \right)^{1/2} \right\|_{L^p({\mathbb{R}^3})} \leq c \| f \|_{L^p({\mathbb{R}^3})}.
\end{equation}

Similarly, if we start with an even function $\tilde{\Phi} \in C_c^\infty({\mathbb{R}^2})$ with support contained in the annulus $\{ t \in {\mathbb{R}^2} : 2^{-1} \leq |t| \leq 2 \}$ such that $0 \leq \tilde{\Phi} \leq 1$, $\sum_{r \in \mathbb{Z}} \tilde{\Phi}(2^r t) = 1$ for $t \in {\mathbb{R}^2} - \{0\}$, and if we define $\tilde{\Psi}_r(t) = 2^{2r} \tilde{\Phi}(2^{-r} t)$, $r \in \mathbb{Z}$, and $\tilde{T}_r f = (\tilde{\Psi}_r \otimes \delta) * f$, we now see that $f = \sum_{r \in \mathbb{Z}} \tilde{T}_r f$, $f \in S({\mathbb{R}^3})$, with convergence in $S'({\mathbb{R}^3})$ and that (4.1) is also true for the family $\{ \tilde{T}_r \}_{r \in \mathbb{Z}}$ in place of $\{ T_r \}_{r \in \mathbb{Z}}$. $\blacksquare$

**Theorem 4.2.** Let $\phi : {\mathbb{R}^2} \to {\mathbb{R}}$ be a homogeneous polynomial function of degree $m \geq 2$ such that $\det \phi''(x) \equiv 0$. Then $((2m+1)/(2m+2), 1/2) \in E$.

**Proof.** As in Theorem 3.3 it is enough to prove the theorem for the case $\phi(x) = x_2^m$. In this case the van der Corput lemma applied with the $m$th derivative gives

$$\left| \int_{-1}^1 e^{-i(y \xi_2 + y^m \xi_3)} dy \right| \leq c |\xi_3|^{-1/m},$$

so $|\sigma^\wedge(\xi)| \leq c(1 + |\xi_3|)^{-1/m}$. Thus the theorem follows from Remark 2.6. $\blacksquare$

**Lemma 4.3.** Let $\phi : {\mathbb{R}^2} \to {\mathbb{R}}$ be a homogeneous polynomial function such that $\det \phi''(x)$ is not identically zero, let $m^*$ be as in Lemma 3.1 and for $\delta > 0$ let $V_\delta$ be defined by (3.1). Then for $\delta$ positive and small enough, $((m^*+1)/(m^*+2), 1/2) \in E_{V_\delta, \phi}$.

**Proof.** We first consider the case $\alpha < m - 2$. Then $m^* = m$. Let $\delta_0$ and $k(\delta_0)$ be as in the proof of Lemma 3.1 and for $k \in \mathbb{Z} \cup \{0\}$ let $Q_k$ be defined by (3.3). For $\theta \in [0, 1]$ let $p_\theta$ be defined by $1/p_\theta = 3\theta/4 + 1 - \theta$. For $f \in S({\mathbb{R}^3})$, from (3.5) and $\| f \|_{L^2({\mathbb{S}^k})} \leq c 2^{-k/2} \| f \|_{L^1({\mathbb{R}^3})}$ (see Remark 2.1(a)), the Riesz–Thorin theorem gives

\begin{equation}
(4.2) \quad \| \widehat{f} \|_{L^2({\mathbb{S}^k})} \leq c 2^{k(\alpha - 4\theta/8 - k(1-\theta)/2)} \| f \|_{L^{p_\theta}({\mathbb{R}^3})}.
\end{equation}

Then for $\delta$ positive and small enough we get

$$\| \widehat{f} \|_{L^2({\mathbb{S}^k})} \leq \sum_{k \geq k(\delta_0)} \| \widehat{f} \|_{L^2({\mathbb{S}^k})} \leq c \| f \|_{L^{m+2}/m+1}({\mathbb{R}^3})$$
with $A_{0,\delta}$ defined by (3.8). Thus, from Remark 2.1(c) we find
\[(4.3) \quad \| \hat{f} \|^2_{L^2(\Sigma_{A_{j,\delta},\varphi})} \leq c \| f \|^2_{L^{(m+2)/(m+1)}(\mathbb{R}^3)}
\]
for some $c > 0$ and all $j \in \mathbb{Z}$, $f \in S(\mathbb{R}^3)$.

For $r \in \mathbb{Z}$ let $\tilde{T}_r$, $\tilde{\Psi}_r$ be as in Remark 4.1. Then for $f \in S(\mathbb{R}^3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ such that $(\xi_1, \xi_2) \neq 0$ we have
\[\hat{f}(\xi) = \sum_{r \in \mathbb{Z}} \tilde{\Psi}_r(\xi_1, \xi_2) \hat{f}(\xi) = \sum_{r \in \mathbb{Z}} (\tilde{T}_r f)^\wedge(\xi)\]
with convergence in $L^2(\mathbb{R}^3)$. Also, $(\tilde{T}_r f)^\wedge(\xi) = \tilde{\Phi}(2^r (\xi_1, \xi_2)) \hat{f}(\xi)$ and so, for each $\xi$, the set $\{r \in \mathbb{Z} : (\tilde{T}_r f)^\wedge(\xi) \neq 0\}$ has at most three elements. Then
\[\left| \sum_{r \in \mathbb{Z}} (\tilde{T}_r f)^\wedge(x, \varphi(x)) \right|^2 \leq 3 \sum_{r \in \mathbb{Z}} \left| (\tilde{T}_r f)^\wedge(x, \varphi(x)) \right|^2.
\]
Let $A'_{r, \delta} = \bigcup_{j=r-1}^{r+1} A_{j,\delta}$. From (4.3) we have
\[\int_{A'_{r, \delta}} \left| (\tilde{T}_r f)^\wedge(x, \varphi(x)) \right|^2 dx \leq c \| \tilde{T}_r f \|^2_{L^{(m+2)/(m+1)}(\mathbb{R}^3)}
\]
with $c$ independent of $f$ and $r$. Also, if $0 < \delta < 1$ then
\[\{x \in V_{\delta} : (\tilde{T}_r f)^\wedge(x, \varphi(x)) \neq 0\} \subset A'_{r, \delta}.
\]
From these facts we deduce that there exist positive constants $c$, $c'$ and $c''$ independent of $f$ such that
\[\left( \sum_{r \in \mathbb{Z}} \left| (\tilde{T}_r f)^\wedge(x, \varphi(x)) \right|^2 \right)^{1/2} = \int_{V_{\delta}} \left| \sum_{r \in \mathbb{Z}} (\tilde{T}_r f)^\wedge(x, \varphi(x)) \right|^2 dx
\]
\[\leq 3 \sum_{r \in \mathbb{Z}} \int_{A'_{r, \delta}} \left| (\tilde{T}_r f)^\wedge(x, \varphi(x)) \right|^2 dx \leq c \sum_{r \in \mathbb{Z}} \| \tilde{T}_r f \|^2_{L^{(m+2)/(m+1)}(\mathbb{R}^3)}
\]
\[\leq c' \left( \sum_{r \in \mathbb{Z}} \| \tilde{T}_r f \|^2 \right)^{1/2} \leq c'' \| f \|^2_{L^{(m+2)/(m+1)}(\mathbb{R}^3)}
\]
where the last inequality follows from the Littlewood–Paley inequality and the previous one from Minkowski’s inequality. Thus the lemma follows for $\alpha < m - 2$.

If $\alpha \geq m - 2$ then $m^* = \alpha + 2$ and so from (4.2) we obtain $\| \hat{f} \|^2_{L^2(\Sigma_{Q_{k,\varphi}})} \leq c \| f \|^2_{L^{(m+2)/(m+1)}(\mathbb{R}^3)}$ for some $c > 0$, for all $k \geq k(\delta_0)$ and $f \in S(\mathbb{R}^3)$.

For $r \in \mathbb{Z}$ let $T_r$, $\Psi_r$ be as in Remark 4.1. Then for $f \in S(\mathbb{R}^3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$ such that $\xi_2 \neq 0$ we have
\[\hat{f}(\xi) = \sum_{r \in \mathbb{Z}} \tilde{\Psi}_r(\xi_2) \hat{f}(\xi) = \sum_{r \in \mathbb{Z}} (T_r f)^\wedge(\xi)\]
with convergence in $L^2(\mathbb{R}^3)$. Now we proceed as in the first part of the proof, but with $\tilde{T}_r$ replaced by $T_r$, to obtain
\[
\|\hat{f}\|_{L^2(\Sigma_{A_0, \delta}, \varphi)} \leq c \|f\|_{L^{(m^*+2)/(m^*+1)}(\mathbb{R}^3)}.
\]
From Remark 2.1(c) we get
\[
(4.4) \quad \|\hat{f}\|_{L^2(\Sigma_{A_j, \delta}, \varphi)} \leq c 2^{-j[1+(m+2)(m^*+1)/(m^*+2) - (m+2)]} \|f\|_{L^{(m^*+2)/(m^*+1)}(\mathbb{R}^3)}
\]
\[
= c 2^{-j(m^* - m)/(m^*+2)} \|f\|_{L^{(m^*+2)/(m^*+1)}(\mathbb{R}^3)}.
\]
If $\alpha > m - 2$ then $m^* > m$, so we can perform the sum on $j$ to obtain the conclusion.

Finally, for $\alpha = m - 2$ the estimates in (4) are uniform on $j$ and then we proceed as in the case $\alpha < m - 2$ to get the assertion. ■

From this lemma we proceed as in the proof of Theorem 3.4 to obtain the following

**Theorem 4.4.** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial function such that $\det \varphi''(x)$ is not identically zero, and let $\tilde{m}$ be as in Theorem 3.4. Then $((\tilde{m} + 1)/(\tilde{m} + 2), 1/2) \in E$.

**References**


FaMAF, Universidad Nacional de Córdoba
and CIEM-CONICET
Ciudad Universitaria
5000 Córdoba, Argentina
E-mail: eferrey@mate.uncor.edu
godoy@mate.uncor.edu
urciuolo@mate.uncor.edu

Received May 6, 2002