Order convexity and concavity of Lorentz spaces
$\Lambda_{p,w}$, $0 < p < \infty$

by

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Abstract. We study order convexity and concavity of quasi-Banach Lorentz spaces $\Lambda_{p,w}$, where $0 < p < \infty$ and w is a locally integrable positive weight function. We show first that $\Lambda_{p,w}$ contains an order isomorphic copy of l^p. We then present complete criteria for lattice convexity and concavity as well as for upper and lower estimates for $\Lambda_{p,w}$. We conclude with a characterization of the type and cotype of $\Lambda_{p,w}$ in the case when $\Lambda_{p,w}$ is a normable space.

0. Introduction. The purpose of this paper is to characterize order convexity and concavity in quasi-Banach Lorentz spaces $\Lambda_{p,w}$, where w is a locally integrable arbitrary weight and $0 < p < \infty$. First results on this topic in Lorentz spaces belong to Creekmore [7], who has studied the spaces $L_{p,q}$ with $1 < p, q < \infty$, as well as to Carothers [4] and Reisner [24], who considered the Lorentz spaces $\Lambda_{p,w}$ with a decreasing weight w and $p \geq 1$. These spaces have been further investigated by the authors in [14], where convexity and concavity properties as well as the type and cotype of $\Lambda_{p,w}$ have been characterized by means of several equivalent integral inequalities as well as by indices of w and its integral $W(t) = \int_0^t w$. It is well known that $\Lambda_{p,w}$ are Banach spaces whenever w is decreasing and $p \geq 1$ ([18]). The present article is a continuation of [14]. We extend our study to $0 < p < \infty$ and an arbitrary weight w such that $\Lambda_{p,w}$ is a quasi-Banach space. In this general setting, when the weight is not decreasing and $0 < p < \infty$ is arbitrary, different methods and techniques must be used.

The paper is organized as follows. In the preliminaries we set up notations and we recall the definitions, notions and results which will be used later on. Among other results, we recall that $\Lambda_{p,w}$ is a quasi-normed space whenever

W satisfies a growth condition called Δ_2 (see [10]). Moreover, in Theorem A, we present several equivalent conditions for normability of $\Lambda_{p,w}$, that is, for the existence of a norm equivalent to the original quasi-norm of the space ([1, 23, 25]).

The first main result, proved in the second section, states that $\Lambda_{p,w}$, $0 < p < \infty$, contains an order isomorphic copy of l^p. This is an extension of the analogous result for w decreasing and $p \geq 1$ due to Figiel, Johnson and Tzafriri [9]. We then apply this fact in the next sections, proving as corollaries that $\Lambda_{p,w}$ cannot be normable for $0 < p < 1$ and that it cannot be r-convex (resp. r-concave) for $r > p$ (resp. $r < p$).

In Section 3, we present criteria for r-convexity and for an upper r-estimate in $\Lambda_{p,w}$. While these criteria are different for $r \neq p$, they coincide for $r = p$. We show for instance that for $r < p$, $\Lambda_{p,w}$ is r-convex whenever the Hardy operator $H^{(r)}$ is bounded, which is equivalent to the fact that the upper Matuszewska–Orlicz index $\beta(W)$ is strictly less than p/r, that is, for some $\varepsilon > 0$, $W(t)/t^{p/r-\varepsilon}$ is pseudo-decreasing. A consequence of this fact is that $\Lambda_{p,w}$ is L-convex ([11]). For the same $r < p$, $\Lambda_{p,w}$ satisfies an upper r-estimate if and only if $W(t)/t^{p/r}$ is pseudo-decreasing. If $r = p$, then we prove that $\Lambda_{p,w}$ is p-convex if and only if it satisfies an upper p-estimate, which is equivalent to the fact that $W(t)/t$ is pseudo-decreasing.

Section 4 contains criteria for r-concavity and for a lower r-estimate in $\Lambda_{p,w}$. While characterizations of lower r-estimate or r-concavity for $r = p$ are counterparts of the corresponding theorems in Section 3 for convexity, the integral characterization of r-concavity with $r \neq p$ requires more effort. We introduce here a new integral condition called D_p, which plays an analogous role in studying concavity to the role of condition B_p in studying convexity of $\Lambda_{p,w}$. It is well known (Theorem A) that $\Lambda_{p,w}$, $1 < p < \infty$, is 1-convex if and only if w satisfies condition B_p, that is, for some $B > 0$,

$$\int_x^\gamma t^{-p} w(t)\,dt \leq B x^{-p} W(x) \quad \text{for all } x \in I,$$

where $I = (0,1]$ or $I = (0,\infty)$, and $\gamma = 1$ or ∞, respectively. We show (Theorem 5) that $\Lambda_{p,w}$, $0 < p < 1$, is 1-concave if and only if w satisfies condition D_p, that is, there exists $A > 0$ such that

$$\int_0^x t^{-p} w(t)\,dt \leq A x^{-p} W(x) \quad \text{for all } x \in I.$$

We then apply some modifications of this condition to the study of r-concavity of these spaces. We conclude the paper with characterizations of type and cotype in normable $\Lambda_{p,w}$ spaces. Most results contained in this article were announced in the research report [13].

1. Preliminaries. We begin with some introductory definitions and results. Recall first that a *quasi-Banach space* X ([12]) is a complete metrizable real vector space whose topology is given by a quasi-norm $\| \ \|$ satisfying the following three conditions: $\|x\| > 0$, $x \in X$, $x \neq 0$; $\|\lambda x\| = |\lambda| \, \|x\|$, $\lambda \in \mathbb{R}$, $x \in X$; and $\|x_1 + x_2\| \leq C(\|x_1\| + \|x_2\|)$, $x_1, x_2 \in X$, where $C \geq 1$ is a constant independent of x_1 and x_2. Given $0 < p \leq 1$, if $\| \ \|$ satisfies the first two conditions and $\|x_1 + x_2\|^p \leq \|x_1\|^p + \|x_2\|^p$, $x_1, x_2 \in X$, then $\| \ \|$ is called a *p-norm*. A quasi-Banach space $(X, \| \ \|)$ is said to be *p-normable*, $0 < p \leq 1$, if there exists an equivalent p-norm in X. A 1-normable space is simply called *normable*. If a quasi-Banach space X is a vector lattice and $\| \ \|$ is monotone, i.e. $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, we say that X is a *quasi-Banach lattice*. A quasi-Banach lattice X is said to be *p-convex*, $0 < p < \infty$, respectively *q-concave*, $0 < q < \infty$, ([8, 11, 17]) if there is a constant $K > 0$ such that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leq K \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p},$$

respectively

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leq K \left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q} \right\|,$$

for every choice of vectors $x_1, \ldots, x_n \in X$. We also say that X satisfies an *upper p-estimate*, $0 < p < \infty$ (resp. a *lower q-estimate*, $0 < q < \infty$), if the above condition defining p-convexity (resp. q-concavity) is satisfied by elements $x_1, \ldots, x_n \in X$ with disjoint supports ([11, 17]). We easily observe that for $0 < p \leq 1$, p-convexity implies p-normability and this in turn yields an upper p-estimate.

Given $0 < p < \infty$ and a quasi-Banach lattice X let $X^{(p)} = \{x : |x|^p \in X\}$ denote the *p-convexification* of X equipped with the quasi-norm $\|x\|_{X^{(p)}} = \||x|^p\|^{1/p}$. It is clear that $X^{(p)}$ is 1-convex (resp. 1-concave) iff X is $1/p$-convex (resp. $1/p$-concave). Notice also that a quasi-Banach space is normable iff it is 1-convex.

Let further $I = (0, 1]$ or $I = (0, \infty)$, and $L^0 \equiv L^0(I, |\ |)$ be the set of all Lebesgue measurable functions $f : I \to \mathbb{R}_+$, where $|\ |$ denotes the Lebesgue measure on I. By γ we will always denote 1 in the case when $I = (0, 1]$ and ∞ when $I = (0, \infty)$. We also agree that "decreasing" or "increasing" will mean "non-increasing" or "non-decreasing" respectively. For $f \in L^0$ we define its decreasing rearrangement as $f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}$, $t > 0$, where $d_f(s) = |\{t : |f(t)| > s\}|$ is the distribution function of f. Given $0 < r < \infty$, the Hardy operators on L^0 are defined as follows:

$$H^{(r)}f(t) = \left(\frac{1}{t} \int_0^t f^{*r}(s) \, ds \right)^{1/r}, \qquad H_{(r)}f(t) = \left(\frac{1}{t} \int_t^\gamma f^{*r}(s) \, ds \right)^{1/r}.$$

For $r = 1$, $H^{(1)}f$ will be denoted by f^{**} and $H_{(1)}f$ by f_{**}. It is well known ([2, 16]) that $(f+g)^{**} \leq f^{**}+g^{**}$. One can also show that for any $0 < r < \infty$, and $f_1, \ldots, f_n \in L^0$,

$$H^{(r)}\Big(\Big(\sum_{i=1}^{n}|f_i|^r\Big)^{1/r}\Big) \leq \Big(\sum_{i=1}^{n}(H^{(r)}(|f_i|))^r\Big)^{1/r},$$

$$H_{(r)}\Big(\Big(\sum_{i=1}^{n}|f_i|^r\Big)^{1/r}\Big) \geq \Big(\sum_{i=1}^{n}(H_{(r)}(|f_i|))^r\Big)^{1/r}.$$

The *Lorentz space* $\Lambda_{p,w}$, $0 < p < \infty$, is a subspace of L^0 such that

$$\|f\| = \|f\|_{p,w} := \Big(\int_I f^{*p}w\Big)^{1/p} = \Big(\int_I f^{*p}(t)w(t)\,dt\Big)^{1/p} < \infty,$$

where a measurable *weight* function $w : I \to (0, \infty)$ satisfies the conditions

$$W(t) := \int_0^t w < \infty \quad \text{for all } t \in I,$$

$$\int_0^\infty w = \infty \quad \text{in the case when } I = (0, \infty).$$

The functional $\| \ \|_{p,w}$ admits several equivalent formulas ([2, 4, 5, 16]). In fact for $f \in \Lambda_{p,w}$, $0 < p < \infty$,

$$\|f\| = \Big(\int_I f^{*p}w\Big)^{1/p} = \Big(\int_0^\infty W(d_f(t))\,d(t^p)\Big)^{1/p} = \Big(-\int_I W(t)\,d(f^{*p}(t))\Big)^{1/p}.$$

If in addition w is decreasing then

$$\|f\| = \Big(\sup_\tau \int_I |f(\tau(t))|^p w(t)\,dt\Big)^{1/p},$$

and if w is increasing then

$$\|f\| = \Big(\inf_\tau \int_I |f(\tau(t))|^p w(t)\,dt\Big)^{1/p},$$

where both the supremum and infimum are taken over all measure preserving transformations $\tau : I \to I$.

It is well known that the functional $\| \ \|_{p,w}$ is a norm if and only if w is decreasing and $1 \leq p < \infty$ ([18]). Moreover, for any $0 < p < \infty$, $\| \ \|_{p,w}$ is a quasi-norm if and only if W satisfies condition Δ_2, i.e. $W(2t) \leq KW(t)$ for all $t \in \frac{1}{2}I$ and some $K > 0$ ([10]). Indeed, if W satisfies condition Δ_2 then

for any $f, g \in \Lambda_{p,w}$,

$$\|f + g\| \leq 2^{1/p}\left(\left(-\int_I W(t)\, df^{*p}(t/2)\right)^{1/p} + \left(-\int_I W(t)\, dg^{*p}(t/2)\right)^{1/p}\right)$$

$$\leq 2^{1/p}K\left(\left(-\int_I W(t/2)\, df^{*p}(t/2)\right)^{1/p} + \left(-\int_I W(t/2)\, dg^{*p}(t/2)\right)^{1/p}\right)$$

$$\leq C(\|f\| + \|g\|).$$

On the other hand, assuming that $\| \ \|$ is a quasi-norm and choosing $f = \chi_{(0,x/2]}$ and $g = \chi_{(x/2,x]}$ we obtain

$$W^{1/p}(x) = \|\chi_{(0,x]}\| \leq C(\|\chi_{(0,x/2]}\| + \|\chi_{(x/2,x]}\|) = 2CW^{1/p}(x/2)$$

for all $x \in I$, that is, W satisfies condition Δ_2.

Since we deal further with the functional $\| \ \|_{p,w}$ which is at least a quasi-norm, we will assume in what follows that W satisfies condition Δ_2.

Given an arbitrary $F : I \to \mathbb{R}_+$ we define the *Matuszewska–Orlicz lower* and *upper indices* as follows:

$$\alpha(F) = \sup\{p \in \mathbb{R} : F(at) \leq Ca^p F(t)$$
$$\text{for some } C > 0 \text{ and all } t \in I, \ 0 < a \leq 1\},$$

$$\beta(F) = \inf\{p \in \mathbb{R} : F(at) \leq Ca^p F(t)$$
$$\text{for some } C > 0 \text{ and all } at \in I, \ a \geq 1\}.$$

If F is increasing then $0 \leq \alpha(F) \leq \beta(F) \leq \infty$. A function $F : I \to \mathbb{R}_+$ is called *pseudo-increasing* (resp. *pseudo-decreasing*) if there exists a constant $K > 0$ such that $F(s) \leq KF(t)$ (resp. $F(s) \geq KF(t)$) for any $0 < s < t$; $s, t \in I$. As usual two functions $F, G : I \to \mathbb{R}_+$ are said to be *equivalent* if there exist positive constants C_1, C_2 such that $C_1 F(t) \leq G(t) \leq C_2 F(t)$ for every $t \in I$.

Recall also the following result due to Matuszewska and Orlicz [21].

LEMMA A ([21, 16]). *If $F : I \to (0, \infty)$ is pseudo-increasing (resp. pseudo-increasing and F satisfies condition Δ_2) and $F(t)/t$ is pseudo-decreasing (resp. pseudo-increasing) on I, then there exists a positive concave (resp. convex) function on I equivalent to F.*

In the theorem below we summarize all known results characterizing normability of $\Lambda_{p,w}$.

THEOREM A. *Let $1 < p < \infty$. The following conditions are equivalent.*

(i) $\Lambda_{p,w}$ *is normable.*

(ii) *The Hardy operator $f^{**}(t) = t^{-1}\int_0^t f^*(s)\, ds$ is bounded in $\Lambda_{p,w}$.*

(iii) *The weight w satisfies condition S_p, that is, for some $A > 0$ and*

$1/p + 1/p' = 1$,

$$\left(\int\limits_0^x w\right)^{1/p} \left(\int\limits_0^x \left(\frac{1}{t}\int\limits_0^t w\right)^{1-p'} dt\right)^{1/p'} \le Ax \quad \text{for all } x \in I.$$

(iv) *The weight w satisfies condition B_p, that is, there exists $B > 0$ such that*

$$\int\limits_x^\gamma t^{-p} w(t)\,dt \le Bx^{-p}W(x) \quad \text{for all } x \in I.$$

(v) *There is $C > 0$ such that*

$$\int\limits_x^\gamma t^{-p-1}W(t)\,dt \le Cx^{-p}W(x) \quad \text{for all } x \in I.$$

(vi) *$\beta(W) < p$, i.e. there exists $\varepsilon > 0$ such that $W(x)/x^{p-\varepsilon}$ is pseudo-decreasing.*

Notice that Boyd proved equivalence of (ii) and (vi) in [3] when w is decreasing but his techniques work just as well for general w (see also [16, Ch. II, Th. 6.6]). Ariño and Muckenhoupt introduced condition B_p and showed the equivalence of (ii) and (iv) in [1]; Sawyer proved in [25] that conditions (i)–(iv) are equivalent, and Raynaud in [23] demonstrated the equivalence of (ii) and (vi). We add here condition (v), which is an easy consequence of (iv) and integration by parts. A direct proof of the equivalence of (iv) and (vi) together with a simple proof of the fact that B_p implies $B_{p-\varepsilon}$ was given by Maligranda in [20]. The equivalence of (v) and (vi) was proved in [19].

2. Copies of l^p in $\Lambda_{p,w}$. We start by showing that $\Lambda_{p,w}$ contains an order copy of l^p. In the case when w is decreasing and $1 \le p < \infty$, this fact has been proved in [9] (for increasing weight see [5]; see also [6]). Notice that the proof we provide here is different than the one in [9] or [5], because of the lack of monotonicity of w.

THEOREM 1. *Let $0 < p < \infty$. The Lorentz space $\Lambda_{p,w}$ contains an order isomorphic copy of l^p.*

Proof. Let $I = (0,1]$ and $\varepsilon > 0$. We shall construct a sequence $\{f_j\}_{j=1}^\infty \subset \Lambda_{p,w}$ with disjoint supports which spans an isomorphic copy of l^p in $\Lambda_{p,w}$. Let $k_1 = 1$, $\mathbb{N} \ni N_1 > k_1$ and $b_1 > 0$ be such that $b_1^p W(2^{-k_1}) = 1$ and $b_1^p W(2^{-N_1}) < \varepsilon/4$. Setting

$$f_1 = b_1 \chi_{(1-2^{-k_1},1)},$$

we have

$$\|f_1\| = 1, \quad \int\limits_0^{2^{-N_1}} f_1^{*p} w < \varepsilon/4.$$

By induction we find sequences (N_j) and (k_j) of natural numbers and a sequence (b_j) of positive numbers such that letting

$$f_j = b_j \chi_{(1 - \sum_{i=1}^{j} 2^{-k_i}, \, 1 - \sum_{i=1}^{j-1} 2^{-k_i})}$$

for $j = 1, 2, \ldots$ with $k_1 = 1$ and $\sum_{\emptyset} = 0$, we have

$$\|f_j\| = b_j^p W(2^{-k_j}) = 1,$$

$$\int_0^{2^{-N_j}} f_j^{*p} w = b_j W(2^{-N_j}) < \varepsilon / 4^j, \quad k_j < N_j < k_{j+1} - 1,$$

for $j = 1, 2, \ldots$ It is clear that $b_j < b_{j+1}$ and

$$\Big(\sum_{j=1}^{\infty} f_j^p \Big)^* = \sum_{j=1}^{\infty} b_j^p \chi_{B_j},$$

where B_j are intervals with $|B_j| = 2^{-k_j}$, $j = 1, 2, \ldots$, and $\bigcup_{j=1}^{\infty} B_j$ is an interval $(0, \alpha)$ with $\alpha \leq 1$.

Let now $(a_i) \in l^p$ with $\|(a_i)\|_p^p = \sum_{i=1}^{\infty} |a_i|^p = 1$. Setting $N_0 = 0$ and $C = \max(2^{p-1}, 1)$ we have

$$\Big\| \sum_{i=1}^{\infty} a_i f_i \Big\|^p = \sum_{j=0}^{\infty} \int_{2^{-N_{j+1}}}^{2^{-N_j}} \Big(a_{j+1} f_{j+1} + \sum_{i \neq j+1} a_i f_i \Big)^{*p} (t) w(t) \, dt$$

$$\leq C \sum_{j=0}^{\infty} \int_{2^{-N_{j+1}}}^{2^{-N_j}} (a_{j+1} f_{j+1})^{*p} (t/2) w(t) \, dt$$

$$+ C \sum_{j=0}^{\infty} \int_{2^{-N_{j+1}}}^{2^{-N_j}} \Big(\sum_{i \neq j+1} a_i f_i \Big)^{*p} (t/2) w(t) \, dt.$$

We will estimate separately each term of the right side of the above inequality. Since $k_{j+1} - 1 > N_j$ the following equality holds for $j = 0, 1, \ldots$:

$$P_{j+1} := \int_{2^{-N_{j+1}}}^{2^{-N_j}} (a_{j+1} f_{j+1})^{*p} (t/2) w(t) \, dt = |a_{j+1}|^p \int_{2^{-N_{j+1}}}^{2^{-(k_{j+1}-1)}} b_{j+1}^p w(t) \, dt.$$

By condition Δ_2 for W,

$$\int_{2^{-N_{j+1}}}^{2^{-(k_{j+1}-1)}} b_{j+1}^p w \leq b_{j+1}^p W(2^{-(k_{j+1}-1)}) \leq K b_{j+1}^p W(2^{-k_{j+1}}) = K,$$

and so

$$P_{j+1} \leq K |a_{j+1}|^p, \quad j = 0, 1, \ldots$$

It is clear that

$$Q_{j+1} := \int_{2^{-N_{j+1}}}^{2^{-N_j}} \Big(\sum_{i \neq j+1} a_i f_i \Big)^{*p}(t/2)w(t)\,dt \leq \int_{2^{-N_{j+1}}}^{2^{-N_j}} \Big(\sum_{i \neq j+1} f_i \Big)^{*p}(t/2)w(t)\,dt$$

for $j = 0, 1, \ldots$ Moreover,

$$\Big(\sum_{i \neq j+1} f_i \Big)^{*p} = \sum_{i=j+2}^{\infty} b_i^p \chi_{B_i} + \sum_{i=1}^{j} b_i^p \chi_{E_i},$$

where E_i are intervals such that $|E_i| = 2^{-k_i}, i = 1, \ldots, j$, and $\bigcup_{i=j+2}^{\infty} B_i \cup \bigcup_{i=1}^{j} E_i$ is also an interval. As $k_{j+2} - 2 \geq N_{j+1}$,

$$\Big| \bigcup_{i=j+2}^{\infty} 2B_i \Big| \leq 2 \sum_{i=k_{j+2}}^{\infty} 2^{-i} = 2^{-(k_{j+2}-2)} \leq 2^{-N_{j+1}},$$

whence

$$\int_{2^{-N_{j+1}}}^{2^{-N_j}} \Big(\sum_{i=j+2}^{\infty} b_i^p \chi_{B_i} \Big)(t/2)w(t)\,dt = 0, \quad j = 0, 1, \ldots$$

Now, since $N_j > k_j$ and $|E_i| = 2^{-k_i}$, we have $\chi_{2E_i \cap (0, 2^{-N_j})} \leq \chi_{(0, 2^{-k_j})}$, $i = 1, \ldots, j$. Thus

$$\int_{2^{-N_{j+1}}}^{2^{-N_j}} \Big(\sum_{i=1}^{j} b_i^p \chi_{E_i} \Big)(t/2)w(t)\,dt \leq \sum_{i=1}^{j} \int_0^{2^{-N_j}} b_j^p \chi_{2E_i}(t)w(t)\,dt$$

$$\leq j \int_0^{2^{-N_j}} f_j^{*p}(t)w(t)\,dt < j\varepsilon/4^j < \varepsilon/2^j.$$

Therefore, $Q_1 = 0$ and

$$Q_{j+1} < \varepsilon/2^j, \quad j = 1, 2, \ldots$$

Combining the above inequalities, for any $(a_i) \in l^p$ with $\|(a_i)\|_p \leq 1$ we obtain

$$\Big\| \sum_{i=1}^{\infty} a_i f_i \Big\| \leq (KC)^{1/p} \Big(\sum_{j=0}^{\infty} |a_{j+1}|^p + \varepsilon \Big)^{1/p},$$

and thus for all $(a_i) \in l^p$ and some $M > 0$,

$$\Big\| \sum_{i=1}^{\infty} a_i f_i \Big\| \leq M \|(a_i)\|_p.$$

On the other hand, since f_j have disjoint supports, for any $(a_i) \in l^p$,

$$\Big\| \sum_{i=1}^{\infty} a_i f_i \Big\|^p = \sum_{j=0}^{\infty} \int_{2^{-N_{j+1}}}^{2^{-N_j}} \Big(|a_{j+1} f_{j+1}| + \Big| \sum_{i \neq j+1} a_i f_i \Big| \Big)^{*p} w$$

$$\geq \sum_{j=0}^{\infty} |a_{j+1}|^p \int_{2^{-N_{j+1}}}^{2^{-N_j}} f_{j+1}^{*p} w,$$

and as $N_j < k_{j+1} < N_{j+1}$,

$$\int_{2^{-N_{j+1}}}^{2^{-N_j}} f_{j+1}^{*p} w = \int_I f_{j+1}^{*p} w - \int_0^{2^{-N_{j+1}}} f_{j+1}^{*p} w \geq 1 - \varepsilon/4^{j+1},$$

and so

$$\Big\| \sum_{i=1}^{\infty} a_i f_i \Big\| \geq (1 - \varepsilon/4)^{1/p} \|(a_i)\|_p.$$

This completes the proof.

3. Convexity of $\Lambda_{p,w}$. In view of Theorems A and 1, we obtain the following characterization of r-convexity of $\Lambda_{p,w}$ for $r \neq p$. Notice that conditions (i), (ii) and (iv) have also been studied in [23].

THEOREM 2. *If $0 < r < p < \infty$ then the following assertions are equivalent.*

(i) *$\Lambda_{p,w}$ is r-convex.*

(ii) *The Hardy operator $H^{(r)}$ is bounded in $\Lambda_{p,w}$, that is, the quasi-norms $\|f\|$ and $\|H^{(r)}f\|$ are equivalent.*

(iii) *The weight w satisfies condition $B_{p/r}$, that is, for some $B > 0$,*

$$\int_x^{\gamma} t^{-p/r} w(t)\, dt \leq B x^{-p/r} W(x) \quad \text{for all } x \in I.$$

(iv) *$\beta(W) < p/r$ or equivalently for some $\varepsilon > 0$, $W(t)/t^{p/r - \varepsilon}$ is pseudo-decreasing.*

If $r > p$ then $\Lambda_{p,w}$ is not r-convex.

REMARK. (a) If $0 < p < 1$ then $\Lambda_{p,w}$ is not normable. This fact, already noticed in [10], is a consequence of Theorem 1 as well.

(b) The space $\Lambda_{p,w}$, $0 < p < \infty$, is L-convex (for the definition and consequences see [11]). Indeed, by Δ_2-condition for W, $\beta(W) < \infty$ ([21, 19]) and so $\beta(W) < p/r$ for some $r > 0$. Hence in view of Theorem 2, $\Lambda_{p,w}$ is r-convex and thus L-convex by Theorem 2.2 of [11].

PROPOSITION 1. *Let $0 < p < \infty$. Given two weight functions w_1 and w_2, the Lorentz spaces Λ_{p,w_1} and Λ_{p,w_2} coincide and there exists a constant C such that $C^{-1}\|f\|_{p,w_1} \leq \|f\|_{p,w_2} \leq C\|f\|_{p,w_1}$ for any f, if and only if W_1 and W_2 are equivalent.*

Proof. The sufficiency follows from the formula $\|f\| = (-\int_I W\, d(f^{*p}))^{1/p}$. The necessity is a consequence of the equality $\|\chi_{(0,x]}\|^p_{p,w_i} = W_i(x)$, $i = 1, 2$.

THEOREM 3. *Given $0 < p, r < \infty$, the following conditions are equivalent.*

   (i) *$\Lambda_{p,w}$ satisfies an upper r-estimate.*
   (ii) *$W(t)/t^{p/r}$ is pseudo-decreasing and $r \leq p$.*
   (iii) *$W(t^{r/p})$ is equivalent to a concave function and $r \leq p$.*

If in addition $0 < r < 1$, then each of the above conditions is equivalent to

   (iv) *$\Lambda_{p,w}$ is r-normable.*

Proof. By Theorem 1, $r \leq p$. Moreover, the equivalence of (ii) and (iii) is a consequence of Lemma A. In view of Kalton's result (Theorem 2.3(ii) in [11]) any L-convex quasi-Banach lattice is r-normable for $0 < r < 1$ iff it satisfies an upper r-estimate. Hence (i) is equivalent to (iv). Now it remains to show that (i) is equivalent to (ii).

Since $(\Lambda_{p,w})^{(1/r)} = \Lambda_{p/r,w}$, it is enough to prove that $\Lambda_{p,w}$, $1 \leq p < \infty$, satisfies an upper 1-estimate iff $W(t)/t^p$ is pseudo-decreasing. Let us begin with the sufficiency part, taking any $\{f_i\}^n_{i=1} \subset \Lambda_{p,w}$ with disjoint supports. By Lemma A, $W^{1/p}(t)$ is equivalent to a concave function and in view of Proposition 1 we assume without loss of generality that $W^{1/p}$ is concave. Since also $W^{1/p}(0) = 0$, $W^{1/p}$ is subadditive ([16, p. 51]). Then $d_{\sum^n_{i=1} f_i} = \sum^n_{i=1} d_{f_i}$, and by subadditivity of $W^{1/p}$ and the Minkowski inequality,

$$\left\|\sum_{i=1}^{n} f_i\right\| = \left(\int_0^\infty \left(W^{1/p}\left(\sum_{i=1}^n d_{f_i}(t)\right)\right)^p d(t^p)\right)^{1/p}$$

$$\leq \left(\int_0^\infty \left(\sum_{i=1}^n W^{1/p}(d_{f_i}(t))\right)^p d(t^p)\right)^{1/p}$$

$$\leq \sum_{i=1}^n \left(\int_0^\infty W(d_{f_i}(t))\, d(t^p)\right)^{1/p} = \sum_{i=1}^n \|f_i\|.$$

In order to prove the necessity, given $0 < s \leq t$, $t \in I$, set $n = [t/s]$ and

$$f_i = \chi_{\left(\frac{(i-1)t}{2n}, \frac{it}{2n}\right]}, \quad i = 1, \dots, 2n.$$

By an upper 1-estimate for $\Lambda_{p,w}$, there exists $C > 0$ such that

$$W^{1/p}(t) = \|\chi_{(0,t]}\| = \Big\| \sum_{i=1}^{2n} |f_i| \Big\| \le C \sum_{i=1}^{2n} \|f_i\| = C \sum_{i=1}^{2n} W^{1/p}(t/2n)$$

$$= 2CnW^{1/p}(t/2n) \le 2C(t/s)W^{1/p}(t/2n) \le 2C(t/s)W^{1/p}(s),$$

which means that $W(t)/t^p$ is pseudo-decreasing.

THEOREM 4. *Given* $0 < p < \infty$ *the following properties are equivalent.*

(i) $\Lambda_{p,w}$ *is p-convex.*
(ii) $\Lambda_{p,w}$ *satisfies an upper p-estimate.*
(iii) $W(t)/t$ *is pseudo-decreasing.*
(iv) *There exists a decreasing weight* w_0 *such that* $\Lambda_{p,w} = \Lambda_{p,w_0}$ *and* $\| \ \|_{p,w}$ *and* $\| \ \|_{p,w_0}$ *are equivalent.*

Moreover, for $0 < p \le 1$ *each of the above properties is equivalent to*

(v) $\Lambda_{p,w}$ *is p-normable.*

Proof. Since (i)$\Rightarrow$(ii) is obvious and (ii)$\Rightarrow$(iii) follows from Theorem 3, we start with (iii)$\Rightarrow$(iv). By Lemma A, there exists a positive concave function W_0 on I equivalent to W. Thus W_0 is absolutely continuous on I and so $W_0(t) = \int_0^t w_0$ for some decreasing positive function w_0, and then in view of Proposition 1, $\Lambda_{p,w} = \Lambda_{p,w_0}$ and $\| \ \|_{p,w}$ and $\| \ \|_{p,w_0}$ are equivalent. In order to prove that (iv) yields (i), we assume without loss of generality that w is decreasing and applying a suitable formula for $\| \ \|_{p,w}$, for any finite sequence $\{f_i\}_{i=1}^n \subset \Lambda_{p,w}$ we obtain

$$\Big\| \Big(\sum_{i=1}^n |f_i|^p \Big)^{1/p} \Big\| \le \Big(\sum_{i=1}^n \sup_\tau \int_I |f_i(\tau(t))|^p w(t) \, dt \Big)^{1/p} = \Big(\sum_{i=1}^n \|f_i\|^p \Big)^{1/p}.$$

For $0 < p \le 1$, the immediate implications (i)$\Rightarrow$(v) and (v)$\Rightarrow$(ii) complete the proof.

4. Concavity of $\Lambda_{p,w}$**.** We begin the discussion of concavity of $\Lambda_{p,w}$ with a lemma which provides a number of equivalent integral conditions. Notice that the equivalence of the first three conditions does not require assuming Δ_2 for W.

LEMMA 1. *Let* $0 < p < \infty$. *The following conditions are equivalent.*

(i) $\alpha(W) > p$ *or equivalently* $W(x)/x^{p+\varepsilon}$ *is pseudo-increasing for some* $\varepsilon > 0$.

(ii) *The weight* w *satisfies condition* D_p, *that is, there exists* $A > 0$ *such that*

$$\int_0^x t^{-p} w(t) \, dt \le Ax^{-p} W(x) \quad \text{for all } x \in I.$$

(iii) *There is $B > 0$ such that*

$$\int\limits_0^x t^{-p-1} W(t)\, dt \le B x^{-p} W(x) \quad \text{for all } x \in I.$$

(iv) *The weight w satisfies condition D_p', that is, for some $C > 0$,*

$$\int\limits_0^x (x-t)^p t^{-p} w(t)\, dt \le C W(x) \quad \text{for all } x \in I.$$

If $0 < p < 1$, then each of the conditions (i)–(iv) *is equivalent to*

(v) *There is $D > 0$ such that*

$$\left(\int\limits_0^x t^{-\frac{1}{1-p}} W(t)^{\frac{1}{1-p}}\, dt \right)^{1-p} \le D x^{-p} W(x) \quad \text{for all } x \in I.$$

(vi) *There is $E > 0$ such that*

$$\left(\int\limits_0^x t^{-\frac{p}{1-p}} W(t)^{\frac{p}{1-p}} w(t)\, dt \right)^{1-p} \le E x^{-p} W(x) \quad \text{for all } x \in I.$$

Proof. (i)$\Rightarrow$(ii). Since $W(2^{-k}x)/(2^{-k}x)^{p+\varepsilon} \le CW(x)/x^{p+\varepsilon}$ or equivalently $W(2^{-k}x) \le C 2^{-k(p+\varepsilon)} W(x)$, $x > 0$, $k = 0, 1, \ldots$, it follows that

$$\int\limits_0^x t^{-p} w(t)\, dt = \sum_{k=0}^\infty \int\limits_{2^{-(k+1)}x}^{2^{-k}x} t^{-p} w(t)\, dt \le \sum_{k=0}^\infty (2^{-(k+1)}x)^{-p} \int\limits_{2^{-(k+1)}x}^{2^{-k}x} w$$

$$\le x^{-p} \sum_{k=0}^\infty 2^{(k+1)p} W(2^{-k}x) \le C 2^p x^{-p} \sum_{k=0}^\infty 2^{kp} 2^{-k(p+\varepsilon)} W(x)$$

$$= C 2^p x^{-p} W(x) \sum_{k=0}^\infty 2^{-k\varepsilon} = K x^{-p} W(x).$$

(ii)$\Rightarrow$(i). First we show that if w satisfies D_p then it also satisfies $D_{p+\varepsilon}$ for some $\varepsilon > 0$; in fact, for all $\varepsilon < p/(A-1)$. Multiplying the inequality in (ii) by $x^{-\varepsilon-1}$ and integrating it from 0 to r, we obtain

$$\int\limits_0^r x^{-\varepsilon-1} \left(\int\limits_0^x t^{-p} w(t)\, dt \right) dx \le A \int\limits_0^r x^{-\varepsilon-1} x^{-p} \left(\int\limits_0^x w \right) dx.$$

After changing the order of integration in both integrals we get

$$\int\limits_0^r t^{-p} w(t) \left(\int\limits_t^r x^{-\varepsilon-1}\, dx \right) dt \le A \int\limits_0^r w(t) \left(\int\limits_t^r x^{-\varepsilon-1} x^{-p}\, dx \right) dt$$

or

$$\frac{1}{\varepsilon} \int_0^r t^{-p-\varepsilon} w(t) \, dt - \frac{r^{-\varepsilon}}{\varepsilon} \int_0^r t^{-p} w(t) \, dt$$

$$\le A \left(\frac{1}{\varepsilon + p} \int_0^r t^{-p-\varepsilon} w(t) \, dt - \frac{r^{-p-\varepsilon}}{\varepsilon + p} \int_0^r w(t) \, dt \right),$$

and applying condition D_p gives

$$\left(\frac{1}{\varepsilon} - \frac{A}{\varepsilon + p} \right) \int_0^r t^{-p-\varepsilon} w(t) \, dt \le \frac{r^{-\varepsilon}}{\varepsilon} \int_0^r t^{-p} w(t) \, dt - \frac{A}{\varepsilon + p} r^{-p-\varepsilon} \int_0^r w$$

$$\le \left(\frac{A}{\varepsilon} - \frac{A}{\varepsilon + p} \right) r^{-p-\varepsilon} \int_0^r w.$$

Hence, if $0 < \varepsilon < p/(A - 1)$ then

$$\int_0^r t^{-p-\varepsilon} w(t) \, dt \le \frac{Ap}{p + \varepsilon - A\varepsilon} r^{-p-\varepsilon} \int_0^r w \quad \text{for all } r \in I,$$

i.e. w satisfies condition $D_{p+\varepsilon}$. Now, if $0 < x < y$ then

$$\frac{W(x)}{x^{p+\varepsilon}} \le \int_0^x t^{-p-\varepsilon} w(t) \, dt \le \int_0^y t^{-p-\varepsilon} w(t) \, dt \le K y^{-p-\varepsilon} \int_0^y w = K \frac{W(y)}{y^{p+\varepsilon}}.$$

(iii)$\Rightarrow$(ii). Since $\int_0^x t^{-p-1} W(t) \, dt \ge (x^{-p}/2) W(x/2)$, it follows that $\lim_{x \to 0^+} x^{-p} W(x) = 0$. Thus, integration by parts yields

$$\int_0^x t^{-p} w(t) \, dt = \int_0^x t^{-p} \, dW(t) = t^{-p} W(t) \big|_0^x - \int_0^x W(t) \, d(t^{-p})$$

$$= x^{-p} W(x) + p \int_0^x t^{-p-1} W(t) \, dt \le (1 + pB) x^{-p} W(x).$$

(ii)$\Rightarrow$(iii). From the inequality $x^{-p} W(x) \le \int_0^x t^{-p} w(t) \, dt$ we see that $\lim_{x \to 0^+} x^{-p} W(x) = 0$, and so by integration by parts,

$$\int_0^x t^{-p-1} W(t) \, dt = \frac{1}{p} \left(\int_0^x t^{-p} w(t) \, dt - x^{-p} W(x) \right) \le \frac{A-1}{p} x^{-p} W(x).$$

(ii)$\Leftrightarrow$(iv). If condition D_p holds then

$$\int_0^x (x - t)^p t^{-p} w(t) \, dt \le \int_0^x x^p t^{-p} w(t) \, dt \le A \int_0^x w.$$

On the other hand, if (iv) is satisfied then in view of Δ_2 for W,

$$CKW(x/2) \ge CW(x) \ge \int_0^x (x - t)^p t^{-p} w(t) \, dt \ge (x/2)^p \int_0^{x/2} t^{-p} w(t) \, dt,$$

which is condition D_p.

(i)$\Rightarrow$(v). For an appropriate $\varepsilon > 0$,

$$\left(\int_0^x t^{-\frac{1}{1-p}} W(t)^{\frac{1}{1-p}}\, dt\right)^{1-p} = \left(\int_0^x t^{-\frac{1}{1-p}} (t^{-p-\varepsilon} W(t))^{\frac{1}{1-p}} t^{\frac{p+\varepsilon}{1-p}}\, dt\right)^{1-p}$$

$$\leq c^{1-p} x^{-p-\varepsilon} W(x) \left(\int_0^x t^{\frac{p+\varepsilon}{1-p}}\, dt\right)^{1-p} = dx^{-p-\varepsilon} W(x) x^{\varepsilon} = dx^{-p} W(x).$$

(v)$\Rightarrow$(i). In view of condition Δ_2 we have

$$K^{-1} 2^{p-1} \leq 2^{p-1} W(x/2) \leq f(x) := x^p \left(\int_0^x t^{-\frac{1}{1-p}} W(t)^{\frac{1}{1-p}}\, dt\right)^{1-p} \leq DW(x),$$

and so $\alpha(W) = \alpha(f)$. Moreover, for sufficiently small $\varepsilon > 0$,

$$\frac{d}{dx}(x^{-p-\varepsilon} f(x))$$

$$= \left(-\varepsilon x^{-\varepsilon-1} \int_0^x t^{-\frac{1}{1-p}} W(t)^{\frac{1}{1-p}}\, dt + x^{-\varepsilon}(1-p) x^{-\frac{1}{1-p}} W(x)^{\frac{1}{1-p}}\right)$$

$$\times \left(\int_0^x t^{-\frac{1}{1-p}} W(t)^{\frac{1}{1-p}}\, dt\right)^{-p}$$

$$\geq (1-p-\varepsilon D^{\frac{1}{1-p}}) x^{-\varepsilon-\frac{1}{1-p}} W(x)^{\frac{1}{1-p}} \left(\int_0^x t^{-\frac{1}{1-p}} W(t)^{\frac{1}{1-p}}\, dt\right)^{-p} > 0.$$

Therefore $\alpha(W) = \alpha(f) \geq p + \varepsilon > p$.

(vi)$\Leftrightarrow$(v). For $x \in I$ we have

$$\int_0^x t^{-\frac{p}{1-p}} W(t)^{\frac{p}{1-p}} w(t)\, dt \geq x^{\frac{-p}{1-p}} \int_0^x W(t)^{\frac{p}{1-p}} w(t)\, dt = (1-p) x^{\frac{-p}{1-p}} W(x)^{\frac{1}{1-p}},$$

$$\int_0^x t^{-\frac{1}{1-p}} W(t)^{\frac{1}{1-p}}\, dt \geq x^{-\frac{1}{1-p}} \int_{x/2}^x W^{\frac{1}{1-p}}(t)\, dt \geq \frac{1}{2} x^{-\frac{p}{1-p}} W(x/2)^{\frac{1}{1-p}}.$$

Thus integrating by parts we obtain

$$\int_0^x t^{-\frac{p}{1-p}} W(t)^{\frac{p}{1-p}} w(t)\, dt = (1-p)(x^{-p} W(x))^{\frac{1}{1-p}} + p \int_0^x t^{-\frac{1}{1-p}} W(t)^{\frac{1}{1-p}}\, dt,$$

which in fact yields the equivalence of (v) and (vi).

REMARK. Recall that for a r.i. quasi-Banach space X over $(I, |\ |)$, the lower and upper Boyd indices are defined as for a r.i. Banach space ([17, 22]), that is,

$$p(X) = \sup\{p > 0 : \text{there exists } C > 0 \text{ such that}$$

$$\|D_a\| \leq Ca^{-1/p} \text{ for all } 0 < a < 1\},$$

$q(X) = \inf\{q > 0 : \text{there exists } C > 0 \text{ such that}$
$$\|D_a\| \leq Ca^{-1/q} \text{ for all } a > 1\},$$

where $D_a : X \to X$ is a dilation operator defined on $I = (0, \infty)$ as $D_a f(t) = f(at)$, and on $I = (0, 1]$ as

$$D_a f(t) = \begin{cases} f(at) & \text{for } 0 \leq t \leq \min(a^{-1}, 1), \\ 0 & \text{for } \min(a^{-1}, 1) < t \leq 1. \end{cases}$$

For any our weight function w, we have

$$p(\Lambda_{p,w}) = p/\beta(W), \qquad q(\Lambda_{p,w}) = p/\alpha(W).$$

We shall sketch a proof only for $I = (0, \infty)$. Let $\|D_a\|$ be the operator norm of D_a on $\Lambda_{p,w}$. Then for any $a > 0$,

$$\|D_a\| \geq \sup_{x>0} \frac{\|\chi_{(0,x/a)}\|}{\|\chi_{(0,x)}\|} = \sup_{x>0} \left(\frac{W(x/a)}{W(x)}\right)^{1/p}.$$

On the other hand, since $d_{D_a f}(t) = a^{-1} d_f(t)$, $a > 0$, we get for any $f \in \Lambda_{p,w}$,

$$\|D_a f\| = \left(\int_0^\infty W(a^{-1} d_f(t)) \, d(t^{1/p})\right)^{1/p} \leq \sup_{x>0} \left(\frac{W(x/a)}{W(x)}\right)^{1/p} \|f\|.$$

These estimates yield directly the above formulas on Boyd indices.

THEOREM 5. *Let* $0 < p < 1$. *The following assertions are equivalent.*

(i) $\Lambda_{p,w}$ *is 1-concave.*

(ii) *The Hardy operator* $f_{**}(t) = t^{-1} \int_t^\gamma f^*(s) \, ds$ *is bounded in* $\Lambda_{p,w}$.

(iii) $\alpha(W) > p$ *or equivalently for some* $\varepsilon > 0$, $W(t)/t^{p+\varepsilon}$ *is pseudo-increasing.*

(iv) *The weight* w *satisfies condition* D_p, *i.e. there exists* $A > 0$ *such that*

$$\int_0^x t^{-p} w(t) \, dt \leq Ax^{-p} W(x) \quad \text{for all } x \in I.$$

Proof. (i)$\Rightarrow$(iv). For $x, y \in I$ and for any Lebesgue measurable function $f : I \to \mathbb{R}$, we have

$$F(y) := \int_0^x f^*(|y - t|) \, dt \, \chi_{(0,x)}(y)$$

$$= \left(\int_0^y f^* + \int_0^{x-y} f^*\right) \chi_{(0,x)}(y) \leq 2\left(\int_0^x f^*\right) \chi_{(0,x)}(y),$$

whence

$$\|F\| \leq 2 \int_0^x f^* \left(\int_0^x w\right)^{1/p}.$$

(We adopt here the convention that $\|F\| = \infty$ if $F \notin \Lambda_{p,w}$.) On the other hand, by the reverse Minkowski inequality which follows from 1-concavity of $\Lambda_{p,w}$,

$$\|F\| = \left\| \int\limits_0^x f^*(|y-t|)\,dt\,\chi_{(0,x)}(y) \right\| \geq C \int\limits_0^x \|f^*(|y-t|)\chi_{(0,x)}(y)\|\,dt$$

for all $x \in I$. Setting

$$g_{t,x}(y) = f^*(|y-t|)\chi_{(0,x)}(y) \quad (0 < t < x,\ x, y \in I),$$

we obtain

$$\|g_{t,x}\| \geq \left(\int\limits_0^x f^{*p}w \right)^{1/p},$$

which follows from the formula

$$g_{t,x}^*(s) = \begin{cases} f^*(s/2)\chi_{(0,2x-2t)}(s) + f^*(s-x+t)\chi_{[2x-2t,x)}(s) & \text{if } t < x \leq 2t, \\ f^*(s/2)\chi_{(0,2t)}(s) + f^*(s-t)\chi_{[2t,x)}(s) & \text{if } x > 2t. \end{cases}$$

Combining the above estimates we obtain

$$\left(\int\limits_0^x f^{*p}w \right)^{1/p} \leq Kx^{-1}\int\limits_0^x f^*\left(\int\limits_0^x w \right)^{1/p} \quad \text{for all } x \in I,$$

for any measurable function f. We can rewrite this inequality, with the convention $\infty/\infty = 0$, as

$$\frac{\int_0^\gamma f^*w\chi_{(0,x)}}{(\int_0^\gamma f^{*(1/p)}\chi_{(0,x)})^p} \leq Kx^{-p}W(x) \quad \text{for all } x \in I.$$

Now, recall Sawyer's duality formula ([25, Theorem 1] and [15, Corollary 1.4]). If $1 < r < \infty$, then for any non-negative $g \in L^0$ and any locally integrable function $h \geq 0$, we have

$$\sup_{\substack{0 \leq f \\ \text{decreasing}}} \frac{\int_0^\gamma fg}{(\int_0^\gamma f^r h)^{1/r}} \approx \left(\int\limits_0^\gamma \left(\frac{\int_0^t g}{\int_0^t h} \right)^{r'} h \right)^{1/r'} + \frac{\int_0^\gamma g}{(\int_0^\gamma h)^{1/r}},$$

where $\approx$ denotes the equivalence of both sides with constants independent of g and h, and the convention $\infty/\infty = 0$ applies. Letting in the above formula $r = 1/p$, $f = f^*$, $g = w\chi_{(0,x)}$ and $h = \chi_{(0,x)}$, we get for $x \in I$,

$$\sup_{f^*} \frac{\int_0^\gamma f^*w\chi_{(0,x)}}{(\int_0^\gamma f^{*(1/p)}\chi_{(0,x)})^p} \approx \left(\int\limits_0^x t^{-\frac{1}{1-p}}W(t)^{\frac{1}{1-p}}w(t)\,dt \right)^{1-p} + \frac{W(x)}{W(\gamma)^p}.$$

Thus, for every $x \in I$,

$$\left(\int\limits_0^x t^{-\frac{1}{1-p}}W(t)^{\frac{1}{1-p}}w(t)\,dt \right)^{1-p} \leq Kx^{-p}W(x).$$

This is however condition (v) of Lemma 1 and so w satisfies condition D_p.

The implication (iv)⇒(iii) has been proved in Lemma 1.

(iii)⇒(ii). We shall apply here Theorem 2 of [22] stating that $H_{(1)}$ is bounded on a r.i. quasi-Banach space X iff $q(X) < 1$. Although this theorem has been proved there only for $I = (0, \infty)$, it is easily verified also for $I = (0, 1]$. Now the implication follows since $q(\Lambda_{p,w}) = p/\alpha(W)$ by the remark before the theorem.

(ii)⇒(i). First we show that for any $f \in \Lambda_{p,w}$, $\|f_{**}\| \geq K^{-p}\|f\|$, where K is the constant in condition Δ_2. This will be done only for $I = (0, 1]$. In fact, it is clear that $f_{**}(t) \geq f^*(2t)\chi_{(0,1/2]}(t)$ for $t \in (0, 1]$. Hence

$$\|f_{**}\| \geq \Big(\int_0^{1/2} f^{*p}(2t)w(t)\, dt \Big)^{1/p}.$$

Setting $f = \chi_{(0,a)}$, $0 < a \leq 1$, in view of condition Δ_2, we have

$$\int_0^{1/2} f^{*p}(2t)w(t)\, dt = \int_0^{a/2} w = W(a/2) \geq K^{-1}W(a) = K^{-1}\int_0^1 f^{*p}w.$$

Thus, by approximating f^* by simple functions of the form $\sum \alpha_i \chi_{(0,a_i)}$, the above formula also holds for any function f in $\Lambda_{p,w}$, and so $\|f_{**}\| \geq K^{-p}\|f\|$.

Finally, we have

$$C\Big\|\Big(\sum_{i=1}^n |f_i|\Big)\Big\| \geq \Big\|\Big(\sum_{i=1}^n |f_i|\Big)_{**}\Big\| \geq \Big\|\sum_{i=1}^n |f_i|_{**}\Big\|$$

$$\geq \sum_{i=1}^n \|(f_i)_{**}\| \geq K^{-p}\sum_{i=1}^n \|f_i\|,$$

where the first inequality follows from the boundedness of $H_{(1)}$, the second from $(|f| + |g|)_{**} \geq f_{**} + g_{**}$, and the third from the reverse Minkowski inequality for $0 < p < 1$.

The next result is a corollary of Theorems 5 and 1. Observe that conditions (i)–(iii) have also been investigated in [23].

THEOREM 6. *If $r > p > 0$ then the following assertions are equivalent.*

(i) *$\Lambda_{p,w}$ is r-concave.*

(ii) *The Hardy operator $H_{(r)}$ is bounded in $\Lambda_{p,w}$.*

(iii) *$\alpha(W) > p/r$, or equivalently $W(t)/t^{p/r+\varepsilon}$ is pseudo-increasing for some $\varepsilon > 0$.*

(iv) *The weight w satisfies condition $D_{p/r}$, that is, there exists $C > 0$ such that*

$$\int_0^x t^{-p/r}w(t)\, dt \leq Cx^{-p/r}\int_0^x w \quad \text{for all } x \in I.$$

If $0 < r < p < \infty$ then $\Lambda_{p,w}$ is not r-concave.

Finally, the last two theorems completing our discussion are companions of Theorems 3 and 4, respectively.

THEOREM 7. *Given $0 < p, r < \infty$, the following properties are equivalent.*

(i) $\Lambda_{p,w}$ *satisfies a lower r-estimate.*
(ii) $W(t)/t^{p/r}$ *is pseudo-increasing and $r \geq p$.*
(iii) $W(t^{r/p})$ *is equivalent to a convex function.*

Proof. The proof, applying the reverse Minkowski inequality for $0 < p \leq 1$, is analogous to that of Theorem 3.

THEOREM 8. *Given $0 < p < \infty$, the following properties are equivalent.*

(i) $\Lambda_{p,w}$ *is p-concave.*
(ii) $\Lambda_{p,w}$ *satisfies a lower p-estimate.*
(iii) $W(t)/t$ *is pseudo-increasing.*
(iv) *There exists an increasing weight w_0 such that $\Lambda_{p,w} = \Lambda_{p,w_0}$ and $\| \ \|_{p,w}$ and $\| \ \|_{p,w_0}$ are equivalent.*

Proof. The proof is analogous to the one of Theorem 4. In fact we obtain reverse inequalities. For instance, assuming that w is increasing, for any $\{f_i\}_{i=1}^n \subset \Lambda_{p,w}$,

$$\Big\| \sum_{i=1}^n |f_i|^p \Big\| = \Big(\inf_\tau \int_I \sum_{i=1}^n |f_i(\tau(t))|^p w(t)\, dt \Big)^{1/p} \geq \Big(\sum_{i=1}^n \|f_i\|^p \Big)^{1/p},$$

which shows that (iv) yields (i).

We conclude the paper with a corollary on the type and cotype of normable $\Lambda_{p,w}$ spaces. This is a consequence of the above characterizations as well as the well known relations between type (resp. cotype) and upper estimation (resp. lower estimation) [17] (see also Theorems 11, 14 and 15 in [14]).

COROLLARY 1. *Let $1 < p < \infty$ and $w \in B_p$.*

(a) *Let $1 < r < 2$. Then $\Lambda_{w,p}$ has type r if and only if $\alpha(W) > 0$, $r \leq p$ and $W(t)/t^{p/r}$ is pseudo-decreasing.*

(b) *Let $2 < r < \infty$. Then $\Lambda_{w,p}$ has cotype r if and only if $r \geq p$ and $W(t)/t^{p/r}$ is pseudo-increasing.*

(c) *Let $p \neq 2$. Then $\Lambda_{p,w}$ has cotype 2 (resp., type 2) if and only if $1 \leq p < 2$ and $\alpha(W) > p/2$ (resp., $2 < p < \infty$ and $\beta(W) < p/2$ and $\alpha(W) > 0$).*

(d) *Let $p = 2$. Then $\Lambda_{2,w}$ has cotype 2 (resp., type 2) if and only if $W(t)/t$ is pseudo-increasing (resp., $W(t)/t$ is pseudo-decreasing and $\alpha(W) > 0$).*

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