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Tiling and spectral properties of near-cubic domains

by

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Abstract. We prove that if a measurable domain tiles $\mathbb{R}$ or $\mathbb{R}^2$ by translations, and if it is "close enough" to a line segment or a square respectively, then it admits a lattice tiling. We also prove a similar result for spectral sets in dimension 1, and give an example showing that there is no analogue of the tiling result in dimensions 3 and higher.

1. Introduction. Let E be a measurable set in $\mathbb{R}^n$ such that $0<|E|<\infty$. We will say that E *tiles* $\mathbb{R}^n$ *by translations* if there is a set $T\subset\mathbb{R}^n$ such that, up to sets of measure 0, the sets $E+t$, $t\in T$, are mutually disjoint and $\bigcup_{t\in T}(E+t)=\mathbb{R}^n$. We call any such T a *translation set* for E, and write $E+T=\mathbb{R}^n$. A tiling $E+T=\mathbb{R}^n$ is called *periodic* if it admits a period lattice of rank n; it is a *lattice tiling* if T itself is a lattice. Here and below, a *lattice* in $\mathbb{R}^n$ will always be a set of the form $T\mathbb{Z}^n$, where T is a linear transformation of rank n.

It is known ([19], [18]) that if a convex set E tiles $\mathbb{R}^n$ by translations, it also admits a lattice tiling. A natural question is whether a similar result holds if E is "sufficiently close" to being convex, e.g. if it is close enough (in an appropriate sense) to an n-dimensional cube. In this paper we prove that this is indeed so in dimensions 1 and 2; we also construct a counterexample in dimensions $n\geq 3$.

A major unresolved problem in the mathematical theory of tilings is the *periodic tiling conjecture*, which asserts that any E which tiles $\mathbb{R}^n$ by translations must also admit a periodic tiling. (See [3] for an overview of this and other related questions.) The conjecture has been proved for all bounded measurable subsets of $\mathbb{R}$ ([16], [12]) and for topological discs in $\mathbb{R}^2$ ([2], [8]). Our Theorem 2 and Corollary 1 prove the conjecture for near-square domains in $\mathbb{R}^2$. We emphasize that no assumptions on the topology of E are needed; in particular, E is not required to be connected and may have infinitely many connected components.

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Our work was also motivated in part by a conjecture of Fuglede [1]. We call a set E *spectral* if there is a discrete set $\Lambda \subset \mathbb{R}^n$, which we call a *spectrum* for E, such that $\{e^{2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(E)$. Fuglede conjectured that E is spectral if and only if it tiles $\mathbb{R}^n$ by translations, and proved it under the assumption that either the translation set T or the spectrum Λ is a lattice. This problem was addressed in many recent papers (see e.g. [4], [7], [10], [13]–[17]), and in particular the conjecture has been proved for convex regions in $\mathbb{R}^2$ ([9], [5], [6]).

It follows from our Theorem 1 and from Fuglede's theorem that the conjecture is true for $E \subset \mathbb{R}$ such that E is contained in an interval of length strictly less than $3|E|/2$. (This was proved in [15] in the special case when E is a union of finitely many intervals of equal length.) In dimension 2, we obtain the "tiling $\Rightarrow$ spectrum" part of the conjecture for near-square domains. Namely, if $E \subset \mathbb{R}^2$ tiles $\mathbb{R}^2$ and satisfies the assumptions of Theorem 2 or Corollary 1, it also admits a lattice tiling, hence it is a spectral set by Fuglede's theorem on the lattice case of his conjecture. We do not know how to prove the reverse implication.

Our main results are the following.

Theorem 1. *Suppose $E \subseteq [0, L]$ is measurable with measure 1 and $L = 3/2 - \varepsilon$ for some $\varepsilon > 0$. Let $\Lambda \subset \mathbb{R}$ be a discrete set containing 0. Then*

(a) *if $E + \Lambda = \mathbb{R}$ is a tiling, it follows that $\Lambda = \mathbb{Z}$;*
(b) *if Λ is a spectrum of E, it follows that $\Lambda = \mathbb{Z}$.*

The upper bound $L < 3/2$ in Theorem 1 is optimal: the set $[0, 1/2] \cup [1, 3/2]$ is contained in an interval of length $3/2$, tiles $\mathbb{Z}$ with the translation set $\{0, 1/2\} + 2\mathbb{Z}$, and has the spectrum $\{0, 1/2\} + 2\mathbb{Z}$, but does not have either a lattice translation set or a lattice spectrum. This example has been known to many authors; an explicit calculation of the spectrum is given e.g. in [14].

Theorem 2. *Let $E \subset \mathbb{R}^2$ be a measurable set such that $[0, 1]^2 \subset E \subset [-\varepsilon, 1 + \varepsilon]^2$ for $\varepsilon > 0$ small enough. Assume that E tiles $\mathbb{R}^2$ by translations. Then E also admits a tiling with a lattice $\Lambda \subset \mathbb{R}^2$ as the translation set.*

Our proof works for $\varepsilon < \varepsilon_0 \approx 0.05496$; we do not know what is the optimal upper bound for ε.

Corollary 1. *Let $E \subset \mathbb{R}^2$ be a measurable set such that $|E| = 1$ and E is contained in a square of sidelength $1 + \varepsilon$ for $\varepsilon > 0$ small enough. If E tiles $\mathbb{R}^2$ by translations, then it also admits a lattice tiling.*

Theorem 3. *Let $n \geq 3$. Then for any $\varepsilon > 0$ there is a set $E \subset \mathbb{R}^n$ with $[0, 1]^n \subset E \subset [-\varepsilon, 1 + \varepsilon]^n$ such that E tiles $\mathbb{R}^n$ by translations, but does not admit a lattice tiling.*

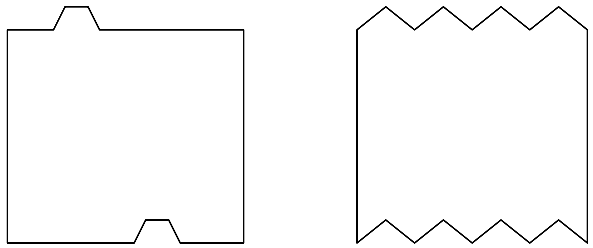

Fig. 1. Examples of near-square regions which tile $\mathbb{R}^2$. Note that the second region also admits aperiodic (hence non-lattice) tilings.

2. The one-dimensional case. In this section we prove Theorem 1. We shall need the following crucial lemma.

Lemma 1. *Suppose that $E \subseteq [0, L]$ is measurable with measure 1 and that $L = 3/2 - \varepsilon$ for some $\varepsilon > 0$. Then*

$$|E \cap (E + x)| > 0 \quad \textit{whenever } 0 \le x < 1. \tag{1}$$

Proof. We distinguish the cases (i) $0 \le x \le 1/2$, (ii) $1/2 < x \le 3/4$, and (iii) $3/4 < x < 1$.

(i) This is the easy case as $E \cup (E + x) \subseteq [0, L + 1/2] = [0, 2 - \varepsilon]$. Since this interval has length less than 2, the sets E and $E + x$ must intersect in positive measure.

(ii) Let $x = 1/2 + \alpha$, $0 < \alpha \le 1/4$. Suppose that $|E \cap (E + x)| = 0$. Then $1 + 2\alpha \le 3/2$ and

$$|(E \cap [0, x]) \cup (E \cap [x, 2x])| \le x,$$

as the second set does not intersect the first when shifted back by x. This implies that

$$|E| \le x + (3/2 - \varepsilon - 2x) = 3/2 - \varepsilon - x = 1 - \varepsilon - \alpha < 1,$$

a contradiction as $|E| = 1$.

(iii) Let $x = 3/4 + \alpha$, $0 < \alpha < 1/4$. Suppose that $|E \cap (E + x)| = 0$. Then

$$|(E \cap [0, 3/4 - \alpha - \varepsilon]) \cup (E \cap [3/4 + \alpha, 3/2 - \varepsilon])| \le 3/4 - \alpha - \varepsilon,$$

for the second set translated to the left by x does not intersect the first. This implies that

$$|E| \le (3/4 - \alpha - \varepsilon) + 2\alpha + \varepsilon = 3/4 + \alpha < 1,$$

a contradiction. ∎

We need to introduce some terminology. If f is a non-negative integrable function on $\mathbb{R}^d$ and Λ is a subset of $\mathbb{R}^d$, we say that $f + \Lambda$ is a *packing* if, almost everywhere,

$$\sum_{\lambda\in\Lambda} f(x-\lambda) \le 1. \tag{2}$$

We say that $f+\Lambda$ is a *tiling* if equality holds almost everywhere. When $f=\chi_E$ is the indicator function of a measurable set, this definition coincides with the classical geometric notions of packing and tiling.

We shall need the following theorem from [10].

THEOREM 4. *If $f,g\ge 0$, $\int f(x)\,dx=\int g(x)\,dx=1$ and both $f+\Lambda$ and $g+\Lambda$ are packings of $\mathbb{R}^d$, then $f+\Lambda$ is a tiling if and only if $g+\Lambda$ is a tiling.*

Proof of Theorem 1. (a) Suppose $E+\Lambda$ is a tiling. From Lemma 1 it follows that any two elements of Λ differ by at least 1. This implies that $\chi_{[0,1]}+\Lambda$ is a packing, hence it is also a tiling by Theorem 4. Since $0\in\Lambda$, we have $\Lambda=\mathbb{Z}$.

(b) Suppose that Λ is a spectrum of E. Write

$$\delta_\Lambda=\sum_{\lambda\in\Lambda}\delta_\lambda$$

for the measure of one unit mass at each point of Λ. Our assumption that Λ is a spectrum for E implies that

$$|\widehat{\chi}_E|^2+\Lambda=\mathbb{R}$$

is a tiling (see, for example, [10]). This, in turn, implies that Λ had density 1. Here and below, we say that a set $A\subset\mathbb{R}$ has *density* ϱ if

$$\lim_{N\to\infty}\frac{\#(A\cap[-N,N])}{2N}=\varrho.$$

NOTATION. The definition of the Fourier transform we use is

$$\widehat{f}(\xi)=\int_{\mathbb{R}} e^{-2\pi i\xi\cdot x} f(x)\,dx$$

for an L^1 function f. If T is a tempered distribution (a bounded linear functional on the Schwarz space $\mathcal{S}$) then its Fourier transform is defined by duality as the tempered distribution $\widehat{T}$ given by

$$\widehat{T}(\phi)=T(\widehat{\phi}),\qquad \phi\in\mathcal{S}.$$

We now use the following result from [10]:

THEOREM 5. *Suppose that $f\ge 0$ is not identically* 0, *that $f\in L^1(\mathbb{R}^d)$, $\widehat{f}\ge 0$ has compact support and $\Lambda\subset\mathbb{R}^d$. If $f+\Lambda$ is a tiling then*

$$\operatorname{supp}\widehat{\delta}_\Lambda\subseteq\{\widehat{f}=0\}\cup\{0\}. \tag{3}$$

Let us emphasize here that the object $\widehat{\delta_\Lambda}$, the Fourier transform of the tempered measure δ_Λ, is in general a tempered distribution and need not be a measure.

For $f = |\widehat{\chi}_E|^2$ Theorem 5 implies

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \{0\} \cup \{\chi_E * \widetilde{\chi}_E = 0\}, \tag{4}$$

since $\chi_E * \widetilde{\chi}_E$ is the Fourier transform of $|\widehat{\chi}_E|^2$ (where $\widetilde{g}(x) = \overline{g(-x)}$). But

$$\{\chi_E * \widetilde{\chi}_E = 0\} = \{x : |E \cap (E + x)| = 0\}.$$

This and Lemma 1 imply that

$$\operatorname{supp}\widehat{\delta_\Lambda} \cap (-1, 1) = \{0\}.$$

Let

$$K_\delta(x) = \max\{0, 1 - (1+\delta)|x|\} = (1+\delta)\chi_{I_\delta} * \widetilde{\chi}_{I_\delta}(x),$$

where $I_\delta = [0, 1/(1+\delta)]$, be a Fejér kernel (we will later take $\delta \to 0$). Then

$$\widehat{K}_\delta = (1+\delta)|\widehat{\chi}_{I_\delta}|^2 = \frac{1+\delta}{\pi^2 x^2} \sin^2 \frac{\pi x}{1+\delta}$$

is a non-negative continuous function and it follows that

$$\widehat{K}_\delta(0) = \frac{1}{1+\delta}$$

and

$$\{x : \widehat{K}_\delta(x) = 0\} = (1+\delta)(\mathbb{Z} \setminus \{0\}). \tag{5}$$

Next, we use the following result from [11] (proved there in a more general setting):

THEOREM 6. *Suppose that $\Lambda \in \mathbb{R}$ is a set with density ϱ, that $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$, and $\widehat{\delta_\Lambda}$ is a measure in a neighborhood of* 0. *Then $\widehat{\delta_\Lambda}(\{0\}) = \varrho$.*

REMARK. The proof of Theorem 6 shows that the assumption of $\widehat{\delta_\Lambda}$ being a measure in a neighborhood of zero is superfluous, if one knows a priori that $\widehat{\delta_\Lambda}$ is supported only at zero, in a neighborhood of zero. Indeed, what is shown in that proof is that, as $t \to \infty$, the quantity $\widehat{\delta_\Lambda}(\phi(tx))$ remains bounded, for any C_c^∞ test function ϕ. If $\widehat{\delta_\Lambda}$ were not a measure near 0 but had support only at 0, locally, this quantity would grow like a polynomial in t of degree equal to the degree of the distribution at 0.

Applying Theorem 6 and the Remark following it we deduce that $\widehat{\delta_\Lambda}$ is equal to δ_0 in a neighborhood of 0, since Λ has density 1.

Next, we claim that

$$\sum_{\lambda \in \Lambda} \widehat{K}_\delta(x - \lambda) = 1 \quad \text{for all } x \in \mathbb{R}.$$

Indeed, take ψ_ε to be an even, smooth, positive-definite approximate identity, supported in $(-\varepsilon,\varepsilon)$, and take $\varepsilon=\varepsilon(\delta)$ to be small enough so that $\operatorname{supp}\psi_\varepsilon * K_\delta \subset (-1,1)$. We then have, for fixed x,

$$\begin{aligned}
&\sum_{\lambda\in\Lambda}\widehat{K}_\delta(x-\lambda)\\
&\qquad= \lim_{\varepsilon\to 0}\sum_{\lambda\in\Lambda}\widehat{\psi}_\varepsilon(x-\lambda)\widehat{K}_\delta(x-\lambda)\\
&\qquad= \lim_{\varepsilon\to 0}\delta_\Lambda((\widehat{\psi}_\varepsilon\widehat{K}_\delta)(x-\cdot)) &&\text{(by definition of } \delta_\Lambda)\\
&\qquad= \lim_{\varepsilon\to 0}\widehat{\delta}_\Lambda(e^{2\pi ixt}(\psi_\varepsilon * K_\delta)(t)) &&\text{(by definition of the FT of } \delta_\Lambda)\\
&\qquad= \lim_{\varepsilon\to 0}\delta_0(e^{2\pi ixt}(\psi_\varepsilon * K_\delta)(t)) &&\text{(for } \varepsilon \text{ small enough)}\\
&\qquad= \lim_{\varepsilon\to 0}(\psi_\varepsilon * K_\delta)(0) = K_\delta(0) = 1,
\end{aligned}$$

which establishes the claim. Applying this for $x=0$ and isolating the term $\lambda=0$ we get

$$1=\frac{1}{1+\delta}+\sum_{0\neq\lambda\in\Lambda}\widehat{K}_\delta(-\lambda).$$

Letting $\delta\to 0$ we obtain $\widehat{K}_\delta(-\lambda)\to 0$ for each $\lambda\in\Lambda\setminus\{0\}$, which implies that each such λ is an integer, as $\mathbb{Z}\setminus\{0\}$ is the limiting set of the zeros of $\widehat{K}_\delta$.

To get $\Lambda=\mathbb{Z}$ notice that $\chi_{[0,1]}+\Lambda$ is a packing. By Theorem 4 we again deduce that $\chi_{[0,1]}+\Lambda$ is in fact a tiling, hence $\Lambda=\mathbb{Z}$. ∎

3. Planar regions

Proof of Theorem 2. We denote the coordinates in $\mathbb{R}^2$ by (x_1,x_2). For $0\le a\le b\le 1$ we define

$$\begin{aligned}
E_1(a,b)&=(E\cap\{a\le x_1\le b,\ x_2\le 0\})\cup\{a\le x_1\le b,\ x_2\ge 0\},\\
E_2(a,b)&=(E\cap\{a\le x_1\le b,\ x_2\ge 0\})\cup\{a\le x_1\le b,\ x_2\le 0\},\\
F_1(a,b)&=(E\cap\{a\le x_2\le b,\ x_1\le 0\})\cup\{a\le x_2\le b,\ x_1\ge 0\},\\
F_2(a,b)&=(E\cap\{a\le x_2\le b,\ x_1\ge 0\})\cup\{a\le x_2\le b,\ x_1\le 0\}.
\end{aligned}$$

We will also use $S_{a,b}$ to denote the vertical strip $[a,b]\times\mathbb{R}$. Let $v=(v_1,v_2)\in\mathbb{R}^2$. We will say that $E_2(a,b)$ *complements* $E_1(a',b')+v$ if $E_1(a',b')+v$ is positioned above $E_2(a,b)$ so that (up to sets of measure 0) the two sets are disjoint and their union is $S_{a,b}$. In particular, we must have $a'+v_1=a$ and $b'+v_1=b$. We will also say that $F_2(a,b)$ *complements* $F_1(a',b')+v$ if the obvious analogue of the above statement holds. We will write $\widetilde{E}_1(a,b)=$

$S_{a,b} \setminus E_1(a,b)$, and similarly for E_2. Finally, we write $A \sim B$ if the sets A and B are equal up to sets of measure 0.

LEMMA 2. *Let* $0 < s'' < s' < s < 2s''$. *Suppose that* $E_1(a,a+s)+v$, $E_1(a,a+s')+v'$, $E_1(a,a+s'')+v''$ *complement* $E_2(b-s,b)$, $E_2(b-s',b)$, $E_2(b-s'',b)$ *respectively. Then the points* v,v',v'' *are collinear. Moreover, the absolute value of the slope of the line through* v,v'' *is bounded by* $\varepsilon(2s''-s)^{-1}$.

Applying the lemma to the symmetric reflection of E about the line $x_2 = 1/2$, we find that the conclusions of the lemma also hold if we assume that $E_2(a,a+s)+v$, $E_2(a,a+s')+v'$, $E_2(a,a+s'')+v''$ complement $E_1(b-s,b)$, $E_1(b-s',b)$, $E_1(b-s'',b)$ respectively. Furthermore, we may interchange the x_1 and x_2 coordinates and obtain the analogue of the lemma with E_1, E_2 replaced by F_1, F_2.

Proof of Lemma 2. Let $v=(v_1,v_2)$, $v'=(v_1',v_2')$, $v''=(v_1'',v_2'')$. We first observe that if $v_1 = v_1''$, it follows from the assumptions that $v = v''$ and there is nothing to prove. We may therefore assume that $v_1 \neq v_1''$. We do, however, allow $v'=v$ or $v'=v''$.

It follows from the assumptions that $E_2(b-s'',b)$ complements each of $E_1(a,a+s'')+v''$, $E_1(a+s'-s'',a+s')+v'$, $E_1(a+s-s'',a+s)+v$. Hence

$$E_1(a+s'-s'',a+s') \sim E_1(a,a+s'')+(v''-v'),$$
$$E_1(a+s-s'',a+s) \sim E_1(a,a+s'')+(v''-v).$$

Let n be the unit vector perpendicular to $v-v''$ and such that $n_2>0$. For $t\in\mathbb{R}$, let $P_t=\{x : x\cdot n \le t\}$. We define for $0\le c\le c'\le 1$:

$$\alpha_{c,c'} = \inf\{t\in\mathbb{R} : |E_1(c,c')\cap P_t|>0\},$$
$$\beta_{c,c'} = \sup\{t\in\mathbb{R} : |\widetilde{E}_1(c,c')\setminus P_t|>0\}.$$

We will say that x is a *low point* of $E_1(c,c')$ if $x\in S_{c,c'}$, $x\cdot n=\alpha_{c,c'}$, and for any open disc D centered at x we have

$$|D\cap E_1(c,c')|>0. \tag{6}$$

Similarly, we call y a *high point* of $\widetilde{E}_1(c,c')$ if $y\in S_{c,c'}$, $y\cdot n=\beta_{c,c'}$, and for any open disc D centered at y we have

$$|D\cap \widetilde{E}_1(c,c')|>0. \tag{7}$$

It is easy to see that such points x,y actually exist. Indeed, by the definition of $\alpha_{c,c'}$ and an obvious covering argument, for any $\alpha>\alpha_{c,c'}$ there are points x' such that $x'\cdot n\le\alpha$ and that (6) holds for any disc D centered at x'. Thus the set of such points x' has at least one accumulation point x on the line $x\cdot n=\alpha_{c,c'}$. It follows that any such x is a low point of $E_1(c,c')$. The same argument works for y.

The low and high points need not be unique; however, all low points x of $E_1(c,c')$ lie on the same line $x\cdot n=\alpha_{c,c'}$ parallel to the vector $v-v''$, and similarly for high points. Furthermore, the low and high points of $E_1(c,c')$ do not change if $E_1(c,c')$ is modified by a set of measure 0.

Let now $A=E_1(a,a+s'')$, and let x be a low point of A. Since $s<2s''$, we have

$$B:=E_1(a,a+s)=E_1(a,a+s'')\cup E_1(a+s-s'',a+s)\sim A\cup(A+v''-v),$$

hence x is also a low point of B with respect to $v-v''$. Now note that

$$E_1(a+s'-s'',a+s')\sim A+(v''-v')$$

intersects any open neighborhood of $x+(v''-v')$ in positive measure. But on the other hand, $E_1(a+s'-s'',a+s')\subset B$. By the extremality of x in B, $x+(v''-v')$ lies on or above the line segment joining x and $x+(v''-v)$, hence $v''-v'$ lies on or above the line segment joining 0 and $v''-v$.

Repeating the argument in the last paragraph with x replaced by a high point y of $\widetilde{E}_1(a,a+s'')$, we deduce that $v''-v'$ lies on or below the line segment joining 0 and $v''-v$. Hence v,v',v'' are collinear.

Finally, we estimate the slope of the line through v,v''. We have to prove that

$$\frac{2s''-s}{s-s''}\,|v_2''-v_2|\le\varepsilon \tag{8}$$

(recall that $v_1''-v_1=s-s''$). Define x as above, and let $k\in\mathbb{Z}$. Iterating translations by $v-v''$ (in both directions), we find that $x+k(v-v'')$ is a low point of B as long as it belongs to B, i.e. as long as

$$a\le x_1+k(s-s'')\le a+s.$$

The number of such k's is at least $s/(s-s'')-1$. On the other hand, all low points of B lie in the rectangle $a\le x_1\le a+s$, $-\varepsilon\le x_2\le 0$. Hence

$$\left(\frac{s}{s-s''}-2\right)|v_2''-v_2|\le\varepsilon,$$

which is (8). ∎

We return to the proof of Theorem 2. Since E is almost a square, we know roughly how the translates of E can fit together. Locally, any tiling by E is essentially a tiling by a "solid" 1×1 square with "margins" of width between 0 and 2ε (see Fig. 2).

We first locate a "corner". Namely, we may assume that the tiling contains E and its translates $E+u$, $E+v$, where

$$1\le u_1\le 1+2\varepsilon,\qquad -2\varepsilon\le u_2\le 2\varepsilon, \tag{9}$$

$$0\le v_1\le \frac{1}{2}+\varepsilon,\qquad 1\le v_2\le 1+2\varepsilon. \tag{10}$$

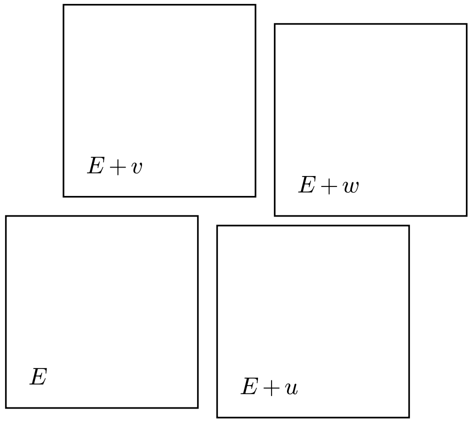

Fig. 2. A "corner" and a fourth near-square

This can always be achieved by translating the tiled plane and taking symmetric reflections of it if necessary.

Let $E + w$ be the translate of E which fits into this corner:

$$v_1 + 1 \leq w_1 \leq v_1 + 1 + 2\varepsilon, \quad u_2 + 1 \leq w_2 \leq u_2 + 1 + 2\varepsilon. \tag{11}$$

We will prove that $w = u + v$ (without the ε-errors).

From (11), (9), (10) we have

$$1 \leq w_1 \leq \frac{3}{2} + 3\varepsilon, \quad -4\varepsilon \leq w_2 - v_2 \leq 4\varepsilon.$$

Observe also that any points (x_1, x_2) between $E+u$ and $E+w$ that belong to tiles other than $E+u$ or $E+w$ must have $x_1 \leq w+\varepsilon$ or $x_1 \geq u+1-\varepsilon$, since otherwise the solid square belonging to the same tile would overlap at least one of the solid squares belonging to $E + u$ or $E + w$. A similar statement holds for $E + v$ and $E + w$. Hence w satisfies both of the following.

(A) $E_1(\varepsilon, 1-(w_1-u_1)-\varepsilon)$ complements $E_2(w_1-u_1+\varepsilon, 1-\varepsilon)+(u-w)$, and

$$1 - (w_1 - u_1) - 2\varepsilon \geq 1 + 1 - \left(\frac{3}{2} + 3\varepsilon\right) - 2\varepsilon = \frac{1}{2} - 5\varepsilon, \quad |w_1 - v_1 - 1| \leq 2\varepsilon.$$

(B) $-4\varepsilon \leq w_2 - v_2 \leq 4\varepsilon$, $u_2 + 1 \leq w_2 \leq u_2 + 1 + 2\varepsilon$, and $F_2(r, t)$ complements $F_1(\widetilde{r}, \widetilde{t}\,) + (w - v)$, where

$$r = \max(0, w_2 - v_2) + \varepsilon, \qquad \widetilde{r} = \max(0, v_2 - w_2) + \varepsilon,$$
$$t = 1 - \max(0, v_2 - w_2) - \varepsilon, \quad \widetilde{t} = 1 - \max(0, w_2 - v_2) - \varepsilon.$$

If $w = u + v$, we have $w - u = v$, $w - v = u$, hence by considering the "corner" $E, E + u, E + v$ we see that both (A) and (B) hold. Assuming that ε is small enough, we shall prove that:

1° All points w satisfying (A) lie on a fixed straight line l_1 with slope m_1, where $|m_1| \le \varepsilon(1/2 - 9\varepsilon)^{-1}$.

2° All points w satisfying (B) lie on a fixed straight line l_2 with slope m_2, where $|m_2| \ge \varepsilon^{-1}(1 - 8\varepsilon)$.

If $\varepsilon < (13 - 3\sqrt{3})/142 \approx 0.05496$ (the smaller root of the equation $71\varepsilon^2 - 13\varepsilon + 1/2 = 0$), the upper bound for $|m_1|$ is less than the lower bound for $|m_2|$. It follows that there can be at most one w which satisfies both (A) and (B), since l_1 and l_2 intersect only at one point. Consequently, if $E + w$ is the translate of E chosen as above, we must have $w = u + v$. Now it is easy to see that $E + \Lambda$ is a tiling, where Λ is the lattice $\{ku + mv : k, m \in \mathbb{Z}\}$.

We first prove 1°. Suppose that $w, w', w'', \ldots$ (not necessarily all distinct) satisfy (A). By the assumptions in (A), we may apply Lemma 2 with E_1 and E_2 interchanged and with $a = 0$, $b = 1$, $s = 1 - (w_1 - u_1)$, $s' = 1 - (w_1' - u_1), \ldots \ge 1/2 - 5\varepsilon$. From the second inequality in (A) and the triangle inequality we also have $|s - s''| \le 4\varepsilon$. We find that all w satisfying (A) lie on a line l_1 with slope bounded by

$$\frac{\varepsilon}{|2s'' - s|} \le \frac{\varepsilon}{s'' - |s'' - s|} \le \frac{\varepsilon}{1/2 - 9\varepsilon}.$$

To prove 2°, we let w, w', w'' be three (not necessarily distinct) points satisfying (B) and such that $w_2 \le w_2' \le w_2''$. Observe that $r \le r' \le r''$ and $t \ge t' \ge t''$ (the notation is self-explanatory). We then apply the obvious analogue of Lemma 2 with E_1, E_2 replaced by F_1, F_2 and with $a = r''$, $s = t - r''$, $s' = t' - r''$, $s'' = t'' - r''$, $b = \widetilde{t}''$. From the estimates in (B) we have

$$|s - s''| = |t - t''| \le |w_2 - w_2''| \le 2\varepsilon,$$
$$s'' = t'' - r'' = 1 - \max(0, v_2 - w_2'') - \max(0, w_2'' - v_2) - 2\varepsilon \ge 1 - 6\varepsilon,$$

hence $|2s'' - s| \ge s'' - |s - s''| \ge 1 - 8\varepsilon$. We conclude that all w satisfying (B) lie on a line l_2 such that the inverse of the absolute value of its slope is bounded by $\varepsilon/(1 - 8\varepsilon)$. ■

Proof of Corollary 1. Let $Q = [0,1] \times [0,1]$. By rescaling, it suffices to prove that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $E \subset Q$, E tiles $\mathbb{R}^2$ by translations, and $|E| \ge 1 - \delta$, then E contains the square

$$Q_\varepsilon = [\varepsilon, 1 - \varepsilon] \times [\varepsilon, 1 - \varepsilon]$$

(up to sets of measure 0). The result then follows from Theorem 2.

Let E be as above, and suppose that $Q_\varepsilon \setminus E$ has positive measure. Since E tiles $\mathbb{R}^2$, there is a $v \in \mathbb{R}^2$ such that $|E \cap (E + v)| = 0$ and $|Q_\varepsilon \cap (E + v)| > 0$. We then have

$$|E \cup (E + v)| = |E| + |E + v| \ge 2 - 2\delta,$$

but also

$$|E \cup (E + v)| \le |Q \cup (Q + v)| \le 2 - \varepsilon^2,$$

since $E \subset Q$, $E+v \subset Q+v$, and $Q_\varepsilon \cap (Q+v) \neq \emptyset$ so that $|Q \cap (Q+v)| \ge \varepsilon^2$. This is a contradiction if δ is small enough. ∎

4. A counterexample in higher dimensions. In this section we prove Theorem 3. It suffices to construct E for $n = 3$, since then $E \times [0,1]^{n-3}$ is a subset of $\mathbb{R}^n$ with the required properties.

Let (x_1, x_2, x_3) denote the Cartesian coordinates in $\mathbb{R}^3$. It will be convenient to rescale E so that $[\varepsilon, 1]^3 \subset E \subset [0, 1+\varepsilon]^3$.

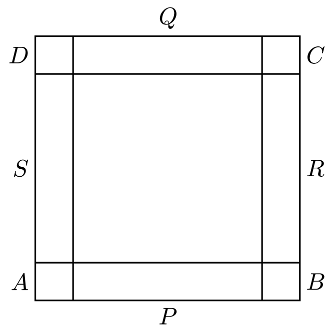

Fig. 3. The construction of E

We construct E as follows. We let E be bounded from below and above by the planes $x_3 = 0$ and $x_3 = 1$ respectively. The planes $x_1 = \varepsilon$, $x_1 = 1$, $x_2 = \varepsilon$, $x_2 = 1$ divide the cube $[0, 1+\varepsilon]^3$ into 9 parts (Figure 3). The middle part is entirely contained in E. We label by A, B, C, D, P, Q, R, S the remaining 8 segments as shown in Figure 3. We then let

$$E \cap P = P \cap \{0 \le x_3 \le 1/8 \text{ or } 1/2 \le x_3 \le 5/8\},$$
$$E \cap R = R \cap \{0 \le x_3 \le 1/8 \text{ or } 1/2 \le x_3 \le 5/8\},$$
$$E \cap Q = Q \cap \{0 \le x_3 \le 1/4 \text{ or } 3/8 \le x_3 \le 3/4 \text{ or } 7/8 \le x_3 \le 1\},$$
$$E \cap S = S \cap \{0 \le x_3 \le 1/4 \text{ or } 3/8 \le x_3 \le 3/4 \text{ or } 7/8 \le x_3 \le 1\},$$
$$E \cap A = A \cap \{0 \le x_3 \le 1/16\},$$
$$E \cap C = A \cap \{1/2 \le x_3 \le 9/16\},$$
$$E \cap B = B \cap \{5/16 \le x_3 \le 3/4\},$$
$$E \cap D = D \cap \{0 \le x_3 \le 1/4 \text{ or } 13/16 \le x_3 \le 1\}.$$

We also define $K = \bigcup_{j \in \mathbb{Z}} (E + (0, 0, j))$.

Let $E + T$ be a tiling of $\mathbb{R}^3$, and assume that $0 \in T$. Suppose that $E + v$ and $E + w$ are neighbors in this tiling so that the vertical sides of $(E \cap P) + v$ and $(E \cap Q) + w$ meet in a set of non-zero two-dimensional measure. Then we must have $v - w = (0, 1, (v - w)_3)$, where $(v - w)_3 \in \{\pm 1/4, \pm 3/4\}$. A similar statement holds with P, Q replaced by R, S and with the x_1, x_2 coordinates interchanged. We deduce that the tiling consists of copies of E stacked into identical vertical "columns" $K_{ij} = K + (i, j, t_{ij})$, arranged in a rectangular grid in the x_1x_2 plane and shifted vertically so that $t_{i+1,j} - t_{ij}$ and $t_{i,j+1} - t_{ij}$ are always $\pm 1/4$. We will use matrices (t_{ij}) to encode such a tiling or portions thereof.

It is easy to see that (t_{ij}), where $t_{ij} = 0$ if $i + j$ is even and $1/4$ if $i + j$ is odd, is indeed a tiling. It remains to show that E does not admit a lattice tiling. Indeed, the four possible choices of the generating vectors in any lattice (t_{ij}) with $t_{ij} = \pm 1/4$ produce the configurations

$$\begin{pmatrix} 0 & t \\ t & 2t \end{pmatrix}, \quad \begin{pmatrix} 2t & t \\ t & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}.$$

But it is easy to see that the corners A, B, C, D do not match if so translated.

References

[1] B. Fuglede, *Commuting self-adjoint partial differential operators and a group-theoretic problem*, J. Funct. Anal. 16 (1974), 101–121.

[2] D. Girault-Beauquier and M. Nivat, *Tiling the plane with one tile*, in: Topology and Category Theory in Computer Science, G. M. Reed, A. W. Roscoe and R. F. Wachter (eds.), Oxford Univ. Press, 1989, 291–333.

[3] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, Freeman, New York, 1987.

[4] A. Iosevich, N. H. Katz and S. Pedersen, *Fourier bases and a distance problem of Erdős*, Math. Res. Lett. 6 (1999), 251–255.

[5] A. Iosevich, N. H. Katz and T. Tao, *Convex bodies with a point of curvature do not have Fourier bases*, Amer. J. Math. 123 (2001), 115–120.

[6] —, —, —, *Fuglede conjecture holds for convex planar domains*, Math. Res. Lett., to appear.

[7] P. Jorgensen and S. Pedersen, *Spectral pairs in Cartesian coordinates*, J. Fourier Anal. Appl. 5 (1999), 285–302.

[8] R. Kenyon, *Rigidity of planar tilings*, Invent. Math. 107 (1992), 637–651.

[9] M. N. Kolountzakis, *Non-symmetric convex domains have no basis of exponentials*, Illinois J. Math. 44 (2000), 542–550.

[10] —, *Packing, tiling, orthogonality and completeness*, Bull. London Math. Soc. 32 (2000), 589–599.

[11] —, *On the structure of multiple translational tilings by polygonal regions*, Discrete Comput. Geom. 23 (2000), 537–553.

[12] M. N. Kolountzakis and J. C. Lagarias, *Structure of tilings of the line by a function*, Duke Math. J. 82 (1996), 653–678.

[13] M. N. Kolountzakis and M. Papadimitrakis, *A class of non-convex polytopes that admit no orthonormal basis of exponentials*, Illinois J. Math. 46 (2002), 1227–1232.

[14] I. Łaba, *Fuglede's conjecture for a union of two intervals*, Proc. Amer. Math. Soc. 121 (2001), 2965–2972.

[15] —, *The spectral set conjecture and multiplicative properties of roots of polynomials*, J. London Math. Soc. (2) 65 (2002), 661–671.

[16] J. Lagarias and Y. Wang, *Tiling the line with translates of one tile*, Invent. Math. 124 (1996), 341–365.

[17] —, —, *Spectral sets and factorization of finite abelian groups*, J. Funct. Anal. 73 (1997), 122–134.

[18] P. McMullen, *Convex bodies which tile the space by translation*, Mathematika 27 (1980), 113–121.

[19] B. A. Venkov, *On a class of Euclidean polyhedra*, Vestnik Leningrad. Univ. Ser. Mat. Fiz. Khim. 9 (1954), no. 2, 11–31 (in Russian).

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