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Left quotients of a C^*-algebra, III: Operators on left quotients

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Abstract. Let L be a norm closed left ideal of a C^*-algebra A. Then the left quotient A/L is a left A-module. In this paper, we shall implement Tomita's idea about representing elements of A as left multiplications: $\pi_p(a)(b+L) = ab + L$. A complete characterization of bounded endomorphisms of the A-module A/L is given. The double commutant $\pi_p(A)''$ of $\pi_p(A)$ in $B(A/L)$ is described. Density theorems of von Neumann and Kaplansky type are obtained. Finally, a comprehensive study of relative multipliers of A is carried out.

1. Introduction. Let A be a C^*-algebra with Banach dual A^* and double dual A^{**}. We also consider A^{**} as the enveloping W^*-algebra of A, as usual. Let L be a norm closed left ideal of A. The quotient A/L is a Banach space. Let $B(A/L) = B(A/L, A/L)$ be the Banach algebra of bounded linear operators from A/L into A/L. In [17, 18], Tomita initiated a program to study the left regular representation π_p of A on the Banach space A/L. More precisely, he considered the *Banach algebra* representation of A,

$$\pi_p : A \to B(A/L),$$

defined by

$$\pi_p(a)(b+L) = ab + L, \qquad a, b \in A.$$

The objective of this paper is to answer the following three questions raised by Tomita [18].

Q1: How do we describe $\pi_p(A)$? In other words, which properties of an operator T in $B(A/L)$ characterize that $T = \pi_p(t)$ for some t in A?

Q2: How do we describe the commutant $\pi_p(A)'$ and the double commutant $\pi_p(A)''$ of $\pi_p(A)$ in $B(A/L)$? Note that the commutant

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$\pi_p(A)' = \{T \in B(A/L) : T\pi_p(a) = \pi_p(a)T$ for all $a \in A\}$ is the Banach algebra of bounded A-module maps when we consider A/L as a left A-module.

Q3: Do we have density theorems of von Neumann and Kaplansky type in this context? In other words, is it true that $\pi_p(A)$ (resp. its unit ball) is dense in $\pi_p(A)''$ (resp. its unit ball)?

In [17, 18], Tomita tried to represent elements of A/L as vector sections (he called them "vector fields") over a compact subset of the state space $S(A)$ (assuming that the C^*-algebra A has an identity). In [17], he defined the notion of a "vector field" as "a mapping of a state space into the dual space of the algebra which satisfies a suitable norm condition". However, due to insufficient tools, "unlike in abelian case, even in a compact space of pure states, the corresponding quotient space of non-commutative algebra A may not generally be represented as the totality of continuous fields on that space". Thus, his treatment in [18] of the left regular representation π_p based on his vector section representation does not work in general.

In Part I [20] of this series of papers, the second author offered another approach. It is well-known that closed left ideals L of a C^*-algebra A are in one-to-one correspondence with closed projections p in A^{**} such that A/L is isometrically isomorphic to Ap as Banach spaces and also as left A-modules (see Section 3). For an arbitrary closed projection p in A^{**} (and thus for an arbitrary closed left ideal L of A), we use the weak* closed face $F(p)$ of the quasi-state space $Q(A)$ of A supported by p as the base space. We implement, in addition to the norm conditions of Tomita, an affine structure of vector sections. Then it was established that the quotient space A/L ($\cong Ap$) is isometrically isomorphic to the Banach space of all continuous admissible vector sections over $F(p)$ (see Theorem 3.4). Based on these new techniques, we are able to provide in this paper more satisfactory answers to the above three questions.

We begin with the W^*-algebra version in Section 2, in which we completely answer all three questions stated above. For example, if p is a (necessarily closed) projection in a W^*-algebra M then $\pi_p(M)'$ consists of right multiplications induced by elements of pMp and $\pi_p(M)'' = \pi_p(M)$ (Theorem 2.3). In particular, all M-module maps T in $B(Mp)$ are of the form $T(xp) = xptp$ for some t in M.

However, the C^*-algebra case is much more difficult (due to lack of projections) and we need to develop some new tools. In [20], elements bp of the Banach space Ap are interpreted as Hilbert space vector sections over $F(p)$. The main idea in this paper is to represent Banach space operators $\pi_p(a)$ in $B(Ap)$ as Hilbert space operator sections (Definition 3.7), which is developed in Section 3. In particular, an operator T in $B(Ap)$ is said to be *decomposable* if T can be represented by an operator section (Definition 3.10). A simple way to verify the decomposability of T is to check if the condition $\varphi(a^*a) = 0$ ensures $\varphi((Tap)^*(Tap)) = 0$ whenever φ is a pure state supported by p and $a \in A$ (Theorem 3.13). In this case, T has to be a $\pi_p(t)$ for some t in $\mathrm{LM}(A, p) = \{x \in A^{**} : xAp \subseteq Ap\}$ (Corollary 3.14). This answers our first question **Q1**.

Various relative multipliers of A associated to p play important roles in the theory of left regular representations. Beside $\mathrm{LM}(A, p)$, we shall introduce and study $\mathrm{RM}(A, p)$, $\mathrm{M}(A, p)$ and $\mathrm{QM}(A, p)$ in Section 4. They behave in a similar way as the sets $\mathrm{LM}(A)$, $\mathrm{RM}(A)$, $\mathrm{M}(A)$ and $\mathrm{QM}(A)$ of classical multipliers of A. For example, they are closures of A in A^{**} under corresponding relative strict topologies (Theorem 4.3). The object studied by Tomita in [18] is essentially the closure of $\pi_p(A)$ in $B(Ap)$ with respect to the so-called quotient(-double) strong topology, or Q^*-topology. In fact, the Q^*-topology is induced by the relative strict topology of A^{**}. Thus, the closure of the Banach algebra $\pi_p(A)$ in $B(Ap)$ in the Q^*-topology is the image of the *C^*-algebra* $\mathrm{M}(A, p) = \{x \in A^{**} : xAp \subseteq Ap,\ pAx \in pA\}$ under π_p (see Remark 4.5). Tomita expected that the double commutant $\pi_p(A)''$ of $\pi_p(A)$ in $B(Ap)$ coincides with $\pi_p(\mathrm{M}(A, p))$. This is, however, not always true for an arbitrary projection p. In some important cases, we do have $\pi_p(A)'' = \pi_p(\mathrm{LM}(A, p))$ (Theorem 4.8). A counterexample is Example 4.9. This partially answers our second question **Q2**.

The classical density theorems of von Neumann and Kaplansky have counterparts in this context. Also in Section 4, we show that $\pi_p(A)$ (resp. its unit ball) is dense in $\pi_p(\mathrm{LM}(A, p))$ (resp. its unit ball) in the strong operator topology (SOT) as well as the weak operator topology (WOT) of $B(Ap)$ (Theorem 4.4). This answers our last question **Q3**.

It is then interesting and useful to find a C^*-subalgebra $\mathcal{A}$ of A^{**} such that $\mathrm{LM}(A, p) = \mathrm{LM}(\mathcal{A})$, $\mathrm{RM}(A, p) = \mathrm{RM}(\mathcal{A})$, $\mathrm{M}(A, p) = \mathrm{M}(\mathcal{A})$ and $\mathrm{QM}(A, p) = \mathrm{QM}(\mathcal{A})$, and thus all good tools of multipliers apply (see e.g. [5]). Several examples and results are provided in Section 5 for the investigation of what $\mathcal{A}$ should consist of (see especially Theorem 5.3).

Finally, we remark that the atomic part of Ap is studied in Part II [9] of this series of papers. Some interesting and new results in this direction are obtained in Section 6. For example, we show that if x is in A^{**} and $\pi_p(x)$ preserves continuous atomic parts, i.e., $z_{\mathrm{at}}xAp \subseteq z_{\mathrm{at}}Ap$, then $z_{\mathrm{at}}xc(p) \in z_{\mathrm{at}}\,\mathrm{LM}(A, p)$, where z_{at} is the maximal atomic projection in A^{**}, and $c(p)$ is the central support of p in A^{**} (Theorem 6.2). In particular, when $p = 1$, we have $z_{\mathrm{at}}x = z_{\mathrm{at}}l$ for some left multiplier l of A whenever $z_{\mathrm{at}}xA \subseteq z_{\mathrm{at}}A$ (Corollary 6.3). This supplements results of Shultz [16] and Brown [7]. Similar results are obtained for other relative multipliers as well.

2. The left regular representation of a W^*-algebra. We provide a new elementary proof of the following result of Tomita [18].

THEOREM 2.1 ([18]). *Let π be a bounded homomorphism from a C^*-algebra A into a Banach algebra B. Then $\pi(A)$ is topologically isomorphic to $A/\ker\pi$. If $\|\pi\| \le 1$, then $\pi(A)$ is isometrically isomorphic to $A/\ker\pi$.*

Proof. The kernel of π is a closed two-sided ideal of A. Since closed two-sided ideals of a C^*-algebra are automatically self-adjoint, by passing to the quotient, we can assume π is one-to-one. Assume that k is a positive number such that

$$\|\pi(a)\| \le k\|a\|$$

for all a in A. It suffices to show that $\|\pi(a)\| \ge \frac{1}{k}\|a\|$ for all a in A. If $k = 1$, then π is an isometry.

First assume that a is a positive element of A. We claim that $\|\pi(a)\| \ge \|a\|$. Since A is a C^*-algebra and B is a Banach algebra,

$$\|a\| = r_\sigma(a) \quad \text{and} \quad \|\pi(a)\| \ge r_\sigma(\pi(a)),$$

where r_σ denotes the spectral radius. We shall verify for the spectra that $\sigma(a) \subseteq \sigma(\pi(a)) \cup \{0\}$. For any positive λ in $\sigma(a)$ and $0 < \varepsilon < \lambda$, let f be a continuous real-valued function on the compact set $\sigma(a)$ such that $f = 1$ on $[\lambda - \varepsilon/2, \lambda + \varepsilon/2] \cap \sigma(a)$, $f = 0$ outside $(\lambda - \varepsilon, \lambda + \varepsilon)$ and $0 \le f \le 1$. In a similar manner, we can choose another continuous real-valued function g on $\sigma(a)$ such that $fg = g \ne 0$. Let $x = f(a)$ and $y = g(a)$. We have $x, y \in A$ and $xy = y \ne 0$. It follows that $\pi(x)\pi(y) = \pi(y) \ne 0$. Therefore, $\|\pi(x)\| \ge 1$. Now, $\|(a-\lambda)x\| < \varepsilon$ implies $\|(\pi(a) - \lambda)\pi(x)\| = \|\pi((a-\lambda)x)\| < k\varepsilon$. The fact that ε can be arbitrarily small ensures $\lambda \in \sigma(\pi(a))$, as asserted. Hence,

$$\|\pi(a)\| \ge r_\sigma(\pi(a)) \ge r_\sigma(a) = \|a\|$$

for all positive a in A.

In general, if $a \in A$ and $a \ne 0$,

$$\|\pi(a)\| \ge \frac{\|\pi(a^*a)\|}{\|\pi(a^*)\|} \ge \frac{\|a^*a\|}{\|\pi(a^*)\|} \ge \frac{\|a\|^2}{k\|a\|} = \frac{1}{k}\|a\|. \ \blacksquare$$

Let p be a projection (all projections in this paper are assumed self-adjoint) in a W^*-algebra M. Let $c(p)$ be the central support of p in M. In other words, $c(p)$ is the minimum central projection in M such that $pc(p) = c(p)p = p$. Recall that π_p is the left regular representation of M into $B(Mp)$, i.e.,

$$\pi_p(x)yp = xyp, \quad y \in M.$$

Clearly, $\pi_p(c(p)) = 1$ in $B(Mp)$. Hence, $\pi_p(t) = \pi_p(tc(p))$ for all t in M, and in fact $\ker\pi_p = M(1 - c(p))$.

LEMMA 2.2. *Suppose $T \in B(Mp)$. Then T commutes with all right multiplications R_{pxp} for x in M if and only if there is a t in M such that $T = \pi_p(t)$. In this case, $\|T\| = \|tc(p)\|$.*

Proof. We shall just verify necessity. Assume $T \in B(Mp)$ such that $TR_{pxp} = R_{pxp}T$ for all $x \in M$. For every central projection z in M, we have

$$\begin{aligned} T(zxp) &= T(xp(pzp)) = T(R_{pzp}(xp)) \\ &= R_{pzp}(T(xp)) = (Txp)pzp = z(Txp), \qquad x \in M. \end{aligned}$$

In particular, $T(zMp) \subseteq zMp$. By passing to $c(p)M$, we can assume $c(p) = 1$ and π_p is an isometry by Theorem 2.1.

Let

$$\mathcal{S} = \{S \in B(Mp) : SR_{pxp} = R_{pxp}S, \forall x \in M\},$$
$$\mathcal{Q} = \{q \in M : q \text{ is a projection and } S\pi_p(q) \in \pi_p(M), \forall S \in \mathcal{S}\}.$$

CLAIM 1. $p \in \mathcal{Q}$.

For S in $\mathcal{S}$, let $s = S(p) \in Mp$. We have

$$\begin{aligned} \pi_p(s)(xp) &= sxp = S(p)(pxp) = R_{pxp}S(p) \\ &= S(R_{pxp}(p)) = S(pxp) = S\pi_p(p)(xp) \end{aligned}$$

for all xp in Mp. Therefore, $S\pi_p(p) = \pi_p(s) \in \pi_p(M)$. Hence, $p \in \mathcal{Q}$.

CLAIM 2. *$\mathcal{Q}$ is hereditary under the quasi-ordering $\precsim$ of projections.*

Suppose $q \in \mathcal{Q}$ and $r \precsim q$. In other words, $r = v^*v$ and $vv^* \le q$ for some partial isometry v in M. Note that $r = v^*qv$. Since $\pi_p(v^*)$ is in $\mathcal{S}$, the operator $S\pi_p(v^*)$ belongs to $\mathcal{S}$ whenever S does. As $q \in \mathcal{Q}$, for each S in $\mathcal{S}$ there is an s' in M such that

$$(S\pi_p(v^*))\pi_p(q) = \pi_p(s').$$

Consequently,

$$S(rxp) = S(v^*qvxp) = S\pi_p(v^*)\pi_p(q)(vxp) = s'vxp, \qquad \forall x \in M.$$

Set $s'' = s'v$. We have

$$S\pi_p(r) = \pi_p(s'') \in \pi_p(M).$$

Hence $r \in \mathcal{Q}$. Therefore, $\mathcal{Q}$ is hereditary under $\precsim$ and, in particular, $\mathcal{Q}$ contains all projections q such that $q \precsim p$ by Claim 1.

CLAIM 3. *$\mathcal{S}$ is directed under the ordering $\le$ of projections.*

We are going to show that $\mathcal{Q}$ is even a lattice. First, it is clear that if $q_1, \dots, q_n$ in Q are mutually orthogonal then $q_1 + \cdots + q_n \in \mathcal{Q}$. Moreover, if $q_1, q_2 \in \mathcal{Q}$, we have

$$q_1 \vee q_2 - q_1 \sim q_2 - q_1 \wedge q_2 \le q_2.$$

Hence $q_1 \vee q_2 - q_1 \in \mathcal{Q}$ by Claim 2, and consequently we have $q_1 \vee q_2 = (q_1 \vee q_2 - q_1) + q_1 \in \mathcal{Q}$.

Associate to each q in $\mathcal{Q}$ a t_q in M such that

$$T\pi_p(q) = \pi_p(t_q).$$

Then $\|t_q\| = \|\pi_p(t_q)\| \leq \|T\|$ because π_p is an isometry. Since the net $\{t_q : q \in \mathcal{Q}\}$ is bounded in the W^*-algebra M, some subnet (t_{q_λ}) converges to some t in M with respect to the $\sigma(M, M_*)$ topology. For every xp in Mp, let q_x be the range projection of xp. Then $q_x \in \mathcal{Q}$ since $q_x \lesssim p$. Consequently, for large enough λ, we have $q_x \leq q_\lambda$ and thus

$$T(xp) = T(q_\lambda xp) = T\pi_p(q_\lambda)(xp) = t_{q_\lambda} xp.$$

It follows that

$$txp = \lim t_{q_\lambda} xp = T(xp), \qquad \forall x \in M.$$

Hence $\pi_p(t) = T$. Finally, $\|t\| = \|\pi_p(t)\| = \|T\|$ since π_p is an isometry. ∎

THEOREM 2.3. *Let M be a W^*-algebra, p a projection in M and π_p the left regular representation of M on Mp. Then the commutant of $\pi_p(M)$ in $B(Mp)$ is*

$$\pi_p(M)' = \{R_{ptp} : t \in M\},$$

and the double commutant is

$$\pi_p(M)'' = \overline{\pi_p(M)}^{\mathrm{SOT}} = \overline{\pi_p(M)}^{\mathrm{WOT}} = \pi_p(M).$$

Proof. Suppose $T \in \pi_p(M)'$. Let $Tp = tp \in Mp$. Now

$$Txp = T\pi_p(x)p = \pi_p(x)Tp = \pi_p(x)(tp) = xtp, \qquad \forall x \in M.$$

Since $(1-p)p = 0$, we must have $(1-p)tp = 0$, i.e., $tp = ptp$. Consequently, $T = R_{ptp}$. The opposite inclusion is obvious and thus we have $\pi_p(M)' = \{R_{ptp} : t \in M\}$. Since the double commutant of any subset of $B(Mp)$ is closed in both the strong operator topology (SOT) and the weak operator topology (WOT) of $B(Mp)$, the second assertion follows from Lemma 2.2. ∎

3. The left regular representation of a C^*-algebra.

Let

$$S(A) = \{\varphi \in A^* : \varphi \geq 0,\ \|\varphi\| = 1\}$$

be the state space and

$$Q(A) = \{\varphi \in A^* : \varphi \geq 0,\ \|\varphi\| \leq 1\}$$

be the quasi-state space of A equipped with the weak* topology. $Q(A)$ is a weak* compact convex set. A convex subset F of $Q(A)$ is called a *face* if both φ and ψ belong to F whenever $\varphi, \psi \in Q(A)$ and $\lambda\varphi + (1-\lambda)\psi \in F$ for some $0 < \lambda < 1$.

Recall that a projection p in A^{**} is *closed* if and only if the face

$$F(p) = \{\varphi \in Q(A) : \varphi(1-p) = 0\}$$

of $Q(A)$ supported by p is weak* closed. The relation

$$L = A^{**}(1-p) \cap A$$

establishes a one-to-one correspondence between closed projections in A^{**} and norm closed left ideals of A. Also, $L^{**} = A^{**}(1-p)$. Moreover, we have isometrical isomorphisms

$$a + L \mapsto ap \quad \text{and} \quad x + L^{**} \mapsto xp$$

under which

$$A/L \cong Ap \quad \text{and} \quad (A/L)^{**} \cong A^{**}/L^{**} \cong A^{**}p,$$

respectively, as Banach spaces and also as left A-modules ([12, 15, 1], see also [14, 3.11.9]).

From now on, p is always the unique closed projection in A^{**} associated to the norm closed left ideal $L = A^{**}(1-p) \cap A$. For simplicity of notation, we write Ap for the left quotient A/L of the C^*-algebra A by L. Consequently, its Banach double dual $A^{**}p$ is the quotient A^{**}/L^{**}. Denote by π_p the left regular representation of A on Ap defined by $\pi_p(a)bp = abp$ (or equivalently, $\pi_p(a)(b+L) = ab + L$). As usual, π_p can be extended to the left regular representation of A^{**} into $B(A^{**}p)$, denoted again by π_p, such that $\pi_p(x)yp = xyp$ (or equivalently, $\pi_p(x)(y + L^{**}) = xy + L^{**}$).

We note that

$$\varphi(x) = \varphi(px) = \varphi(xp) = \varphi(pxp), \qquad \forall x \in A^{**}, \forall \varphi \in F(p).$$

Let $\varphi \in F(p) \setminus \{0\}$. The GNS construction yields a cyclic representation $(\pi_\varphi, H_\varphi, \omega_\varphi)$ of A such that $\overline{\pi_\varphi(A)\omega_\varphi} = H_\varphi$ and $\varphi(x) = \langle \pi_\varphi(x)\omega_\varphi, \omega_\varphi \rangle_\varphi$ for all x in A^{**}. Here π_φ also denotes the canonical extension of π_φ to A^{**}, and $\langle \cdot, \cdot \rangle_\varphi$ is the inner product of the Hilbert space H_φ (see, e.g., [10, 2.4.4]). Set $H_\varphi = \{0\}$ for $\varphi = 0$.

NOTATION. Write $x\omega_\varphi$ for $\pi_\varphi(x)\omega_\varphi$ in H_φ for all $x \in A^{**}$ and $\varphi \in F(p)$.

There is a linear embedding of $A^{**}p$ into the product space $\prod_{\varphi \in F(p)} H_\varphi$ defined by associating to each xp in $A^{**}p$ the vector section $(x\omega_\varphi)_{\varphi \in F(p)}$ in $\prod_{\varphi \in F(p)} H_\varphi$. Note that the fiber Hilbert spaces H_φ are not totally independent. In fact, we have

LEMMA 3.1 ([20, 2.3]). *For φ, ψ in $F(p)$ such that $0 \leq \psi \leq \lambda\varphi$ for some $\lambda > 0$, we can define a bounded linear map*

$$T_{\psi\varphi} : H_\varphi \to H_\psi$$

by sending $a\omega_\varphi$ to $a\omega_\psi$ for all $a \in A$. Moreover, $\|T_{\psi\varphi}\|^2 \leq \lambda$ and

$$T_{\psi\varphi}(x\omega_\varphi) = x\omega_\psi, \qquad \forall x \in A^{**}.$$

Definition 3.2 ([20, 2.4]). A vector section $(x_\varphi)_\varphi$ in $\prod_{\varphi\in F(p)} H_\varphi$ is said to be *admissible* if

$$T_{\psi\varphi}x_\varphi = x_\psi$$

whenever $\varphi, \psi \in F(p)$ and $0 \leq \psi \leq \lambda\varphi$ for some $\lambda > 0$.

Clearly, each xp in $A^{**}p$ induces an admissible vector section $(x\omega_\varphi)_\varphi$ in $\prod_{\varphi\in F(p)} H_\varphi$. They are exactly all of them.

Theorem 3.3 ([20, 3.1]). *The image of the linear embedding $xp \mapsto (x\omega_\varphi)_\varphi$ of $A^{**}p$ into $\prod_{\varphi\in F(p)} H_\varphi$ coincides with the set of all admissible vector sections in $\prod_{\varphi\in F(p)} H_\varphi$. Moreover,*

$$\|xp\| = \sup_{\varphi\in F(p)} \|x\omega_\varphi\|_{H_\varphi}.$$

In particular, admissible vector sections are automatically bounded.

It is natural to ask which properties characterize those admissible vector sections arising from elements of Ap. Recall the notion of a continuous field of Hilbert spaces [13, 11]. We equip $F(p)$ with the weak* topology inherited from A^*. Note that $\{a\omega_\varphi : a \in A\}$ is norm dense in H_φ for all $\varphi \in F(p)$, and the norm functions $\varphi \mapsto \|a\omega_\varphi\|_\varphi = \varphi(a^*a)^{1/2}$ are continuous on $F(p)$ for a in A. Consequently, the image of Ap under the embedding $A^{**}p \hookrightarrow \prod_{\varphi\in F(p)} H_\varphi$ defines a continuous structure of the field of Hilbert spaces $(F(p), \{H_\varphi\}_\varphi)$ with base space $F(p)$ and fiber Hilbert spaces H_φ for all $\varphi \in F(p)$. In this context:

- A vector section $(x_\varphi)_{\varphi\in F(p)}$ in $\prod_{\varphi\in F(p)} H_\varphi$ is *bounded* if
$$\sup_{\varphi\in F(p)} \|x_\varphi\|_{H_\varphi} < \infty.$$
- A bounded vector section $(x_\varphi)_{\varphi\in F(p)}$ is *weakly continuous* if
$$\varphi \mapsto \langle x_\varphi, a\omega_\varphi\rangle_\varphi \quad \text{is continuous on } F(p) \text{ for all } ap \text{ in } Ap.$$
- A weakly continuous vector section $(x_\varphi)_{\varphi\in F(p)}$ is *continuous* if
$$\varphi \mapsto \langle x_\varphi, x_\varphi\rangle_\varphi \quad \text{is also continuous on } F(p).$$

Let us denote the continuous field of Hilbert spaces thus obtained by $(F(p), \{H_\varphi\}_\varphi, Ap)$. The following result says that there are no more continuous *admissible* vector sections in $(F(p), \{H_\varphi\}_\varphi, Ap)$ other than those arising from elements of Ap.

Theorem 3.4 ([20, 3.2]). *The image of Ap under the linear embedding $xp \mapsto (x\omega_\varphi)_\varphi$ of $A^{**}p$ into $\prod_{\varphi\in F(p)} H_\varphi$ coincides with the set of all continuous admissible vector sections in the continuous field of Hilbert spaces*

$(F(p), \{H_\varphi\}_\varphi, Ap)$. *Consequently,*

$$Ap = \{xp \in A^{**}p : \varphi \mapsto \langle x\omega_\varphi, x\omega_\varphi\rangle_\varphi = \varphi(x^*x) \text{ and } \varphi \mapsto \langle x\omega_\varphi, a\omega_\varphi\rangle_\varphi = \varphi(a^*x) \text{ are continuous on } F(p),\ \forall a \in A\}.$$

Let $\mathcal{W}_p$ be the set of weakly continuous admissible vector sections in $(F(p), \{H_\varphi\}_\varphi, Ap)$. In other words,

$$\mathcal{W}_p = \{xp \in A^{**}p : \varphi \mapsto \langle x\omega_\varphi, a\omega_\varphi\rangle_\varphi = \varphi(a^*x) \text{ is continuous on } F(p),\ \forall a \in A\}.$$

The following extension of Kadison function representation is useful for our work. The classical one deals with the case $p = 1$ (see, e.g., [14, 3.10.3]). In the following, A_{sa} (resp. A^{**}_{sa}) denotes the set of all self-adjoint elements of A (resp. A^{**}).

Proposition 3.5 ([5, 3.5]). *$pA_{\mathrm{sa}}p$ (resp. $pA^{**}_{\mathrm{sa}}p$) is isometrically linear and order isomorphic to the Banach space of all continuous (resp. bounded) real affine functionals of $F(p)$ vanishing at zero. In particular, for any x in A^{**}, we have*

$$pxp \in pAp \quad \text{if and only if} \quad \varphi \mapsto \varphi(pxp) = \varphi(x) \text{ is continuous on } F(p).$$

Corollary 3.6 ([20, 4.1]). *Let $xp \in A^{**}p$.*

(1) $\mathcal{W}_p = \{xp \in A^{**}p : pa^*xp \in pAp$ *for all* $a \in A\}$.
(2) $Ap = \{xp \in A^{**}p : px^*xp \in pAp$ *and* $pa^*xp \in pAp$ *for all* $a \in A\}$.
(3) $Ap = \{xp \in A^{**}p : pw^*xp \in pAp$ *for all* $wp \in \mathcal{W}_p\}$.

Motivated by the fact that elements of $A^{**}p$ are exactly the admissible vector sections in $\prod_{\varphi\in F(p)} H_\varphi$, we make the following definition.

Definition 3.7. Let T_φ be in $B(H_\varphi)$ for each φ in $F(p)$. The operator section $(T_\varphi)_{\varphi\in F(p)}$ is said to be *admissible* if

$$T_{\psi\varphi}T_\varphi = T_\psi T_{\psi\varphi}$$

whenever $\psi, \varphi \in F(p)$ such that $0 \le \psi \le \lambda\varphi$ for some $\lambda > 0$.

Lemma 3.8. *Let $(T_\varphi)_{\varphi\in F(p)}$ be an operator section in $\prod_{\varphi\in F(p)} B(H_\varphi)$. The following are equivalent:*

(1) *$(T_\varphi)_{\varphi\in F(p)}$ is admissible.*
(2) *$(T_\varphi)_{\varphi\in F(p)}$ sends continuous admissible vector sections to admissible vector sections; that is, it induces a linear operator T from Ap into $A^{**}p$.*
(3) *$(T_\varphi)_{\varphi\in F(p)}$ sends admissible vector sections to admissible vector sections; that is, it induces a linear operator T from $A^{**}p$ into $A^{**}p$.*

Proof. Firstly, we note that the assertions in (2) and (3) follow from Theorems 3.3 and 3.4.

(3)⇒(2) is clear.

(2)⇒(1). Suppose that $(T_\varphi(a\omega_\varphi))_{\varphi\in F(p)}$ is admissible for each a in A. Hence there is an xp in $A^{**}p$ such that $x\omega_\varphi = T_\varphi(a\omega_\varphi)$ for all $\varphi \in F(p)$, by Theorem 3.3. Let $\psi, \varphi \in F(p)$ be such that $0 \le \psi \le \lambda\varphi$ for some $\lambda > 0$. Then

$$T_{\psi\varphi}T_\varphi(a\omega_\varphi) = T_{\psi\varphi}(x\omega_\varphi) = x\omega_\psi = T_\psi(a\omega_\psi) = T_\psi T_{\psi\varphi}(a\omega_\varphi).$$

Since $\pi_p(A)\omega_\varphi$ is dense in H_φ, $T_{\psi\varphi}T_\varphi = T_\psi T_{\psi\varphi}$. As a result, $(T_\varphi)_{\varphi\in F(p)}$ is an admissible operator section.

(1)⇒(3). We suppose that $(T_\varphi)_{\varphi\in F(p)}$ is an admissible operator section. We want to show that $y_\varphi = T_\varphi(x\omega_\varphi)$, $\varphi \in F(p)$, defines an admissible vector section for each x in A^{**}. Let $\psi, \varphi \in F(p)$ be such that $0 \le \psi \le \lambda\varphi$ for some $\lambda > 0$. Observe that

$$T_{\psi\varphi}(y_\varphi) = T_{\psi\varphi}(T_\varphi(x\omega_\varphi)) = T_\psi(T_{\psi\varphi}(x\omega_\varphi)) = T_\psi(x\omega_\psi) = y_\psi.$$

This proves the admissibility of $(y_\varphi)_{\varphi\in F(p)}$. ∎

Lemma 3.9. *Every admissible operator section $(T_\varphi)_{\varphi\in F(p)}$ induces a unique bounded linear operator T in $B(A^{**}p)$ such that the vector section representing $T(xp)$ is $(T_\varphi(x\omega_\varphi))_{\varphi\in F(p)}$. In this case, we write $T = (T_\varphi)_{\varphi\in F(p)}$.*

Proof. In view of the proof of Lemma 3.8, we can define $T : A^{**}p \to A^{**}p$ by

$$T(xp)\omega_\varphi = T_\varphi(x\omega_\varphi), \qquad \varphi \in F(p).$$

We apply the closed graph theorem to establish the boundedness of T. Assume $x_np \to xp$ and $T(x_np) \to yp$ in norm. If $yp \neq T(xp)$ then there is a φ in $F(p)$ such that $y\omega_\varphi \neq T(xp)\omega_\varphi = T_\varphi(x\omega_\varphi)$. But they are both the limit of $T_\varphi(x_n\omega_\varphi) = T(x_np)\omega_\varphi$, a contradiction. So $\|T\| < \infty$. ∎

Definition 3.10. A bounded linear operator T in $B(A^{**}p)$ is said to be *decomposable* if for each φ in $F(p)$ there is a T_φ in $B(H_\varphi)$ such that $(Txp)\omega_\varphi = T_\varphi(x\omega_\varphi)$ for all x in A^{**}.

In other words, $T = (T_\varphi)_{\varphi\in F(p)}$ (cf. Lemma 3.9). Note that the operator section $(T_\varphi)_{\varphi\in F(p)}$ must be admissible in this case (Lemma 3.8).

It is clear that all operators T in $\pi_p(A^{**})$ are decomposable. In fact, $T = \pi_p(t)$ for some t in A^{**}, and thus we can set $T_\varphi = \pi_\varphi(t)$ for all $\varphi \in F(p)$. We are going to prove that every decomposable operator in $B(A^{**}p)$ arises in this way.

Lemma 3.11. *Suppose that $(T_\varphi)_{\varphi\in F(p)}$ is an admissible section of operators in $\prod_{\varphi\in F(p)} B(H_\varphi)$. Then T_φ belongs to the double commutant $\pi_\varphi(A)''$ of $\pi_\varphi(A)$ in $B(H_\varphi)$ for each φ in $F(p)$.*

Proof. Let $\varphi \in F(p)$ and q be a projection in $\pi_\varphi(A)' \subseteq B(H_\varphi)$. Define a linear functional ψ on A by

$$\psi(a) = \langle a\omega_\varphi, q\omega_\varphi\rangle_\varphi .$$

It is easy to see that $\psi \in F(p)$ and $0 \le \psi \le \varphi$. Observe that for a, b in A,

$$\begin{aligned}\left\langle T^*_{\psi\varphi}(a\omega_\psi), b\omega_\varphi\right\rangle_\varphi &= \langle a\omega_\psi, T_{\psi\varphi}(b\omega_\varphi)\rangle_\psi = \langle a\omega_\psi, b\omega_\psi\rangle_\psi \\ &= \psi(b^*a) = \langle b^*a\omega_\varphi, q\omega_\varphi\rangle_\varphi = \langle a\omega_\varphi, bq\omega_\varphi\rangle_\varphi \\ &= \langle qa\omega_\varphi, b\omega_\varphi\rangle_\varphi .\end{aligned}$$

We thus have $qa\omega_\varphi = T^*_{\psi\varphi}(a\omega_\psi)$ for all a in A. In particular, $qH_\varphi = \overline{T^*_{\psi\varphi}H_\psi}$. The admissibility condition gives $T_{\psi\varphi}T_\varphi = T_\psi T_{\psi\varphi}$ and so $T^*_\varphi T^*_{\psi\varphi} = T^*_{\psi\varphi}T^*_\psi$. It follows that qH_φ is invariant under T^*_φ. Applying the same argument to $1-q$, we can conclude that qH_φ is a reducing subspace of T^*_φ. Hence $qT^*_\varphi = T^*_\varphi q$ for every projection q in the von Neumann algebra $\pi_\varphi(A)'$. Consequently, $T^*_\varphi \in \pi_\varphi(A)''$ and thus $T_\varphi \in \pi_\varphi(A)''$ for each φ in $F(p)$. ∎

THEOREM 3.12. *Let A be a C^*-algebra, p a closed projection in A^{**} with central support $c(p)$ and $T \in B(A^{**}p)$. Then $T \in \pi_p(A^{**})$ if and only if T is decomposable. In this case, if $T = (T_\varphi)_{\varphi\in F(p)} = \pi_p(t)$ for some t in A^{**} then $\|T\|_{B(A^{**}p)} = \sup_{\varphi\in F(p)} \|T_\varphi\| = \|tc(p)\|$.*

Proof. We check sufficiency only. Suppose that T induces an admissible operator section $(T_\varphi)_{\varphi\in F(p)}$ in $\prod_{\varphi\in F(p)} B(H_\varphi)$. In view of Lemma 2.2, we need only verify that T commutes with the right multiplications R_{pxp} for all x in A^{**}, i.e., for every y in A^{**}, $T(R_{pxp}yp) = R_{pxp}(Typ)$. In other words,

$$T(ypxp) = (Typ)xp,$$

or equivalently,

$$T(ypxp)\omega_\varphi = (T(yp)xp)\omega_\varphi, \quad \forall\varphi \in F(p).$$

By Lemma 3.11, for each φ in $F(p)$ we can choose a t_φ in A^{**} such that

$$\pi_\varphi(t_\varphi) = T_\varphi.$$

The admissibility of $(T_\varphi)_{\varphi\in F(p)}$ says that $T_\psi T_{\psi\varphi} = T_{\psi\varphi}T_\varphi$. Consequently,

$$\pi_\psi(t_\psi)T_{\psi\varphi} = T_{\psi\varphi}\pi_\varphi(t_\varphi)$$

whenever $\varphi, \psi \in F(p)$ satisfy $0 \le \psi \le \lambda\varphi$ for some $\lambda > 0$. In this case,

$$t_\psi y\omega_\psi = \pi_\psi(t_\psi)T_{\psi\varphi}(y\omega_\varphi) = T_{\psi\varphi}\pi_\varphi(t_\varphi)(y\omega_\varphi) = T_{\psi\varphi}(t_\varphi y\omega_\varphi) = t_\varphi y\omega_\psi$$

for every y in A^{**}, and thus

$$\pi_\psi(t_\psi) = \pi_\psi(t_\varphi) \quad \text{in } B(H_\psi). \tag{1}$$

Moreover, we note that

$$p\omega_\varphi = \omega_\varphi \text{ and } T(xp) = (T(xp))p \in A^{**}p, \quad \forall\varphi \in F(p), \forall x \in A^{**}. \tag{2}$$

For each x in A^{**} with $\|x\| \leq 1$ and φ in $F(p)$ we define ψ, ρ in $F(p)$ by

$$\psi(\cdot) = \langle \cdot\, px\omega_\varphi, px\omega_\varphi \rangle_\varphi \quad \text{and} \quad \rho = \frac{\varphi + \psi}{2}.$$

Since $0 \leq \varphi \leq 2\rho$ and $0 \leq \psi \leq 2\rho$, by (1) we have

$$\pi_\varphi(t_\varphi) = \pi_\varphi(t_\rho) \quad \text{and} \quad \pi_\psi(t_\psi) = \pi_\psi(t_\rho). \tag{3}$$

It follows that

$$\begin{aligned} (T(ypxp))\omega_\varphi &= T_\varphi(ypx\omega_\varphi) = \pi_\varphi(t_\varphi)(ypx\omega_\varphi) \\ &= \pi_\varphi(t_\rho)(ypx\omega_\varphi) = (t_\rho ypx)\omega_\varphi. \end{aligned} \tag{4}$$

Observe also that for every y in A^{**}, by (2) and (3) we have

$$\begin{aligned} \langle (Typ)x\omega_\varphi, ypx\omega_\varphi \rangle_\varphi &= \langle (Typ)\omega_\psi, y\omega_\psi \rangle_\psi = \langle T_\psi(y\omega_\psi), y\omega_\psi \rangle_\psi \\ &= \langle \pi_\psi(t_\psi)y\omega_\psi, y\omega_\psi \rangle_\psi = \langle \pi_\psi(t_\rho)y\omega_\psi, y\omega_\psi \rangle_\psi \\ &= \langle t_\rho ypx\omega_\varphi, ypx\omega_\varphi \rangle_\varphi . \end{aligned}$$

Therefore, $((Typ) - t_\rho yp)x\omega_\varphi \in (A^{**}px\omega_\varphi)^\perp$. Hence,

$$(Typ)x\omega_\varphi = t_\rho ypx\omega_\varphi.$$

Consequently, by (4) we have

$$(T(ypxp))\omega_\varphi = t_\rho ypx\omega_\varphi = ((Typ)xp)\omega_\varphi, \quad \forall \varphi \in F(p),$$

and thus $T(ypxp) = (Typ)xp$, as asserted.

For the norm equalities, we choose a t in A^{**} by Lemma 2.2 such that $T = \pi_p(t)$ and

$$\|T\|_{B(A^{**}p)} = \|tc(p)\| = \sup_{\varphi \in F(p)} \|\pi_\varphi(t)\| = \sup_{\varphi \in F(p)} \|T_\varphi\|. \ ■$$

Let

$$\mathrm{QM}(A, p) = \{x \in A^{**} : pAxAp \subseteq pAp\},$$

the Banach space of *relative quasi-multipliers* of A associated to p. By Corollary 3.6(1), for any x in A^{**}, we have $x \in \mathrm{QM}(A, p)$ if and only if $\pi_p(x) \in B(Ap, \mathcal{W}_p)$, that is, $\pi_p(x)$ sends continuous admissible vector sections to weakly continuous admissible vector sections in $(F(p), \{H_\varphi\}_\varphi, Ap)$.

Theorem 3.13. *Let A be a C^*-algebra and p a closed projection in A^{**} with central support $c(p)$. Assume T in $B(Ap, \mathcal{W}_p)$ satisfies the condition*

$$\varphi(a^*a) = 0 \ \Rightarrow \ \varphi((Tap)^*(Tap)) = 0$$

whenever φ is a pure state in $F(p)$ and $a \in A$. Then T can be extended to a decomposable operator in $B(A^{**}p)$, denoted again by T, such that $T = \pi_p(t)$ for some t in $\mathrm{QM}(A, p)$ and $\|T\|_{B(Ap,\mathcal{W}_p)} = \|T\|_{B(A^{**}p)} = \|tc(p)\|$.

Proof. We first recall that

$$\|a\omega_\varphi\|^2 = \langle a\omega_\varphi, a\omega_\varphi \rangle_\varphi = \varphi(a^*a), \quad \forall a \in A, \forall \varphi \in F(p).$$

Let $X = F(p)\cap P(A)$, where $P(A)$ is the pure state space of A. By hypothesis and the Kadison transitivity theorem, for each φ in X we can define a linear map T_φ on $H_\varphi = A\omega_\varphi$ by

$$T_\varphi(a\omega_\varphi) = (T(ap))\omega_\varphi.$$

Let $\varphi \in X$ and $a\omega_\varphi \in H_\varphi$ such that $\|a\omega_\varphi\| = 1$. Again by the Kadison transitivity theorem, there is a b in A such that $b\omega_\varphi = a\omega_\varphi$ and $\|b\| = 1$. Hence

$$\|T_\varphi(a\omega_\varphi)\| = \|T_\varphi(b\omega_\varphi)\| = \|(T(bp))\omega_\varphi\| \le \|T(bp)\| \le \|T\|\,\|bp\| \le \|T\|.$$

Therefore, $\|T_\varphi\| \le \|T\|$ for every φ in X. Consequently, we have

$$\sup_{\varphi\in X} \|T_\varphi\| \le \|T\|.$$

Now assume φ belongs to $\overline{X}$, the weak* closure of X, and $a, b \in A$. Since $T(ap) \in \mathcal{W}_p$, the scalar functions $\psi \mapsto \|a\omega_\psi\|_\psi$, $\psi \mapsto \|b\omega_\psi\|_\psi$ and $\psi \mapsto \langle (T(ap))\omega_\psi, b\omega_\psi\rangle_\psi$ are all continuous on $F(p)$. It follows that

$$|\langle (Tap)\omega_\varphi, b\omega_\varphi\rangle_\varphi| \le \Big(\sup_{\psi\in X} \|T_\psi\|\Big)\|a\omega_\varphi\|_\varphi \|b\omega_\varphi\|_\varphi \le \|T\|\,\|a\omega_\varphi\|_\varphi\|b\omega_\varphi\|_\varphi.$$

Hence there exists T_φ in $B(H_\varphi)$ such that

$$T_\varphi(a\omega_\varphi) = (T(ap))\omega_\varphi, \qquad \forall a \in A,\ \forall\varphi \in \overline{X}. \tag{5}$$

Moreover, $\|T_\varphi\| \le \|T\|$ for every φ in $\overline{X} = \overline{F(p) \cap P(A)}$.

Note that $X \cup \{0\}$ is the extreme boundary of the compact convex set $F(p)$. Consequently, continuous affine functionals of $F(p)$ assume extrema at points in X. From Proposition 3.5, we know that there is an order-preserving linear isometry from $pA_{\mathrm{sa}}p$ into $C_{\mathbb{R}}(\overline{X})$, the Banach space of continuous real-valued functions defined on the compact Hausdorff space $\overline{X}$. Hence each φ in $F(p)$ has a (non-unique) Hahn–Banach positive extension m_φ in the space $M(\overline{X})$ ($\cong C_{\mathbb{R}}(\overline{X})^*$) of regular finite Borel measures on $\overline{X}$. By handling real and imaginary parts separately, for each φ in $F(p)$ we can write

$$\varphi(a) = \varphi(pap) = \int_{\overline{X}} \psi(pap)\, dm_\varphi(\psi) = \int_{\overline{X}} \psi(a)\, dm_\varphi(\psi), \qquad \forall a \in A. \tag{6}$$

For any a, b in A, since $T(ap) \in \mathcal{W}_p$, we have $pb^*(T(ap)) \in pAp$ by Corollary 3.6. Therefore, the barycenter formula (6) applies and gives

$$\begin{aligned}\langle T(ap)\omega_\varphi, b\omega_\varphi\rangle_\varphi &= \varphi(pb^*(T(ap))) = \int_{\overline{X}} \psi(pb^*(T(ap)))\, dm_\varphi(\psi)\\ &= \int_{\overline{X}} \langle (Tap)\omega_\psi, b\omega_\psi\rangle_\psi\, dm_\varphi(\psi), \qquad \forall\varphi \in F(p).\end{aligned}$$

Consequently, by (5) we have

$$\begin{aligned}
&|\langle T(ap)\omega_\varphi, b\omega_\varphi\rangle_\varphi| \\
&\quad = \Big|\int_{\overline{X}} \langle T(ap)\omega_\psi, b\omega_\psi\rangle_\psi \, dm_\varphi(\psi)\Big| = \Big|\int_{\overline{X}} \langle T_\psi(a\omega_\psi), b\omega_\psi\rangle_\psi \, dm_\varphi(\psi)\Big| \\
&\quad \le \int_{\overline{X}} \|T_\psi\|\,\|a\omega_\psi\|\,\|b\omega_\psi\|\, dm_\varphi(\psi) \\
&\quad \le \Big(\sup_{\psi\in\overline{X}} \|T_\psi\|\Big)\Big(\int_{\overline{X}} \|a\omega_\psi\|^2\, dm_\varphi(\psi)\Big)^{1/2}\Big(\int_{\overline{X}} \|b\omega_\psi\|^2\, dm_\varphi(\psi)\Big)^{1/2} \\
&\quad = \Big(\sup_{\psi\in\overline{X}} \|T_\psi\|\Big)\Big(\int_{\overline{X}} \psi(a^*a)\, dm_\varphi(\psi)\Big)^{1/2}\Big(\int_{\overline{X}} \psi(b^*b)\, dm_\varphi(\psi)\Big)^{1/2} \\
&\quad = \Big(\sup_{\psi\in\overline{X}} \|T_\psi\|\Big)\varphi(a^*a)^{1/2}\varphi(b^*b)^{1/2} \le \|T\|\,\|a\omega_\varphi\|_\varphi\|b\omega_\varphi\|_\varphi.
\end{aligned}$$

Hence, a bounded linear operator T_φ in $B(H_\varphi)$ exists such that $T_\varphi(a\omega_\varphi) = (T(ap))\omega_\varphi$ for every a in A. Moreover,

$$\|T_\varphi\| \le \|T\|, \qquad \forall\varphi\in F(p).$$

At this point, we have shown that T can be written as an admissible section of operators $T = (T_\varphi)_{\varphi\in F(p)}$ in $\prod_{\varphi\in F(p)} B(H_\varphi)$ (cf. Lemma 3.8). Extend T to a bounded linear operator on $A^{**}p$ as in Lemma 3.9. Consequently, by Theorem 3.12, there is a t in A^{**} such that $T = \pi_p(t)$ and $\|T\|_{B(A^{**}p)} = \sup_{\varphi\in F(p)} \|T_\varphi\|_{B(H_\varphi)} = \|tc(p)\|$. Since $T(Ap)\subseteq \mathcal{W}_p$, we have $pb^*(Tap)\in pAp$ by Corollary 3.6. Hence $pAtAp\subseteq pAp$. As a result, $t\in \mathrm{QM}(A,p)$. Finally, we note that

$$\|T\|_{B(Ap,\mathcal{W}_p)} \le \|T\|_{B(A^{**}p)} = \sup_{\varphi\in F(p)} \|T_\varphi\|_{B(H_\varphi)} \le \|T\|_{B(Ap,\mathcal{W}_p)}. \ \blacksquare$$

Let

$$\mathrm{LM}(A,p) = \{x\in A^{**} : xAp\subseteq Ap\},$$

the Banach algebra of *relative left multipliers* of A associated to p.

COROLLARY 3.14. *Let A be a C^*-algebra, p a closed projection in A^{**} with central support $c(p)$ and $T\in B(Ap)$. The following are all equivalent:*

(1) *$T\in\pi_p(\mathrm{LM}(A,p))$.*
(2) *T is decomposable.*
(3) *$\varphi(a^*a)=0$ implies $\varphi((Tap)^*(Tap))=0$ whenever φ is a pure state supported by p and a in A.*

In this case, if $t\in\mathrm{LM}(A,p)$ is such that $T=\pi_p(t)$ then $\|T\|_{B(Ap)} = \|tc(p)\|$.

Proof. The implication (1)⇒(2) is trivial as $T_\varphi = \pi_\varphi(t)$ when $T = \pi_p(t)$ with t in $\mathrm{LM}(A,p)$, while (2)⇒(3) is straightforward. The implication (3)⇒(1) follows from Theorem 3.13, which also provides the norm equalities. ■

4. Commutants and density theorems

DEFINITION 4.1. Let A be a C^*-algebra and p a closed projection in A^{**}. Recall that

$$\begin{aligned}\mathrm{LM}(A,p) &= \{x \in A^{**} : xAp \subseteq Ap\},\\ \mathrm{RM}(A,p) &= \{x \in A^{**} : pAx \subseteq pA\},\\ \mathrm{M}(A,p) &= \{x \in A^{**} : xAp \subseteq Ap,\ pAx \subseteq pA\},\\ \mathrm{QM}(A,p) &= \{x \in A^{**} : pAxAp \subseteq pAp\}\end{aligned}$$

are, respectively, the sets of *relative left multipliers, relative right multipliers, relative multipliers* and *relative quasi-multipliers* associated to p. We define the *relative left strict topology, relative right strict topology, relative strict topology* and *relative quasi-strict topology* of A^{**} associated to p by the seminorms $x \mapsto \|xap\|$, $x \mapsto \|pax\|$, $x \mapsto \|xap\| + \|pbx\|$ and $x \mapsto \|paxbp\|$ for a, b in A.

REMARKS 4.2.

(1) It is easy to see that $\mathrm{LM}(A) \subseteq \mathrm{LM}(A,p)$, $\mathrm{RM}(A) \subseteq \mathrm{RM}(A,p), \ldots,$ and all of them are norm closed subspaces of A^{**}.

(2) $\mathrm{QM}(A,p)$ is $*$-invariant whereas $\mathrm{LM}(A,p)^* = \mathrm{RM}(A,p)$. Moreover, both $\mathrm{LM}(A,p)$ and $\mathrm{RM}(A,p)$ are Banach algebras, and $\mathrm{M}(A,p) = \mathrm{LM}(A,p) \cap \mathrm{RM}(A,p)$ is a C^*-algebra.

(3) The relative strict topologies associated to p are Hausdorff if and only if the central support $c(p)$ of p equals 1.

THEOREM 4.3. *Let A be a C^*-algebra and p a closed projection in A^{**}. Then* $\mathrm{LM}(A,p)$ (*resp.* $\mathrm{RM}(A,p)$, $\mathrm{M}(A,p)$ *and* $\mathrm{QM}(A,p)$) *coincides with the closure of A in A^{**} with respect to the relative left strict* (*resp. right strict, strict and quasi-strict*) *topology associated to p.*

Moreover, the unit ball (*resp. its self-adjoint part, positive part*) *of A is dense in the unit ball* (*resp. its self-adjoint part, positive part*) *of* $\mathrm{LM}(A,p)$, $\mathrm{RM}(A,p)$, $\mathrm{M}(A,p)$ *and* $\mathrm{QM}(A,p)$ *in the corresponding relative strict topologies associated to p.*

Proof. We only prove the assertion about relative left multipliers since all others follow in a similar manner. We denote by B_{sa} (resp. B_+, B_1) the set of all self-adjoint elements (resp. positive elements, elements of norm not greater than 1) in B whenever B is a subset of A or A^{**}.

Assume $x \in \mathrm{LM}(A,p)$. We want to show that x belongs to the relative left strict closure of A. Let $a_1,\ldots,a_n \in A$. Consider the convex set V in the direct sum $(Ap)^n = Ap \oplus \cdots \oplus Ap$ given by

$$V = \{(ba_1p,\ldots,ba_np) : b \in A\}.$$

(In case $x \in A_1^{**}$, $x \in A_{\mathrm{sa}}^{**} \cap A_1^{**}$ or $x \in A_+^{**} \cap A_1^{**}$, in the definition of V we replace A by A_1, $A_{\mathrm{sa}} \cap A_1$ or $A_+ \cap A_1$, respectively.) Since $x \in \mathrm{LM}(A,p)$, we have $\tilde{x} = (xa_1p,\ldots,xa_np) \in (Ap)^n$. If $\tilde{x} \notin \overline{V}^{\|\cdot\|}$ then there is an $\tilde{f}$ in $((Ap)^n)^*$ such that

$$\mathrm{Re}\,\tilde{f}(\tilde{x}) < -1 \le \mathrm{Re}\,\tilde{f}(\tilde{b}), \quad \forall \tilde{b} \in V, \tag{7}$$

where $\tilde{b} = (ba_1p,\ldots,ba_np)$. Since $(Ap)^* \cong A^{**}F(p)$ (see, e.g., [12]), we can write $\tilde{f} = f_1 \oplus \cdots \oplus f_n$ such that $f_k = y_k^*\varphi_k$ for some y_k in A^{**} and φ_k in $F(p)$, $k = 1,\ldots,n$. Hence

$$\tilde{f}(\tilde{x}) = \sum_{k=1}^n f_k(xa_kp) = \sum_{k=1}^n \varphi_k(y_k^*xa_k) = \sum_{k=1}^n \langle xa_k\omega_{\varphi_k}, y_k\omega_{\varphi_k}\rangle_{\varphi_k},$$
$$\tilde{f}(\tilde{b}) = \sum_{k=1}^n f_k(ba_kp) = \sum_{k=1}^n \varphi_k(y_k^*ba_k) = \sum_{k=1}^n \langle ba_k\omega_{\varphi_k}, y_k\omega_{\varphi_k}\rangle_{\varphi_k}.$$

Let $\{b_\lambda\}_\lambda$ be a net in A such that b_λ converges to x σ-weakly. (In case $x \in A_1^{**}, x \in A_{\mathrm{sa}}^{**} \cap A_1^{**}$ or $x \in A_+^{**} \cap A_1^{**}$, the Kaplansky density theorem (see, e.g., [14, 2.3.3]) enables us to choose b_λ's from $A_1, A_{\mathrm{sa}} \cap A_1$ or $A_+ \cap A_1$, respectively.) In particular,

$$\langle b_\lambda a_k\omega_{\varphi_k}, y_k\omega_{\varphi_k}\rangle_{\varphi_k} \to \langle xa_k\omega_{\varphi_k}, y_k\omega_{\varphi_k}\rangle_{\varphi_k} \qquad \text{for } k = 1,\ldots n.$$

Therefore, $\tilde{f}(\tilde{b}_\lambda) \to \tilde{f}(\tilde{x})$ where $\tilde{b}_\lambda = (b_\lambda a_1p,\ldots,b_\lambda a_np) \in V$. This contradicts (7) and thus $\tilde{x} \in \overline{V}^{\|\cdot\|}$. This shows that for any positive ε and $a_1,\ldots,a_n$ in A there is a b in A such that

$$\|(x-b)a_kp\| < \varepsilon \quad \text{for } k = 1,\ldots,n.$$

In other words, x belongs to the relative left strict closure of A. (In case x comes from A_1^{**}, $A_{\mathrm{sa}}^{**} \cap A_1^{**}$ or $A_+^{**} \cap A_1^{**}$, we can choose b from A_1, $A_{\mathrm{sa}} \cap A_1$ or $A_+ \cap A_1$, respectively.) Our assertion follows since the opposite inclusion is obvious. ∎

THEOREM 4.4. *The closure of $\pi_p(A)$ in $B(Ap)$ with respect to the strong operator topology (SOT) as well as the weak operator topology (WOT) coincides with $\pi_p(\mathrm{LM}(A,p))$. Moreover, the unit ball of $\pi_p(A)$ is SOT dense as well as WOT dense in the unit ball of $\pi_p(\mathrm{LM}(A,p))$.*

Proof. It is well-known that a linear functional on $B(E)$, for E a Banach space, is continuous with respect to SOT if and only if it is continuous with respect to WOT. Since $\pi_p(A)$ is convex, its closures in $B(Ap)$ with respect to these topologies coincide. We are going to show that they are identical to $\pi_p(\mathrm{LM}(A,p))$.

Let $\{a_\lambda\}_\lambda$ be a net in A such that $\pi_p(a_\lambda)$ converges to some bounded linear operator T in SOT. By Corollary 3.14, to see $T \in \pi_p(\mathrm{LM}(A,p))$ we just need to check whether the condition $\varphi(a^*a) = 0$ implies $\varphi((Tap)^*(Tap)) = 0$ whenever φ is a pure state in $F(p)$ and $a \in A$. In this case, $ap_\varphi = 0$ where p_φ is the support projection of the pure state φ. Now

$$(Tap)p_\varphi = (\lim \pi_p(a_\lambda)ap)p_\varphi = \lim a_\lambda ap_\varphi = 0.$$

Hence $\varphi((Tap)^*(Tap)) = 0$, as asserted. Thus

$$\overline{\pi_p(A)}^{\mathrm{SOT}} \subseteq \pi_p(\mathrm{LM}(A,p)).$$

The opposite inclusion and other assertions follow from Theorem 4.3 since the strong operator topology of $B(Ap)$ restricted to $\pi_p(\mathrm{LM}(A,p))$ coincides with the one induced by the relative left strict topology of A^{**} associated to p. ∎

Remark 4.5. In [18], Tomita defined the notion of Q^*-topology. In fact, it is the double strong operator topology (DSOT) of $\pi_p(\mathrm{M}(A,p))$, which is defined by the seminorms

$$\pi_p(x) \mapsto \|xap\| + \|x^*ap\|, \qquad \forall a \in A.$$

Since $\mathrm{RM}(A,p)^* = \mathrm{LM}(A,p)$ and $\mathrm{M}(A,p) = \mathrm{LM}(A,p) \cap \mathrm{RM}(A,p)$, Theorems 4.3 and 4.4 imply $\overline{\pi_p(A)}^{\mathrm{DSOT}} = \pi_p(\mathrm{M}(A,p))$. Moreover, the unit ball of $\pi_p(A)$ (resp. its self-adjoint part, positive part) is DSOT dense in the unit ball (resp. its self-adjoint part, positive part) of $\pi_p(\mathrm{M}(A,p))$. Another way to look at $\pi_p(\mathrm{M}(A,p))$ is to observe that it coincides with the family of all *adjointable* admissible operator sections $\{T_\varphi\}_\varphi$ in $\prod_{\varphi\in F(p)} B(H_\varphi)$. We say that $\{T_\varphi\}_\varphi$ is adjointable if the operator section $\{T_\varphi^*\}_\varphi$ is admissible (see Corollary 3.14). Tomita expected that in some situations the double commutant $\pi_p(A)''$ of $\pi_p(A)$ in $B(Ap)$ is the *C^*-algebra* $\pi_p(\mathrm{M}(A,p))$. However, as indicated by Theorem 4.8 below, the Banach algebra $\pi_p(\mathrm{LM}(A,p))$ is a more appropriate object to look for.

Recall that a projection r in A^{**} is closed if the face $F(r) = \{\varphi \in Q(A) : \varphi(1-r) = 0\}$ of $Q(A)$ supported by r is weak* closed, and r is *compact* if $F(r) \cap S(A)$ is weak* closed [2]. An element h of $pA^{**}_{\mathrm{sa}}p$ is called *q-continuous* on p (see [4]) if the spectral projection $E_F(h)$ (computed in $pA^{**}p$) is closed for every closed subset F of $\mathbb{R}$. Also, h is called *strongly q-continuous* on p (see [5]) if, in addition, $E_F(h)$ is compact whenever F is closed and $0 \notin F$.

Lemma 4.6 ([5, 3.43]). *Let $h \in pA^{**}_{\mathrm{sa}}p$.*

(1) *h is strongly q-continuous on p if and only if $h = pa = ap$ for some a in A_{sa}.*

(2) *In case A is σ-unital, h is q-continuous on p if and only if $h = px = xp$ for some x in $M(A)_{\mathrm{sa}}$.*

In general, h in $pA^{**}p$ is said to be *q-continuous* or *strongly q-continuous* if both $\operatorname{Re} h$ and $\operatorname{Im} h$ are. Denote by $\mathrm{QC}(p)$ (resp. $\mathrm{SQC}(p)$) the set of all q-continuous elements (resp. strongly q-continuous elements) on p. Observe that $\mathrm{SQC}(p)$ is always a C^*-algebra, and so is $\mathrm{QC}(p)$ if A is σ-unital. We say that p has MQC ("*many q-continuous elements*") or MSQC ("*many strongly q-continuous elements*") if $\mathrm{QC}(p)$ or $\mathrm{SQC}(p)$, respectively, is σ-weakly dense in $pA^{**}p$ (see [8]).

Lemma 4.7 ([8, 3.1 and 3.3]). *The following statements are all equivalent:*

(1) *p has MSQC.*
(2) *$pAp = \mathrm{SQC}(p)$.*
(3) *pAp is an algebra.*
(4) *pAp is a Jordan algebra.*
(5) *$F(p)$ is isomorphic to the quasi-state space of a C^*-algebra.*
(6) *$p \in \mathrm{M}(A,p)$, i.e., $pAp \subseteq pA \cap Ap$.*
(7) *$p \in \mathrm{QM}(A,p)$, i.e., $pApAp \subseteq pAp$.*

In this case,

$$pApAp = pAp = pA \cap Ap = \mathrm{SQC}(p).$$

When the closed projection p has MSQC, it shares many good properties with the projection 1. Moreover, every central closed projection in A^{**} has MSQC.

The first part of the following theorem says that all bounded A-module maps in $B(Ap)$ are right multiplications provided that A is σ-unital.

Theorem 4.8. *Let A be a C^*-algebra, p a closed projection in A^{**} and π_p the left regular representation of A on Ap. Denote by $\pi_p(A)'$ the commutant and by $\pi_p(A)''$ the double commutant of $\pi_p(A)$ in $B(Ap)$. Denote by $\mathcal{Y}$ the set $\{x \in \mathrm{RM}(A) : xp = pxp\}$. If A is σ-unital then*

$$\pi_p(A)' = \{R_{pxp} : x \in \mathcal{Y}\}.$$

If A is σ-unital and p has MQC then also

$$\pi_p(A)'' = \pi_p(\mathrm{LM}(A,p)).$$

Here $R_{pxp}(ap) := apxp = axp$ for all $a \in A$ and $x \in \mathcal{Y}$.

Proof. It is clear that all right multiplications of the form R_{pxp} with x in $\mathcal{Y}$ commute with elements of $\pi_p(A)$. Conversely, assume $T \in \pi_p(A)' \subseteq B(Ap)$. If $\{u_\lambda\}_\lambda$ is a (bounded) approximate unit of A, the bounded net $\{T(u_\lambda p)\}_\lambda$ in Ap has a weak* cluster point xp in $A^{**}p$. For each a in A, we see that axp is a weak* cluster point of $\{aT(u_\lambda p)\}_\lambda = \{T(au_\lambda p)\}_\lambda$. But $T(au_\lambda p) \to T(ap)$ in norm. It follows that $T(ap) = axp \in Ap$. Therefore,

$Axp = T(Ap) \subseteq Ap$. By [5, 3.9], we have $xp \in \mathrm{RM}(A)p$ if A is σ-unital. Moreover, if $a, b \in A$ and $ap = bp$ then $T(ap) = T(bp)$. This is equivalent to $axp = bxp$. Consequently, $Lxp = \{0\}$ where $L = A^{**}(1-p) \cap A$, the norm closed left ideal of A related to the closed projection p. It follows that $L^{**}xp = \{0\}$; i.e., $A^{**}(1-p)xp = \{0\}$. This forces $(1-p)xp = 0$. Therefore $xp = pxp$. Hence $T(ap) = axp = apxp = R_{pxp}(ap)$.

By Theorem 4.4, $\pi_p(\mathrm{LM}(A,p)) \subseteq \pi_p(A)''$. Let $T \in \pi_p(A)'' \subseteq B(Ap)$, $a \in A$ and φ be a pure state in $F(p)$. Assume that $\varphi(a^*a) = 0$, or equivalently $ap_\varphi = 0$, where p_φ is the support projection of φ in A^{**}. Since p is assumed to have MQC and A is σ-unital, there is a net $\{m_\lambda p\}_\lambda$ with m_λ in $\mathrm{M}(A)$ such that

$$m_\lambda p = pm_\lambda \quad \text{and} \quad m_\lambda p \to p_\varphi \quad \sigma\text{-weakly} \tag{8}$$

by Lemma 4.6. Hence, $am_\lambda p \to ap_\varphi = 0$ σ–weakly. In particular, $am_\lambda p \to 0$ with respect to $\sigma(Ap, (Ap)^*)$ since $(Ap)^* \cong (A/L)^* \cong L^\circ$ can be considered as a subspace of A^*, and the σ-weak topology of A^{**} coincides with $\sigma(A^{**}, A^*)$. Here L° is the polar of the left ideal $L = A^{**}(1-p) \cap A$ in A^*. As a bounded Banach space operator, T is $\sigma(Ap, (Ap)^*)$-$\sigma(Ap, (Ap)^*)$ continuous. Therefore, $T(am_\lambda p) \to 0$ in the $\sigma(Ap, (Ap)^*)$ topology of Ap and thus also σ-weakly. On the other hand, the right multiplication $R_{pm_\lambda p}$ belongs to $\pi_p(A)'$. As a result, by (8) we have

$$\begin{aligned} T(am_\lambda p) &= T(apm_\lambda p) = TR_{pm_\lambda p}(ap) = R_{pm_\lambda p}T(ap) \\ &= (Tap)pm_\lambda p \to (Tap)p_\varphi \quad \sigma\text{-weakly}. \end{aligned}$$

Therefore, $(Tap)p_\varphi = 0$, and hence $\varphi((Tap)^*(Tap)) = 0$. Now, Corollary 3.14 implies $T \in \pi_p(\mathrm{LM}(A,p))$. ∎

Although it follows from Theorem 4.4 that we always have $\pi_p(\mathrm{LM}(A,p)) \subseteq \pi_p(A)''$, the following example indicates that the inclusion can be strict in case p does not have MQC.

EXAMPLE 4.9 (Based on an example given in [8, 3.4]). Let $A = C[0,1] \otimes \mathcal{K}$ where $\mathcal{K}$ is the C^*-algebra of all compact operators on a separable infinite-dimensional Hilbert space H. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of H, and E_n the projection on $\mathrm{span}\{e_1, \ldots, e_n\}$. A closed projection in A is given by a projection-valued function $P : [0,1] \to B(H)$ such that if h is any weak cluster point of $P(y)$ as $y \to x$, then $h \leq P(x)$ [5, Section 5.G]. We observe that P describes the atomic part of a closed projection p in $A^{**} \cong C[0,1]^{**} \otimes B(H)$, and P determines p since a closed projection is determined by its atomic part. In our case p will equal its atomic part. We now define P.

For each $n = 0, 1, 2, \ldots$ we construct recursively a countable subset S_n of $[0,1]$ and a unit vector $v(x)$ for each x in S_n with $\|E_n v(x)\| \leq n^{-1/2}$.

Step 0. Take $S_0 = \{1/2\}$ and $v(1/2) = e_1$.

Step 1. Take $S_1 = \{x_1, x_2, \ldots\}$ where the x_j's are distinct, $x_j \neq 1/2$, and $x_j \to 1/2$ as $j \to \infty$. Let $v(x_j) = 2^{-1/2}e_1 + 2^{-1/2}e_{j+1}$ for $j = 1, 2, \ldots$.

Step n ($n > 1$). Write $S_{n-1} = \{x_1, x_2, \ldots\}$. Choose distinct y_{ij}'s from $[0,1]$ but outside $\bigcup_{k=0}^{n-1} S_k$ such that $|y_{ij} - x_i| \leq 2^{-(i+j)}$. Let $S_n = \{y_{ij} : i, j = 1, 2, \ldots\}$ and $v(y_{ij}) = n^{-1/2}v(x_i) + (1 - n^{-1})^{1/2}w_{ij}$, where w_{ij} is a unit vector such that $\langle w_{ij}, v(x_i)\rangle_H = 0$ and $E_{i+j+n}w_{ij} = 0$.

Let $S = \bigcup_{n=0}^{\infty} S_n$. Define a projection-valued function P on $[0,1]$ by setting $P(x)$ to be the projection on span $\{v(x)\}$ if $x \in S$, and $P(x) = 0$ otherwise. It is shown in [8] that P describes a closed projection p in A^{**} which is atomic and abelian. Moreover, if h in $pA^{**}p$ satisfies $h \in pAp$ and $h^2 \in pAp$ then $h = 0$. (In [8], this fact is used to show that $\mathrm{SQC}(p) = \{0\}$.)

Now consider the C^*-algebra $B = C[-1,1] \otimes \mathcal{K}$. Define a projection-valued function Q on $[-1,1]$ by putting $Q(t) := P(|t|)$ for all $t \in [-1,1]$. It is clear that Q determines an atomic, abelian and closed projection q in B^{**} such that $k = 0$ whenever $k \in qB^{**}q$ with $k \in qBq$ and $k^2 \in qBq$.

Let $\tilde{A}$ be the C^*-algebra obtained by adjoining an identity to A and let $\tilde{p} = p + p_\infty$ where $p_\infty = 0 \oplus 1$ in $\tilde{A}^{**} \cong A^{**} \oplus \mathbb{C}$. Thus $\tilde{p} = p \oplus 1$. In [8], it is shown that $\tilde{p}$ is closed, and hence compact, in $\tilde{A}^{**}$ and that $QC(\tilde{p}) = \mathbb{C}\tilde{p}$. Similarly, a compact projection $\tilde{q} = q + q_\infty$ in $\tilde{B}^{**} \cong B^{**} \oplus \mathbb{C}$ can be obtained such that $QC(\tilde{q}) = \mathbb{C}\tilde{q}$ and thus $\tilde{q}$, like $\tilde{p}$, does not have MQC.

We now consider the left regular representation $\pi_{\tilde{q}} : \tilde{B} \to B(\tilde{B}\tilde{q})$. Since $\tilde{B}$ is unital, $\mathrm{RM}(\tilde{B}) = \tilde{B}$ and thus

$$\pi_{\tilde{q}}(\tilde{B})' = \{R_{\tilde{x}} : \tilde{x} = \tilde{r}\tilde{q} = \tilde{q}\tilde{r}\tilde{q} \text{ for some } \tilde{r} \text{ in } \tilde{B}\}$$

by Theorem 4.8. Suppose $\tilde{x} = \tilde{r}\tilde{q} = \tilde{q}\tilde{r}\tilde{q}$ for some $\tilde{r}$ in $\tilde{B}$. Here $\tilde{r} = r + \lambda 1_{\tilde{B}}$ for some r in B and λ in $\mathbb{C}$. It follows from $(r + \lambda 1_{\tilde{B}})(q + q_\infty) = (q + q_\infty)(r + \lambda 1_{\tilde{B}})(q + q_\infty)$ that $rq = qrq \in qBq$. Now $(qrq)^2 = qrqrq = qr^2q \in qBq$ implies $qrq = 0$. Therefore,

$$\tilde{x} = \tilde{q}\tilde{r}\tilde{q} = qrq + \lambda q + \lambda q_\infty = \lambda\tilde{q}.$$

Consequently, $\pi_{\tilde{q}}(\tilde{B})' = \mathbb{C}R_{\tilde{q}}$ and thus $\pi_{\tilde{q}}(\tilde{B})'' = B(\tilde{B}\tilde{q})$, since the right multiplication $R_{\tilde{q}}$ induced by $\tilde{q}$ is the identity in $B(\tilde{B}\tilde{q})$.

It is easy to see that $B(\tilde{B}\tilde{q}) \neq \pi_{\tilde{q}}(\mathrm{LM}(\tilde{B}, \tilde{q}))$. For example, we define an isometry T in $B(\tilde{B}\tilde{q})$ by

$$T((\lambda + a)\tilde{q}) := (\lambda + \overline{a})\tilde{q}, \qquad \lambda \in \mathbb{C},\, a \in B,$$

where

$$\overline{a}(t) := a(-t), \qquad t \in [-1,1].$$

To see that T is not implemented as a left multiplication $\pi_{\tilde{q}}(\tilde{h})$ for any $\tilde{h}$

in $\mathrm{LM}(\tilde{B}, \tilde{q})$, we just need to show that T is not decomposable, by Corollary 3.14. Let $t \in (S \cup (-S)) - \{0\}$, and φ_t be the corresponding pure state in $F(\tilde{q})$. Since there is b in B such that $\varphi_t(b^*b) = 0$ but $\varphi_{-t}(b^*b) \neq 0$, it is clear that T is not decomposable. ∎

5. The C^*-algebra associated to a closed projection. Recall that for a C^*-algebra A and a closed projection p in A^{**}, the Banach space Ap (resp. $\mathcal{W}_p$) consists of all continuous (resp. weakly continuous) admissible vector sections in $A^{**}p$ (see Theorem 3.4). It follows from Corollary 3.6 that for all x in A^{**} we have

$$\pi_p(x)Ap \subseteq Ap \ \Leftrightarrow \ \pi_p(x^*)\mathcal{W}_p \subseteq \mathcal{W}_p.$$

We collect these facts in the following.

$$\begin{aligned} \mathrm{LM}(A,p) &= \{x \in A^{**} : \pi_p(x)Ap \subseteq Ap\}, \\ \mathrm{RM}(A,p) &= \{x \in A^{**} : \pi_p(x)\mathcal{W}_p \subseteq \mathcal{W}_p\}, \\ \mathrm{M}(A,p) &= \{x \in A^{**} : \pi_p(x)Ap \subseteq Ap, \pi_p(x)\mathcal{W}_p \subseteq \mathcal{W}_p\}, \\ \mathrm{QM}(A,p) &= \{x \in A^{**} : \pi_p(x)Ap \subseteq \mathcal{W}_p\}. \end{aligned}$$

Since the kernel of π_p is $A^{**}(1 - c(p))$, the interesting parts of $\mathrm{LM}(A,p)$, $\mathrm{RM}(A,p)$, $\mathrm{M}(A,p)$ and $\mathrm{QM}(A,p)$ are the ones cut down by $c(p)$. It is also interesting and useful to see if there exists a C^*-subalgebra $\mathcal{B}$ of $A^{**}c(p)$ such that

(a) $$\mathrm{LM}(A,p)c(p) = \mathrm{LM}(\mathcal{B}),$$

(b) $$\mathrm{RM}(A,p)c(p) = \mathrm{RM}(\mathcal{B}),$$

(c) $$\mathrm{M}(A,p)c(p) = \mathrm{M}(\mathcal{B}),$$

(d) $$\mathrm{QM}(A,p)c(p) = \mathrm{QM}(\mathcal{B}).$$

Consider

$$\mathcal{A} = \{x \in A^{**} : \pi_p(x)\mathcal{W}_p \subseteq Ap\}.$$

We think of $\mathcal{A}c(p)$ as a natural candidate for $\mathcal{B}$. It is easy to see that $\mathcal{A}$ is an ideal of the C^*–algebra $\mathrm{M}(A,p)$. Moreover, $\mathrm{LM}(A,p)\mathcal{A} \subseteq \mathcal{A}$, $\mathcal{A}\,\mathrm{RM}(A,p) \subseteq \mathcal{A}$, $\mathrm{M}(A,p)\mathcal{A} + \mathcal{A}\,\mathrm{M}(A,p) \subseteq \mathcal{A}$ and $\mathcal{A}\,\mathrm{QM}(A,p)\mathcal{A} \subseteq \mathcal{A}$.

Example 5.1. If p is central, or equivalently if the ideal $L = A^{**}(1-p) \cap A$ is two-sided, then $Ap \cong A/L$ as C^*-algebras. Consequently, we have $\mathcal{A}c(p) = Ap$ and (a)–(d) hold for $\mathcal{B} = \mathcal{A}c(p)$.

It follows from definitions and Corollary 3.6 that we have

Lemma 5.2. *Let $x \in A^{**}$.*

1. *$x \in \mathcal{A}$ if and only if $pv^*xup \in pAp$ for all $up, vp \in \mathcal{W}_p$.*
2. *$x \in \mathrm{LM}(A,p)$ if and only if $pv^*xap \in pAp$ for all $ap \in Ap$ and $vp \in \mathcal{W}_p$.*

(3) *$x \in \mathrm{RM}(A, p)$ if and only if $pb^*xup \in pAp$ for all $up \in \mathcal{W}_p$ and $bp \in Ap$.*
(4) *$x \in \mathrm{M}(A, p)$ if and only if pv^*xap, $pb^*xup \in pAp$ for all $ap, bp \in Ap$ and $up, vp \in \mathcal{W}_p$.*
(5) *$x \in \mathrm{QM}(A, p)$ if and only if $pb^*xap \in pAp$ for all $ap, bp \in Ap$.*

THEOREM 5.3. *The following conditions are all equivalent and each of them implies* (a)–(d) *for $\mathcal{B} = \mathcal{A}c(p)$:*

(1) *$\pi_p(\mathcal{A})Ap$ is norm dense in Ap.*
(2) *$\pi_p(\mathcal{A})\mathcal{W}_p$ is norm dense in Ap.*
(3) *$\mathcal{A}$ is non-degenerately represented on H_{univ}, that is, $\overline{\pi_\varphi(\mathcal{A})H_\varphi} = H_\varphi$ for all $\varphi \in Q(A)$, where $H_{\mathrm{univ}} = \bigoplus_2\{H_\varphi : \varphi \in Q(A)\}$ is the underlying Hilbert space of the universal representation of A.*
(4) *$\mathcal{A}$ is σ-weakly dense in A^{**}.*
(5) *$\pi_\varphi(\mathcal{A}) \neq \{0\}$ for all pure states φ in $F(p)$.*

Proof. (1)⇒(2) is trivial.

(2)⇒(3): Since $\mathcal{A}$ contains $A^{**}(1 - c(p))$, we may assume φ is supported by $c(p)$. Now, since $\pi_p(\mathcal{A})\mathcal{W}_p$ is norm dense in Ap, we see that $\pi_\varphi(\mathcal{A})(\mathcal{W}_pH_\varphi)$ is dense in $\pi_\varphi(Ap)H_\varphi = ApH_\varphi$, which is dense in $A^{**}pH_\varphi$. Let $q = v^*pv$ be a projection for some partial isometry v in A^{**}. We see that $qH_\varphi = v^*pvH_\varphi \subseteq A^{**}pH_\varphi$. Hence $\pi_\varphi(\mathcal{A})H_\varphi$ is also dense in H_φ, and this gives (3).

(3)⇒(4) follows from the fact that $A\mathcal{A} \subseteq \mathcal{A}$.

(4)⇒(5) is obvious.

(5)⇒(1): Suppose the norm closure $\overline{\pi_p(\mathcal{A})Ap} \neq Ap$. Choose a non-zero φ in $(Ap)^*$ such that $\varphi(\pi_p(\mathcal{A})Ap) = \{0\}$. Let $\{v_\lambda\}_\lambda$ be a positive increasing approximate identity in the C^*-subalgebra $\mathcal{A}$ of A^{**}, and note that $v_\lambda \nearrow q$ for some projection q in A^{**}. For every a in A, $pa^*v_\lambda ap \nearrow pa^*qap$. Note that $pa^*v_\lambda ap \in pAp$. It follows from the continuity of $pa^*v_\lambda ap$ that pa^*qap is lower semicontinuous on $F(p)$. Since $A\mathcal{A} \subseteq \mathcal{A}$, we see that $\overline{\pi_\psi(\mathcal{A})H_\psi}$ is an invariant subspace for $\pi_\psi(A)$ for every ψ in $F(p)$. For each pure state ψ in $F(p)$, the hypothesis $\pi_\psi(\mathcal{A}) \neq \{0\}$ implies $\overline{\pi_\psi(\mathcal{A})H_\psi} = H_\psi$ and hence $\pi_\psi(q) = 1$. Therefore, the non-positive lower semicontinuous affine function

$$\psi \mapsto \psi(pa^*(q-1)ap), \qquad \psi \in F(p),$$

vanishes on the extreme boundary $(F(p) \cap P(A)) \cup \{0\}$ of the weak* compact convex set $F(p)$, where $P(A)$ is the pure state space of A. It follows that $pa^*(q-1)ap = 0$. We then have $qap = ap$ for every a in A. Consequently,

$$\varphi(ap) = \varphi(qap) = \lim \varphi(v_\lambda ap) = 0, \qquad \forall a \in A.$$

This contradiction establishes the implication.

From now on, we assume these equivalent conditions are satisfied and we are going to verify (a) to (d). We prove only that $\mathrm{LM}(\mathcal{B}) \subseteq \mathrm{LM}(A, p)c(p)$

since the opposite inclusion is obvious and the other assertions will follow similarly. Note that we can consider $\mathrm{LM}(\mathcal{B})$ as a subset of $A^{**}c(p)$ (cf. [3, 4.3]).

Let x be a non-zero element of $\mathrm{LM}(\mathcal{B})$ and $\varepsilon > 0$. For each a in A, it follows from (2) that there exist $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in $\mathcal{A}$ and $w_1p, \ldots, w_np$ in $\mathcal{W}_p \subseteq A^{**}p$ such that

$$\Big\| ap - \sum_{k=1}^{n} \mathfrak{a}_k w_k p \Big\| < \frac{\varepsilon}{\|x\|}.$$

Hence

$$\Big\| xap - \sum_{k=1}^{n} x\mathfrak{a}_k w_k p \Big\| < \varepsilon.$$

Since $x \in \mathrm{LM}(\mathcal{B}) \subseteq A^{**}c(p)$, $x\mathfrak{a}_k = x(\mathfrak{a}_k c(p)) \in x(\mathcal{A}c(p)) = x\mathcal{B} \subseteq \mathcal{B}$. Note that elements of $\pi_p(\mathcal{B})$ send $\mathcal{W}_p$ into Ap. Consequently, $\pi_p(x\mathfrak{a}_k)w_kp \in Ap$ for $k = 1, \ldots, n$. It follows that $xap \in \overline{Ap} = Ap$. That is, $x \in \mathrm{LM}(A,p)$. Since $x = xc(p)$, we have $x \in \mathrm{LM}(A,p)c(p)$, too. ∎

Corollary 5.4. *If p has MSQC then* (a)–(d) *are satisfied for $\mathcal{B} = \mathcal{A}c(p)$. Moreover, $Ap + pA \subseteq \mathcal{A}$ in this case.*

Proof. By Theorem 5.3, it suffices to show that $\pi_p(\mathcal{A})p = Ap$ (since $p \in \mathcal{W}_p$). One inclusion is easy:

$$\pi_p(\mathcal{A})p \subseteq \pi_p(\mathcal{A})\mathcal{W}_p \subseteq Ap.$$

For the opposite inclusion, as well as the assertion $Ap + pA \subseteq \mathcal{A}$, it suffices to show that $Ap \subseteq \mathcal{A}$. To this end, let $up, vp \in \mathcal{W}_p$ and $a \in A$. Observe that

$$\begin{aligned} pu^*(apvp) &= (pa^*up)^*vp \\ &\in (pAp)^*vp \\ &= pApvp \\ &\subseteq pAvp \quad \text{since } pAp \subseteq pA \text{ as } p \text{ has MSQC} \\ &\subseteq pAp. \end{aligned}$$

Hence $ap \in \mathcal{A}$ by Lemma 5.2. ∎

We remark that the inclusion in Corollary 5.4 does not hold if p fails to have MSQC (see Example 5.7). Even when p does have MSQC, the inclusion can be strict (see Example 5.8). The rest of this section is devoted to a few assorted results and examples about what $\mathcal{A}$ contains.

Proposition 5.5. *Let $B = pA^{**}p \cap \mathrm{QM}(A,p)$. Then $\mathcal{A}$ contains the norm closure of the linear space spanned by ABA.*

Proof. Since $\mathcal{A}$ is a C^*-algebra, we only need to prove that if $a, c \in A$, $b \in B$ then $abc \in \mathcal{A}$. It is equivalent to show that $pu^*abcvp \in pAp$ for every

up, vp in $\mathcal{W}_p$, by Lemma 5.2. In fact,

$$\begin{aligned} pu^*abcvp &= pu^*apbpcvp && \text{since } b \in pA^{**}p \\ &\in pApbpAp && \text{since } up, vp \in \mathcal{W}_p \\ &= pAbAp && \text{since } b \in pA^{**}p \\ &\subseteq pAp && \text{since } b \in \mathrm{QM}(A,p). \ \blacksquare \end{aligned}$$

Corollary 5.6. *Let* $C = \mathrm{SQC}(p) \cap \mathrm{M}(A,p)$. *Then* $\mathcal{A}$ *contains* C *as a* C^**-subalgebra.*

Proof. Note that C is a C^*-algebra. In particular, $C = C^3$. The assertion now follows from Proposition 5.5 since $C \subseteq pA^{**}p \cap \mathrm{QM}(A,p)$ and $C^3 \subseteq ACA$ (see Lemma 4.6). ■

To convince the readers that B and C in Proposition 5.5 and Corollary 5.6 can be non-zero, we present the following example. In particular, the closed span of ABA is the whole of $\mathcal{A}$, and C is only a proper subalgebra of $\mathcal{A}$ in this example.

Example 5.7. In this example, A is a separable scattered C^*-algebra and p is a closed projection in A^{**} with central support $c(p) = 1$. But p does not have MSQC. We shall see that (a)–(d) are all satisfied here. In fact, $\mathcal{A} = A$, $\mathrm{LM}(A,p) = \mathrm{LM}(A)$, $\mathrm{RM}(A,p) = \mathrm{RM}(A)$, $\mathrm{M}(A,p) = \mathrm{M}(A)$ and $\mathrm{QM}(A,p) = \mathrm{QM}(A)$. Moreover, B and C are both non-zero. Furthermore, ABA is norm dense in $\mathcal{A}$ but $Ap \nsubseteq \mathcal{A}$ (cf. Corollary 5.4).

Let A be the C^*-subalgebra of $c \otimes M_2$ consisting of all sequences of 2×2 matrices $x = (x_n)_{n \geq 1}$ such that

$$x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

We observe that A^{**} can be represented as the C^*-algebra of all uniformly bounded sequences of 2×2 matrices plus a copy of $\mathbb{C}$. More precisely, every element of A^{**} is of the form $x = (x_n)_{n=1}^{\infty}$ where

$$x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad n = 1, 2, \ldots, \quad \text{and} \quad x_\infty = a \in \mathbb{C}.$$

The norm of A^{**} (and A) is given by $\|x\| := \sup_{1 \leq n \leq \infty} \|x_n\| < \infty$. Put

$$p_n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad n = 1, 2, \ldots, \quad \text{and} \quad p_\infty = 1 \in \mathbb{C}.$$

Then $p = (p_n)_{n=1}^{\infty}$ is a closed projection in A^{**} and $c(p) = 1$. Let $x = (x_n)_{n=1}^{\infty} \in A^{**}$, with notation as above. We have:

(1) $x \in Ap \Leftrightarrow x_n = \frac{1}{2}\begin{pmatrix} u_n & u_n \\ v_n & v_n \end{pmatrix}$ with $u_n \to a$ and $v_n \to 0$.

(2) $x \in \mathcal{W}_p \Leftrightarrow x_n = \frac{1}{2}\left(\begin{smallmatrix} u_n & u_n \\ v_n & v_n \end{smallmatrix}\right)$ with $u_n \to a$.

(3) $x \in pA^{**}p \Leftrightarrow x_n = \frac{1}{4}\left(\begin{smallmatrix} s_n & s_n \\ s_n & s_n \end{smallmatrix}\right)$ for some uniformly bounded scalars s_n.

(4) $x \in pAp \Leftrightarrow x_n = \frac{1}{4}\left(\begin{smallmatrix} s_n & s_n \\ s_n & s_n \end{smallmatrix}\right)$ for some scalars $s_n \to a$.

(5) $x \in \mathrm{SQC}(p) \Leftrightarrow x_n = \frac{1}{4}\left(\begin{smallmatrix} s_n & s_n \\ s_n & s_n \end{smallmatrix}\right)$ for some scalars $s_n \to a = 0$.

(6) $x \in \mathrm{LM}(A) = \mathrm{LM}(A,p) \;\Leftrightarrow\; a_n \to a$ and $c_n \to 0$.

(7) $x \in \mathrm{RM}(A) = \mathrm{RM}(A,p) \;\Leftrightarrow\; a_n \to a$ and $b_n \to 0$.

(8) $x \in \mathrm{M}(A) = \mathrm{M}(A,p) \;\Leftrightarrow\; a_n \to a$ and $b_n, c_n \to 0$.

(9) $x \in \mathrm{QM}(A) = \mathrm{QM}(A,p) \;\Leftrightarrow\; a_n \to a$.

(10) $x \in A = \mathcal{A} \;\Leftrightarrow\; a_n \to a$ and $b_n, c_n, d_n \to 0$.

Since $pAp \neq \mathrm{SQC}(p)$, we see that p does not have MSQC by Lemma 4.7. It is clear that both $B = \mathrm{QM}(A,p) \cap pA^{**}p$ and $C = \mathrm{SQC}(p) \cap \mathrm{M}(A,p) = \mathrm{SQC}(p)$ are non-zero. In addition, the closed span $\overline{ABA}$ equals $A = \mathcal{A}$. ∎

Example 5.8. In this example we shall see that $\mathrm{LM}(A,p) \neq \mathrm{LM}(A)$ etc., and $\mathcal{A}$ is neither a subset nor a superset of A even when p has MSQC and its central support $c(p)$ is 1. However, (a) to (d) are all satisfied.

Let A be the C^*-subalgebra of $c \otimes M_2$ given by

$$A = \left\{ \left\{ \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right\}_{n \geq 1} : \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \to \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\}.$$

Let $p = (p_n) \in A^{**}$ with

$$p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad n = 1, 2, \ldots, \quad \text{and} \quad p_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then p is a closed projection in A^{**}. Let $x = (x_n) \in A^{**}$ with

$$x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad n = 1, 2, \ldots, \quad \text{and} \quad x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

We have:

(1) $x \in Ap \Leftrightarrow x_n = \left(\begin{smallmatrix} a_n & 0 \\ c_n & 0 \end{smallmatrix}\right)$ with $a_n \to a$ and $c_n \to 0$.

(2) $x \in \mathcal{W}_p \Leftrightarrow x_n = \left(\begin{smallmatrix} a_n & 0 \\ c_n & 0 \end{smallmatrix}\right)$ with $a_n \to a$.

(3) $x \in pAp \Leftrightarrow x_n = \left(\begin{smallmatrix} a_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $a_n \to a$.

(4) $x \in \mathrm{LM}(A,p) \Leftrightarrow a_n \to a$ and $c_n \to 0$.

(5) $x \in \mathrm{RM}(A,p) \Leftrightarrow a_n \to a$ and $b_n \to 0$.

(6) $x \in \mathrm{M}(A,p) \Leftrightarrow a_n \to a$ and $b_n, c_n \to 0$.

(7) $x \in \mathrm{QM}(A,p) \Leftrightarrow a_n \to a$.

(8) $x \in \mathcal{A} \Leftrightarrow a_n \to a$ and $b_n, c_n, d_n \to 0$.

We first note that $c(p) = 1$. Since pAp is an algebra, p has MSQC by Lemma 4.7. Thus, (a)–(d) are satisfied for $\mathcal{B} = \mathcal{A}$. On the other hand, obviously we have $A \nsubseteq \mathcal{A}$. We also want to point out that $\mathcal{A}$ is *not* contained in A, either. For example, the element $x = (x_n)$ of $\mathcal{A} \subseteq A^{**}$ given by $x_n = 0$, $n = 1, 2, \ldots$, and $x_\infty = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ does not belong to A. It is clear that $\mathrm{LM}(A, p) \neq \mathrm{LM}(A) = A$ etc., since A is unital. ∎

Example 5.9. Consider the C^*-algebra $A = c \otimes \mathcal{K}$ and

$$A^{**} = \{(h_n) : h_n \in B(H),\ 1 \leq n \leq \infty,\ \|h\| = \sup \|h_n\| < \infty\}.$$

Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of the Hilbert space H. Let

$$v_n = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_{n+1}, \quad n < \infty, \quad \text{and} \quad v_\infty = e_1,$$

and

$$p_n = v_n \otimes v_n, \qquad n = 1, 2, \ldots, \infty.$$

Then $p = (p_n)$ is a closed projection in A^{**} without MSQC (cf. [8]) and the central support $c(p)$ of p is 1. We have

(1) $Ap = \big\{(x_n p_n) \in A^{**}p : x_n v_n \xrightarrow{\|\cdot\|} \frac{1}{\sqrt{2}} x_\infty e_1\big\}$.

(2) $\mathcal{W}_p = \big\{(x_n p_n) \in A^{**}p : x_n v_n \xrightarrow{\text{weakly}} \frac{1}{\sqrt{2}} x_\infty e_1\big\}$.

(3) $pAp = \big\{(p_n b_n p_n) : \langle b_n v_n, v_n\rangle \to \frac{1}{2}\langle b_\infty e_1, e_1\rangle\big\}$.

(4) $\mathrm{LM}(A) = \mathrm{LM}(A, p) = \{(t_n) \in A^{**} : t_n \xrightarrow{\text{SOT}} t_\infty\}$.

(5) $\mathrm{RM}(A) = \mathrm{RM}(A, p) = \{(t_n) \in A^{**} : t_n^* \xrightarrow{\text{SOT}} t_\infty^*\}$.

(6) $\mathrm{M}(A) = \mathrm{M}(A, p) = \{(t_n) \in A^{**} : t_n \xrightarrow{\text{DSOT}} t_\infty\}$.

(7) $\mathrm{QM}(A) = \mathrm{QM}(A, p) = \{(t_n) \in A^{**} : t_n \xrightarrow{\text{WOT}} t_\infty\}$.

(8) $\mathcal{A} = \{(t_n) \in A^{**} : t_n \xrightarrow{\|\cdot\|} t_\infty,\ t_\infty \in \mathcal{K}\}$.

By Theorem 5.3 and the fact that $A \subseteq \mathcal{A}$, the equations $\mathrm{LM}(A, p) = \mathrm{LM}(\mathcal{A})$ etc. are satisfied in this case. This can also be verified by direct calculation. ∎

Remark 5.10. In [6], it is shown that for two *separable* C^*-algebras A_1 and A_2, the multiplier algebras $\mathrm{M}(A_1)$ and $\mathrm{M}(A_2)$ are isomorphic if and only if A_1 and A_2 are isomorphic. In fact, A_1 (resp. A_2) is the largest separable closed, two-sided ideal of $\mathrm{M}(A_1)$ (resp. $\mathrm{M}(A_2)$). However, in the inseparable case, this may not be true. A perhaps less artificial illustration to this fact than usual is provided by Example 5.9, since $\mathrm{M}(A) = \mathrm{M}(\mathcal{A})$, A is separable and $\mathcal{A}$ is not separable.

6. Atomic parts of relative multipliers. In the following, $z = z_{\mathrm{at}}$ denotes the maximal atomic projection in A^{**}; in other words, z is the smallest central projection in A^{**} supporting all pure states of A.

LEMMA 6.1. *Let xp and yp be in $\mathcal{W}_p$. If $zxp = zyp$ then $xp = yp$. Moreover, we have $\|xp\| = \|zxp\|$. In other words, weakly continuous vector sections are determined by their atomic parts.*

Proof. For each a in A, the continuous affine function $\varphi \mapsto \varphi(a^*(x-y))$ on $F(p)$ vanishes at all pure states in $F(p)$. Consequently, it is identically zero on $F(p)$. As a result, $pA(x-y)p = \{0\}$, and thus $xp = yp$. For the norm equality, we note that the bounded affine function $\varphi \mapsto \varphi(x^*x)$ is lower semicontinuous on the weak* compact convex set $F(p)$ [9, Lemma 2.1]. It follows from the Krein–Milman theorem that

$$\|xp\|^2 \le \sup\{\varphi(x^*x) : \varphi \text{ is a pure state in } F(p)\} = \|zxp\|^2 \le \|xp\|^2. \ \blacksquare$$

The following theorem says that if the operator section $\pi_p(x)$ preserves the continuity of the atomic part of every vector section in $A^{**}p$ then x itself must have an appropriate atomic part.

THEOREM 6.2. *Let x be an element of A^{**}.*

(1) *$zxAp \subseteq zAp$ if and only if $zx \in z\,\mathrm{LM}(A,p)$.*
(2) *$zx\mathcal{W}_p \subseteq z\mathcal{W}_p$ if and only if $zx \in z\,\mathrm{RM}(A,p)$.*
(3) *$zxAp \subseteq zAp$ and $zx\mathcal{W}_p \subseteq z\mathcal{W}_p$ if and only if $zx \in z\,\mathrm{M}(A,p)$.*
(4) *$zxAp \in z\mathcal{W}_p$ if and only if $zx \in z\,\mathrm{QM}(A,p)$.*
(5) *$zx\mathcal{W}_p \subseteq zAp$ if and only if $zx \in z\mathcal{A}$.*

Proof. The sufficiency is obvious and thus we verify the necessity only. Suppose first that $zxAp \subseteq z\mathcal{W}_p$. By Lemma 6.1, we can define a linear map T from Ap into $\mathcal{W}_p$. More precisely, we set $Tap = up$ if $zxap = zup$. Moreover, $\|T\| \le \|x\|$ since $\|zyp\| = \|yp\|$ for all yp in $\mathcal{W}_p$. Suppose that φ is a pure state in $F(p)$ and a is in A such that $\varphi(a^*a) = 0$. Then $\varphi((Tap)^*(Tap)) = \varphi(u^*u) = \varphi((zup)^*(zup)) = \varphi((xap)^*(xap)) = \varphi(pa^*x^*xap) \le \|x\|^2\varphi(a^*a) = 0$. By Theorem 3.13, there is a relative quasi-multiplier q in $\mathrm{QM}(A,p)$ such that $Tap = qap$ for all a in A. Therefore $zxap = zTap = zqap$ for all a in A. Consequently, $z(x-q)Ap = \{0\}$, and thus $zxc(p) = zq \in z\,\mathrm{QM}(A,p)$.

Consider next the case $zxAp \subseteq zAp$. A similar argument yields a bounded linear map T from Ap into Ap (by restricting the co-domain of T). We thus have an l in $A^{**}c(p)$ such that $lap = Tap \in Ap$ for all a in A. Consequently, $l \in \mathrm{LM}(A,p)$, and thus $zxc(p) = zl \in z\,\mathrm{LM}(A,p)$.

For the case $zx\mathcal{W}_p \subseteq z\mathcal{W}_p$, we note that $zx^*Ap \subseteq zAp$. To see this, we observe that $zpy^*x^*ap = (pa^*zxyp)^* \in zpAp$ for all yp in $\mathcal{W}_p$, and quote [9, Theorem 1.7], which says $zup \in zAp$ if and only if $zpAup \subseteq zpAp$ and $zpu^*up \in zpAp$. Hence there is a relative left multiplier l in A^{**} such that $zx^* = zl$. By setting $r = l^*$, we have $zx = zr \in z\,\mathrm{RM}(A,p)$. The case where $zx\mathcal{W}_p \subseteq zAp$ is similar.

Finally, we suppose that $zxAp \subseteq zAp$ and $zx\mathcal{W}_p \subseteq z\mathcal{W}_p$. By the above observation, there is an l in $\mathrm{LM}(A, p)$ and an r in $\mathrm{RM}(A, p)$ such that $zx = zl = zr$. Now, $pa_1(l - r)a_2p$ belongs to pAp and vanishes at each pure state in $F(p)$ for all a_1, a_2 in A. It follows that $pA(l - r)Ap = \{0\}$. Therefore, $lc(p) = rc(p)$, and thus $zx \in \mathrm{M}(A, p)$. ∎

The following is the special case when $p = 1$.

Corollary 6.3. *Let x be an element of A^{**}.*

(1) *If $zxA \subseteq zA$ then $zx = zl$ for some left multiplier l of A in A^{**}.*
(2) *If $zx\,\mathrm{RM}(A) \subseteq z\,\mathrm{RM}(A)$ then $zx = zr$ for some right multiplier r of A in A^{**}.*
(3) *If $zxA \subseteq zA$ and $zx\,\mathrm{RM}(A) \subseteq z\,\mathrm{RM}(A)$ then $zx = zm$ for some multiplier m of A in A^{**}.*
(4) *If $zxA \subseteq z\,\mathrm{RM}(A)$ then $zx = zq$ for some quasi-multiplier q of A in A^{**}.*
(5) *If $zx\,\mathrm{RM}(A) \subseteq zA$ then $zx = za$ for some a in A.*

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