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Locally Lipschitz continuous integrated semigroups

by

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Abstract. This paper is concerned with the problem of real characterization of locally Lipschitz continuous $(n+1)$-times integrated semigroups, where n is a nonnegative integer. It is shown that a linear operator is the generator of such an integrated semigroup if and only if it is closed, its resolvent set contains all sufficiently large real numbers, and a stability condition in the spirit of the finite difference approximation theory is satisfied.

1. Introduction. Let X be a Banach space and $B(X)$ the set of all bounded linear operators from X into itself. Let n be a positive integer. A family $\{U(t);\, t \geq 0\}$ in $B(X)$ is called an *n-times integrated semigroup on X* if the following conditions are satisfied:

(I1) $U(\cdot)x : [0,\infty) \to X$ is continuous for $x \in X$.

(I2) $$U(t)U(s)x = \frac{1}{(n-1)!}\Big(\int_t^{t+s} (t+s-r)^{n-1}U(r)x\,dr - \int_0^s (t+s-r)^{n-1}U(r)x\,dr\Big)$$

for $x \in X$ and $t, s \geq 0$.

(I3) $U(t)x = 0$ for all $t > 0$ implies $x = 0$.

Let $\{U(t); t \geq 0\}$ be an n-times integrated semigroup on X. Then the *generator* A of $\{U(t); t \geq 0\}$ is defined in the following way: $x \in D(A)$ and $y = Ax$ if and only if

$$U(t)x = \int_0^t U(r)y\,dr + \frac{t^n}{n!}x \quad \text{for } t \geq 0. \tag{1.1}$$

The densely defined generators in the exponentially bounded case were characterized by Neubrander [10], and his result was extended to the following two cases:

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(i) Arendt [1] gave a characterization of the generator of an $(n+1)$-times integrated semigroup $\{U(t);\ t \geq 0\}$ on X which is exponentially Lipschitz continuous in the sense that $\|U(t+h) - U(t)\| \leq Me^{\omega(t+h)}h$ for $t \geq 0$ and $h \geq 0$, in the case where the domain of the generator is not necessarily dense in X.

(ii) Tanaka and Okazawa [14] characterized the densely defined generators of n-times integrated semigroups which are not necessarily exponentially bounded.

An integrated semigroup $\{U(t); t \geq 0\}$ on X is said to be *locally Lipschitz continuous* if for each $\tau > 0$ there exists $L_\tau > 0$ such that $\|U(t) - U(s)\| \leq L_\tau|t-s|$ for $t, s \in [0, \tau]$. It is shown in the final part of Section 3 that there exists a non-densely defined operator which is the generator of a locally Lipschitz continuous twice integrated semigroup on l^∞ but not the generator of any exponentially Lipschitz continuous $(n+1)$-times integrated semigroup on l^∞ for every nonnegative integer n. The twice integrated semigroup constructed cannot be dealt with by the above-mentioned results.

We are interested in locally Lipschitz continuous $(n+1)$-times integrated semigroups which are not necessarily exponentially bounded and in characterizing their generators whose domains are not necessarily dense in X. If $n = 0$ then our objective has already been accomplished, since Kellerman and Hieber [5] showed that every locally Lipschitz continuous once integrated semigroup is always exponentially Lipschitz continuous. The main result of this paper is given by

Main Theorem. *Let n be a nonnegative integer. An operator A is the generator of a locally Lipschitz continuous $(n+1)$-times integrated semigroup $\{U(t);\ t \geq 0\}$ on X if and only if the following conditions are satisfied*:

(A1) *A is a closed linear operator in X whose resolvent set contains (ω, ∞) for some $\omega \geq 0$.*

(A2) *For each $\tau > 0$ there exists $M_\tau > 0$ such that*

$$\left\|\prod_{l=1}^{k}(I - h_l A)^{-1}x\right\| \leq M_\tau\|x\|_n$$

for $x \in D(A^n)$ and every finite sequence $\{h_l\}_{l=1}^k$ of positive numbers such that $h_l\omega < 1$ for $1 \leq l \leq k$ and $\sum_{l=1}^k h_l \leq \tau$. Here $\|x\|_n = \sum_{j=0}^n \|A^j x\|$ for $x \in D(A^n)$.

Condition (A2) can be regarded as a stability condition from the viewpoint of the finite difference approximation theory. In fact, if $\{0 = t_0 < t_1 < \cdots < t_k \leq \tau\}$ is a partition of $[0, t_k]$ with $(t_l - t_{l-1})\omega < 1$ for $1 \leq l \leq k$ and $\{x_l\}_{l=1}^k$ is a solution of the finite difference equation

$$(x_l - x_{l-1})/(t_l - t_{l-1}) = Ax_l \quad \text{for } 1 \leq l \leq k$$

with the initial condition $x_0 = x \in D(A^n)$, then condition (A2) gives the estimate $\|x_l\| \le M_\tau \|x\|_n$ for $1 \le l \le k$. For this reason, condition (A2) is natural in the study of the abstract Cauchy problem

$$\text{(ACP; } x\text{)} \qquad u'(t) = Au(t) \quad \text{for } t \ge 0, \qquad u(0) = x.$$

By a *solution u to* (ACP; x) we mean that $u \in C^1([0,\infty); X)$ and u satisfies the equation (ACP; x).

In Section 2 we investigate some basic properties of generators of locally Lipschitz continuous $(n+1)$-times integrated semigroups and prove the necessity part of the main theorem. Section 3 concerns the generation of locally Lipschitz continuous $(n+1)$-times integrated semigroups on X and the relationship between the main theorem and some previous results.

2. Basic properties of generators of locally Lipschitz continuous integrated semigroups. Let n be a nonnegative integer. Let A be the generator of a locally Lipschitz continuous $(n+1)$-times integrated semigroup $\{U(t); t \ge 0\}$ on X. Then it is known [9] that A is a closed linear operator in X with the following properties:

$$U(t)x \in D(A), \quad AU(t)x = U(t)Ax \quad \text{for } x \in D(A) \text{ and } t \ge 0, \tag{2.1}$$

$$\int_0^t U(s)x\,ds \in D(A), \quad A\int_0^t U(s)x\,ds = U(t)x - \frac{t^{n+1}}{(n+1)!}\,x \quad \text{for } x \in X \text{ and } t \ge 0. \tag{2.2}$$

To investigate some properties of the generator of a locally Lipschitz continuous $(n+1)$-times integrated semigroup, we use a method similar to that due to Sanekata [12], but more delicate arguments are required here. For each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, define $R_0(\lambda) \in B(X)$ by

$$R_0(\lambda)x = \lambda^{n+1}\int_0^1 e^{-\lambda t}U(t)x\,dt \quad \text{for } x \in X.$$

LEMMA 2.1. *There exists $M > 0$ such that*

$$\|R_0(\lambda)\| \le M|\lambda|^n \quad \textit{for } \lambda \in \mathbb{C} \textit{ with } \operatorname{Re}\lambda > 0.$$

Proof. Let $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda > 0$. Let $x \in X$ and $x^* \in X^*$. By the local Lipschitz continuity of $\{U(t); t \ge 0\}$, we see that $\langle x^*, U(t)x\rangle$ is Lipschitz continuous on $[0,1]$ and $|(d/dt)\langle x^*, U(t)x\rangle| \le L\|x^*\|\,\|x\|$ for almost all $t \in (0,1)$, where $L > 0$ is a constant. By integration by parts we have

$$\langle x^*, R_0(\lambda)x\rangle = \lambda^n\Big(-e^{-\lambda}\langle x^*, U(1)x\rangle + \int_0^1 e^{-\lambda t}(d/dt)\langle x^*, U(t)x\rangle\,dt\Big).$$

The desired inequality follows readily from this equality. ∎

For each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$, define $Q_0(\lambda) \in B(X)$ by

$$Q_0(\lambda)x = \lambda^{n+1}e^{-\lambda}U(1)x + \sum_{k=0}^{n} e^{-\lambda}\,\frac{\lambda^k}{k!}\,x \quad \text{for } x \in X.$$

LEMMA 2.2. (i) *There exists* $M > 0$ *such that*

$$\|Q_0(\lambda)\| \le Me^{-\operatorname{Re}\lambda}(1+|\lambda|)^{n+1} \quad \textit{for } \lambda \in \mathbb{C} \textit{ with } \operatorname{Re}\lambda > 0.$$

(ii) *For each* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > 0$ *we have*

$$R_0(\lambda)(\lambda I - A)x = (I - Q_0(\lambda))x \quad \textit{for } x \in D(A), \tag{2.3}$$

$$R_0(\lambda)x \in D(A), \quad (\lambda I - A)R_0(\lambda) = (I - Q_0(\lambda))x \quad \textit{for } x \in X. \tag{2.4}$$

Proof. Let $\lambda \in \mathbb{C}$ and $\operatorname{Re}\lambda > 0$. Since $\max_{0\le k\le n}(1/k!) \le e$ and

$$(1+|\lambda|)^{n+1} \ge \sum_{k=0}^{n+1} |\lambda|^k,$$

assertion (i) is easily verified. To prove (ii), let $x \in X$. Since

$$R_0(\lambda)x = \lambda^{n+1}\Big(e^{-\lambda}\int_0^1 U(s)x\,ds + \int_0^1 \lambda e^{-\lambda t}\Big(\int_0^t U(s)x\,ds\Big)dt\Big),$$

it follows from (2.2) that $R_0(\lambda)x \in D(A)$ and

$$\begin{aligned}(\lambda I - A)R_0(\lambda)x = {}& \frac{\lambda^{n+1}}{(n+1)!}\,e^{-\lambda}x + \int_0^1 \lambda e^{-\lambda t}\,\frac{(\lambda t)^{n+1}}{(n+1)!}\,x\,dt \\ & - \lambda^{n+1}e^{-\lambda}U(1)x.\end{aligned} \tag{2.5}$$

Integration by parts yields

$$\int_0^1 \lambda e^{-\lambda t}\,\frac{(\lambda t)^{n+1}}{(n+1)!}\,dt = 1 - e^{-\lambda}\sum_{k=0}^{n+1}\frac{\lambda^k}{k!}.$$

Substituting this equality into (2.5) we obtain (2.4). Since A is closed, by (2.1) we have $R_0(\lambda)z \in D(A)$ and $AR_0(\lambda)z = R_0(\lambda)Az$ for $z \in D(A)$. This fact together with (2.4) implies (2.3). ∎

PROPOSITION 2.3. (i) *The resolvent set of* A *contains a region*

$$\begin{aligned}\Omega = {}& \{\lambda \in \mathbb{C}\setminus\mathbb{R};\ \operatorname{Re}\lambda \ge \alpha\log|\operatorname{Im}\lambda| + \beta,\ \operatorname{Re}\lambda \ge \gamma\} \\ & \cup \{\lambda \in \mathbb{R};\ \lambda \ge \gamma\},\end{aligned}$$

where α, β *and* γ *are positive numbers.*

(ii) *There exists* $M > 0$ *such that*

$$\|(\lambda I - A)^{-1}\| \le M|\lambda|^n \quad \textit{for } \lambda \in \Omega.$$

Proof. Let $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}$ satisfy

$$\eta \neq 0, \qquad \xi \geq \alpha \log|\eta| + \beta, \qquad \xi \geq \gamma,$$

where α, β and γ are yet to be determined. Then we have $|\eta|^\alpha \leq e^{-\beta+\xi}$. If $\alpha > 0$ is chosen such that $\alpha = n+1$, then $e^{-\xi}((1+|\xi|)^{n+1} + |\eta|^{n+1})$ vanishes as $\xi \to \infty$ and $\beta \to \infty$. This fact implies that the positive numbers γ and β can be chosen so large that $\|Q_0(\lambda)\| \leq 1/2$ for $\lambda \in \Omega$, by Lemma 2.2(i). Since $(I - Q_0(\lambda))^{-1} \in B(X)$ exists and $\|(I - Q_0(\lambda))^{-1}\| \leq 2$ for $\lambda \in \Omega$, the proposition follows from Lemmas 2.1 and 2.2. ■

To prove the necessity part of the main theorem, we define

$$T_h(t)x = h^{-1}(U(t+h) - U(t))A^n x + \sum_{k=0}^{n-1} \frac{t^k}{k!} A^k x$$

for $x \in D(A^n)$, $t \geq 0$ and $h \neq 0$, and

$$\varepsilon_h(t) = \frac{t^n}{n!} - \frac{1}{h}\left(\frac{(t+h)^{n+1}}{(n+1)!} - \frac{t^{n+1}}{(n+1)!}\right)$$

for $t \geq 0$ and $h \neq 0$.

LEMMA 2.4. *Let $x \in D(A^n)$ and $\lambda \in \Omega$. Then*

$$\begin{aligned}(2.6)\quad (\lambda I - A)^{-1}x = & \int_0^{\tau_0} e^{-\lambda t} T_h(t)x\,dt + e^{-\lambda\tau_0} T_h(\tau_0)(\lambda I - A)^{-1}x \\ & + \Big(e^{-\lambda\tau_0}\varepsilon_h(\tau_0) + \int_0^{\tau_0} \lambda e^{-\lambda t}\varepsilon_h(t)\,dt\Big)(\lambda I - A)^{-1}A^n x \\ & - h^{-1}\int_0^h U(s)A^{n+1}(\lambda I - A)^{-1}x\,ds\end{aligned}$$

for any $\tau_0 > 0$ and $h \neq 0$.

Proof. Let $x \in D(A^n)$, $\lambda \in \Omega$, $\tau_0 > 0$ and $h \neq 0$. Integration by parts yields

$$\begin{aligned}\int_0^{\tau_0} & e^{-\lambda t}(U(t+h) - U(t))A^n x\,dt \\ & = e^{-\lambda\tau_0}\int_0^{\tau_0}(U(s+h) - U(s))A^n x\,ds \\ & \quad + \int_0^{\tau_0}\lambda e^{-\lambda t}\Big(\int_0^t (U(s+h) - U(s))A^n x\,ds\Big)dt.\end{aligned}$$

By (2.2) we see that the right-hand side belongs to $D(A)$ and

$$
\begin{aligned}
A\int_0^{\tau_0} & e^{-\lambda t}(U(t+h)-U(t))A^n x\,dt \\
&= e^{-\lambda\tau_0}\Big(U(\tau_0+h)A^n x - \frac{(\tau_0+h)^{n+1}}{(n+1)!}A^n x - U(\tau_0)A^n x + \frac{\tau_0^{n+1}}{(n+1)!}A^n x\Big) \\
&\quad + \int_0^{\tau_0}\lambda e^{-\lambda t}\Big(U(t+h)A^n x - \frac{(t+h)^{n+1}}{(n+1)!}A^n x \\
&\qquad\qquad - U(t)A^n x + \frac{t^{n+1}}{(n+1)!}A^n x\Big)\,dt \\
&\quad - \Big(U(h)A^n x - \frac{h^{n+1}}{(n+1)!}A^n x\Big).
\end{aligned}
$$

The last term on the right-hand side is equal to $-A\int_0^h U(s)A^n x\,ds$ by (2.2). By integration by parts we have

$$
A\int_0^{\tau_0} e^{-\lambda t}\sum_{k=0}^{n-1}\frac{t^k}{k!}A^k x\,dt = e^{-\lambda\tau_0}\sum_{k=1}^{n}\frac{(\tau_0 A)^k}{k!}x + \lambda\int_0^{\tau_0} e^{-\lambda t}\sum_{k=1}^{n}\frac{(tA)^k}{k!}x\,dt.
$$

The desired equality is obtained by combining the equalities above. ∎

Let $\tau > 0$ and choose $\tau_0 > 2\tau$. By the local Lipschitz continuity of $\{U(t);\, t \ge 0\}$ there exists $M_\tau > 0$ such that $\|T_h(t)x\| \le M_\tau\|x\|_n$ for $t \in [0,\tau_0]$ and $h \in (0,1]$. Here and below M_τ denotes various constants depending on τ.

Let α, β and γ be the positive numbers in Proposition 2.3(i), and let $\alpha_0 > \max\{\alpha, (n+1)/(\tau_0-2\tau)\}$, $\beta_0 > \beta$, $\omega_0 > \max\{\gamma, \beta_0\}$ and $\eta_0 = \exp((\omega_0-\beta_0)/\alpha_0)$. Then we define $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{\zeta = \xi + i\eta;\, \xi = \alpha_0\log|\eta| + \beta_0,\, |\eta| \ge \eta_0\}$ and $\Gamma_2 = \{\zeta = \omega_0 + i\eta; |\eta| \le \eta_0\}$. Here i stands for the imaginary unit. Notice that Γ is oriented so that $\operatorname{Im}\zeta$ increases along Γ.

Lemma 2.5. *Let $x^* \in X^*$ and $x \in D(A^n)$. Then there exist a measurable function f on $(0,\tau_0)$ and a holomorphic function g on Ω such that*

$$ |f(t)| \le M_\tau\|x^*\|\,\|x\|_n \quad \textit{for almost all } t \in (0,\tau_0), \tag{2.7} $$

$$ |g(\zeta)| \le M_\tau|\zeta|^n\|x^*\|\,\|x\|_n \quad \textit{for } \zeta \in \Omega, \tag{2.8} $$

$$ \langle x^*, (\lambda I - A)^{-1}x\rangle = \int_0^{\tau_0} e^{-\lambda t}f(t)\,dt + \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda-\zeta}e^{-\zeta\tau_0}g(\zeta)\,d\zeta \quad \textit{for } \lambda > \omega_0. \tag{2.9} $$

Proof. Let $x^* \in X^*$ and $x \in D(A^n)$. Since $\langle x^*, U(t)A^n x\rangle$ is Lipschitz continuous on $[0,\tau_0]$, it is differentiable for almost all $t \in (0,\tau_0)$, so that the limit $f(t) := \lim_{h\downarrow 0}\langle x^*, T_h(t)x\rangle$ exists for almost all $t \in (0,\tau_0)$. Clearly,

f is measurable and satisfies (2.7). If $\alpha_0\tau_0 > n+1$ and $\lambda > \omega_0$ then $\int_\Gamma |\lambda-\zeta|^{-1}e^{-(\mathrm{Re}\,\zeta)\tau_0}|\zeta|^n\,|d\zeta| < \infty$ and the integral

$$\int_{R-s_Ri}^{R+s_Ri} |\lambda-\zeta|^{-1}e^{-(\mathrm{Re}\,\zeta)\tau_0}|\zeta|^n\,|d\zeta|$$

tends to zero as $R\to\infty$, where $s_R = \exp((R-\beta_0)/\alpha_0)$. Since $\Gamma\subset\Omega$, by Cauchy's integral formula we have

$$e^{-\lambda\tau_0}T_h(\tau_0)(\lambda I-A)^{-1}x = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda-\zeta}\,e^{-\zeta\tau_0}T_h(\tau_0)(\zeta I-A)^{-1}x\,d\zeta$$

for $\lambda > \omega_0$ and $h\in(0,1]$. Here we have used Proposition 2.3(ii) and the condition $\alpha_0\tau_0 > n+1$. Since $U(\cdot)z : [0,\infty)\to X$ is differentiable for $z\in D(A)$, the limit $g(\zeta) := \lim_{h\downarrow 0}\langle x^*, T_h(\tau_0)(\zeta I-A)^{-1}x\rangle$ exists for all $\zeta\in\Omega$. By Proposition 2.3(ii) we see that g satisfies (2.8). Taking the weak limit in (2.6) as $h\downarrow 0$ we obtain (2.9) by Lebesgue's convergence theorem. Here we have used the strong continuity of $\{U(t);\, t\ge 0\}$ and $U(0)=0$. ■

Set $R(\lambda) = (\lambda I-A)^{-1}$ for $\lambda\in\Omega$. By applying the resolvent equation $(\lambda-\mu)R(\lambda)R(\mu) = R(\mu)-R(\lambda)$ for $\lambda,\mu\in\Omega$, the following lemma can be proved by induction.

LEMMA 2.6. *Let $k\ge 2$. Then*

$$\prod_{l=1}^k R(\lambda_l) = (-1)^{k-1}\int_{D_{k-1}} F_{\lambda_k,\dots,\lambda_1}(\sigma_1,\dots,\sigma_{k-1})\,d\sigma_1\cdots d\sigma_{k-1} \tag{2.10}$$

for every finite sequence $\{\lambda_l\}_{l=1}^k$ with $\lambda_l > \omega_0$ for $1\le l\le k$, where

$$F_{\lambda_k,\dots,\lambda_1}(\sigma_1,\dots,\sigma_{k-1}) = R^{(k-1)}\Big(\lambda_1\sigma_1+\dots+\lambda_{k-1}\sigma_{k-1}+\lambda_k\Big(1-\sum_{l=1}^{k-1}\sigma_l\Big)\Big),$$

$$D_{k-1} = \Big\{(\sigma_1,\dots,\sigma_{k-1});\ \sigma_l\ge 0 \textit{ for } 1\le l\le k-1 \textit{ and } \sum_{l=1}^{k-1}\sigma_l\le 1\Big\}.$$

LEMMA 2.7. *For $x\in D(A^n)$ and every finite sequence $\{h_l\}_{l=1}^k$ of positive numbers with $h_l\omega_0 < 1$ for $1\le l\le k$ and $k\ge 1$, we have*

$$\Big\|\prod_{l=1}^k (I-h_lA)^{-1}x\Big\| \le M_\tau\Big(1+\frac{1}{2\pi}\int_\Gamma\prod_{l=1}^k|1-\zeta h_l|^{-1}e^{-(\mathrm{Re}\,\zeta)\tau_0}|\zeta|^n\,|d\zeta|\Big)\|x\|_n. \tag{2.11}$$

Proof. Let $x \in D(A^n)$ and $x^* \in X^*$. Let $k \geq 2$ and let $\{\lambda_l\}_{l=1}^k$ be any sequence such that $\lambda_l > \omega_0$ for $1 \leq l \leq k$. If $(\sigma_1, \ldots, \sigma_{k-1}) \in D_{k-1}$ then $\lambda_1\sigma_1 + \cdots + \lambda_{k-1}\sigma_{k-1} + \lambda_k(1 - \sum_{l=1}^{k-1} \sigma_l) > \omega_0$. By (2.9) and (2.10) we see that $\langle x^*, \prod_{l=1}^k R(\lambda_l)x\rangle$ can be written as

$$\int_{D_{k-1}} d\sigma_1 \cdots d\sigma_{k-1}\Big(\int_0^{\tau_0} t^{k-1} e^{-(\lambda_1\sigma_1+\cdots+\lambda_{k-1}\sigma_{k-1}+\lambda_k(1-\sum_{l=1}^{k-1}\sigma_l))t} f(t)\, dt$$

$$+\frac{1}{2\pi i}\int_\Gamma (-1)^{k-1} r_\zeta^{(k-1)}\Big(\lambda_1\sigma_1+\cdots+\lambda_{k-1}\sigma_{k-1}+\lambda_k\Big(1-\sum_{l=1}^{k-1}\sigma_l\Big)\Big)e^{-\zeta\tau_0} g(\zeta)\, d\zeta\Big),$$

where $r_\zeta(\lambda) = (\lambda - \zeta)^{-1}$ for $\lambda > \omega_0$ and $\zeta \in \Gamma$. Changing the variable, by Lemma 2.6 we have

$$\int_{D_{k-1}} d\sigma_1 \cdots d\sigma_{k-1} \int_0^\infty t^{k-1} e^{-(\lambda_1\sigma_1+\cdots+\lambda_{k-1}\sigma_{k-1}+\lambda_k(1-\sum_{l=1}^{k-1}\sigma_l))t}\, dt = (\lambda_1 \cdots \lambda_k)^{-1}.$$

Since

$$\int_\Gamma |r_\zeta^{(k-1)}(\lambda_1\sigma_1 + \cdots + \lambda_{k-1}\sigma_{k-1} + \lambda_k(1 - \sigma_1 - \cdots - \sigma_{k-1}))e^{-\zeta\tau_0} g(\zeta)|\, |d\zeta|$$
$$\leq ((-1)^{k-1} r_{\omega_0}^{(k-1)}(\lambda_1\sigma_1 + \cdots + \lambda_{k-1}\sigma_{k-1} + \lambda_k(1 - \sigma_1 - \cdots - \sigma_{k-1})) + 1) \times M_\tau \|x^*\|\, \|x\|_n$$

and the right-hand side is integrable on D_{k-1}, we apply Fubini's theorem to find

$$(-1)^{k-1} \int_{D_{k-1}} d\sigma_1 \cdots d\sigma_{k-1} \int_\Gamma r_\zeta^{(k-1)}\Big(\sum_{l=1}^{k-1}\lambda_l\sigma_l + \lambda_k\Big(1 - \sum_{l=1}^{k-1}\sigma_l\Big)\Big)e^{-\zeta\tau_0} g(\zeta)\, d\zeta = \int_\Gamma \prod_{l=1}^k (\lambda_l - \zeta)^{-1} e^{-\zeta\tau_0} g(\zeta)\, d\zeta.$$

The desired inequality is obtained by combining these equalities. ∎

Once the following lemma is shown, the proof of the necessity part of the main theorem is completed, since the right-hand side of (2.11) is bounded by $M_\tau\|x\|_n$, under the condition $\alpha_0 > (n+1)/(\tau_0 - 2\tau)$.

Lemma 2.8. *There exists $M_\tau > 0$ such that*

$$\prod_{l=1}^k |1 - \zeta h_l|^{-1} \leq M_\tau \exp(2\tau \operatorname{Re}\zeta)$$

for $\zeta \in \Gamma$ and every finite sequence $\{h_l\}_{l=1}^k$ of positive numbers with $h_l\omega_0 \leq 1/2$ for $1 \leq l \leq k$ and $\sum_{l=1}^k h_l \leq \tau$.

Proof. Let $\zeta = \xi + i\eta \in \Gamma$ and let $\{h_l\}_{l=1}^k$ be a finite sequence of positive numbers such that $h_l\omega_0 \le 1/2$ for $1 \le l \le k$ and $\sum_{l=1}^k h_l \le \tau$. We divide the set $\{1, \ldots, k\}$ into the disjoint sets $I_1 = \{l;\, h_l\xi \in [0, 1/2]\}$ and $I_2 = \{l;\, \zeta = \xi + i\eta \in \Gamma_1,\, h_l\xi > 1/2\}$. If $l \in I_1$ then

$$|1 - \zeta h_l|^{-1} \le (1 - (\operatorname{Re}\zeta)h_l)^{-1} \le \exp(2(\operatorname{Re}\zeta)h_l).$$

Here we have used the fact $(1-t)^{-1} \le \exp(2t)$ for $t \in [0, 1/2]$. If $l \in I_2$ then

$$|1 - \zeta h_l|^{-1} \le |\eta|^{-1}h_l^{-1} \le 2\xi/|\eta|.$$

Hence

$$\prod_{l=1}^k |1 - \zeta h_l|^{-1} \le \exp(2\xi\tau)\sup\{(2(\alpha_0 \log|\eta| + \beta_0)/|\eta|)^{|I_2|};\, |\eta| \ge \eta_0\},$$

where $|I_2|$ denotes the number of elements in I_2. To estimate $|I_2|$, let $\xi + i\eta \in \Gamma_1$ and $h_l\xi > 1/2$. Since $\xi = \alpha_0 \log|\eta| + \beta_0 > 0$ we have

$$|I_2|/2 = \sum_{l\in I_2}(1/2) < \xi \sum_{l\in I_2} h_l \le (\alpha_0 \log|\eta| + \beta_0)\tau,$$

so that $|I_2| \le 2(\alpha_0 \log|\eta| + \beta_0)\tau$. Since $\lim_{t\to\infty}(\alpha_0 \log t + \beta_0)/t = 0$, it follows that the set $\{\eta;\, 2(\alpha_0 \log|\eta| + \beta_0)/|\eta| > 1\}$ is bounded in $\mathbb{R}$. These facts together imply that $(2(\alpha_0 \log|\eta| + \beta_0)/|\eta|)^{|I_2|}$ is bounded from above for $|\eta| \ge \eta_0$. The proof is thus complete. ∎

3. Generation of locally Lipschitz continuous integrated semigroups. Let A be an operator in X satisfying conditions (A1) and (A2). For simplicity of notation, we write $J_h = (I - hA)^{-1}$ for $h \in (0, h_0]$, where $h_0 > 0$ is such that $h_0\omega < 1$. Let $\lambda, \mu \in (0, h_0]$ and set $A_{k,l} = J_\lambda^k - J_\mu^l$ for $k, l \ge 0$. To prove the sufficiency part of the main theorem, let $\tau > 0$ and $z \in D(A^{n+1})$. For $0 \le k \le [\tau/\lambda]$ and $0 \le l \le [\tau/\mu]$ we define

$$a_{k,l} = \max_p \|J_\sigma^p A_{k,l} z\|,$$

where $\sigma = \lambda\mu/(\lambda + \mu)$ and the maximum is taken over all nonnegative integers p such that $\sigma p + \lambda k + \mu l \le 2\tau$. The use of the quantity $a_{k,l}$ is a new idea.

LEMMA 3.1. *For* $0 \le k \le [\tau/\lambda]$ *and* $0 \le l \le [\tau/\mu]$ *we have*

$$a_{k,l} \le M_{2\tau}((k\lambda - l\mu)^2 + k\lambda^2 + l\mu^2)^{1/2}\|Az\|_n.$$

Proof. Let $0 \le k \le [\tau/\lambda]$ and consider any nonnegative integer p such that $\sigma p + k\lambda \le 2\tau$. Since

$$J_\sigma^p A_{k,0} z = J_\sigma^p \sum_{j=1}^k (J_\lambda^j z - J_\lambda^{j-1} z) = \lambda \sum_{j=1}^k J_\sigma^p J_\lambda^j A z$$

and $\|J_\sigma^p J_\lambda^j Az\| \le M_{2\tau}\|Az\|_n$ for $1 \le j \le k$ (by condition (A2)), we have $a_{k,0} \le M_{2\tau}\|Az\|_n k\lambda$. Similarly, $a_{0,l} \le M_{2\tau}\|Az\|_n l\mu$.

Now, let $1 \le k \le [\tau/\lambda]$ and $1 \le l \le [\tau/\mu]$, and consider any positive integer p with $p\sigma + k\lambda + l\mu \le 2\tau$. By the resolvent equation we have

$$J_\lambda v = J_\sigma\Big(\frac{\mu}{\lambda+\mu}\, v + \frac{\lambda}{\lambda+\mu}\, J_\lambda v\Big),$$
$$J_\mu w = J_\sigma\Big(\frac{\lambda}{\lambda+\mu}\, w + \frac{\mu}{\lambda+\mu}\, J_\mu w\Big) \quad \text{for } v, w \in X.$$

Using these equalities with $v = J_\lambda^{k-1} z$ and $w = J_\mu^{l-1} z$ we find

$$J_\sigma^p A_{k,l} z = \frac{\lambda}{\lambda+\mu}\, J_\sigma^{p+1} A_{k,l-1} z + \frac{\mu}{\lambda+\mu}\, J_\sigma^{p+1} A_{k-1,l} z.$$

Since $\sigma \le \lambda$ and $\sigma \le \mu$, we notice that $(p+1)\sigma + k\lambda + (l-1)\mu \le 2\tau$ and $(p+1)\sigma + (k-1)\lambda + l\mu \le 2\tau$. By the definition of $a_{k,l-1}$ and $a_{k-1,l}$ we have

$$a_{k,l} \le \frac{\lambda}{\lambda+\mu}\, a_{k,l-1} + \frac{\mu}{\lambda+\mu}\, a_{k-1,l}.$$

The desired inequality is proved by induction. (See also [7] and [16, Chapter XIV, Section 7].) ∎

Proof of the sufficiency part of the main theorem. Let $x \in X$ and $\tau > 0$. Choose $c > \omega$ and set $C = (cI - A)^{-(n+1)}$. By Lemma 3.1 we have

$$\|J_\lambda^{[t/\lambda]} Cx - J_\mu^{[s/\mu]} Cx\| \\ \le M_{2\tau}(([t/\lambda]\lambda - [s/\mu]\mu)^2 + [t/\lambda]\lambda^2 + [s/\mu]\mu^2)^{1/2}\|ACx\|_n$$

for $\lambda, \mu \in (0, h_0]$ and $t \in [0,\tau]$. This implies that $S(t)x = \lim_{\lambda\downarrow 0} J_\lambda^{[t/\lambda]} Cx$ exists for all $t \ge 0$ and the family $\{S(t);\, t \ge 0\}$ is a locally Lipschitz continuous C-regularized semigroup on X.

Let $z \in X$ and $t \ge 0$. Since

$$J_\lambda^{[t/\lambda]} Cz - Cz = \sum_{k=1}^{[t/\lambda]} (J_\lambda^k Cz - J_\lambda^{k-1} Cz) = A\Big(\lambda \sum_{k=1}^{[t/\lambda]} J_\lambda^k Cz\Big)$$
$$= A \int_\lambda^{([t/\lambda]+1)\lambda} J_\lambda^{[s/\lambda]} Cz\, ds,$$

by the closedness of A we have $\int_0^t S(s)z\, ds \in D(A)$ and

$$A\int_0^t S(s)z\, ds = S(t)z - Cz. \tag{3.1}$$

For $k = 1, \dots, n+1$, define a family $\{V_k(t);\, t \ge 0\}$ in $B(X)$ by

$$V_k(t)z = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} S(t_k)z\, dt_k \cdots dt_1$$

for $z \in X$ and $t \ge 0$. Then it is proved similarly to [13, Lemma] that the following hold for $k = 1, \dots, n+1$.

(i) $V_k(t)z \in D(A^k)$ and $\int_0^t (cI - A)^{k-1} V_{k-1}(s)z\, ds \in D(A)$ for $z \in X$ and $t \ge 0$.

(ii) $(cI - A)^k V_k(t) \in B(X)$ and there exist $K_k > 0$ and $\omega_k \ge 0$ such that

$$\|(cI - A)^k V_k(t) - (cI - A)^k V_k(s)\| \le \max(M_{2\tau}, 1) K_k e^{\omega_k t}(t - s) \tag{3.2}$$

for $0 \le s \le t \le \tau$ and $\tau > 0$.

(iii) We have

$$(cI - A)^k V_k(t) = c(cI - A)^{k-1} V_k(t) - (cI - A)^{k-1} V_{k-1}(t) + \frac{t^{k-1}}{(k-1)!}(cI - A)^{k-1} C \quad \text{for } t \ge 0.$$

By the above fact with $k = n+1$, the family $\{U(t);\, t \ge 0\}$ in $B(X)$ defined by $U(t) = (cI - A)^{n+1} V_{n+1}(t)$ for $t \ge 0$ is locally Lipschitz continuous. By [14, Lemma 4.8] we see that $\{U(t);\, t \ge 0\}$ is an $(n+1)$-times integrated semigroup on X.

To prove that A is the generator of $\{U(t);\, t \ge 0\}$, let $u \in D(A)$. The $(n+1)$-fold integration of (3.1) implies

$$U(t)u = \frac{t^{n+1}}{(n+1)!} u + \int_0^t U(s)Au\, ds$$

for $t \ge 0$. Hence $A \subset \mathfrak{A}$, where $\mathfrak{A}$ denotes the generator of $\{U(t);\, t \ge 0\}$. Since the intersection of the resolvent sets of A and $\mathfrak{A}$ is nonempty, we have $A = \mathfrak{A}$. ∎

The following asserts that a densely defined operator in X is the generator of an n-times integrated semigroup on X if and only if it satisfies conditions (A1) and (A2) of this paper.

Corollary 1. *Let n be a nonnegative integer. Let A be a densely defined linear operator in X. Then the following statements are mutually equivalent.*

(i) *A is the generator of an n-times integrated semigroup on X.*

(ii) *A is closed and the resolvent set $\varrho(A)$ of A contains (ω, ∞) for some $\omega \ge 0$. For each $\tau > 0$ there exists $M_\tau > 0$ such that*

$$\Big\|\prod_{l=1}^{k}(I-h_lA)^{-1}x\Big\| \le M_\tau\|x\|_n$$

for $x \in D(A^n)$ *and every finite sequence* $\{h_l\}_{l=1}^k$ *of positive numbers with* $h_l\omega < 1$ *for* $1 \le l \le k$ *and* $\sum_{l=1}^k h_l \le \tau$.

(iii) *A is closed and* $\varrho(A) \supset (\beta,\infty)$ *for some* $\beta \ge 0$. *For each* $\tau > 0$ *there exists* $K_\tau > 0$ *such that*

$$\sup\{\|\lambda^k(\lambda I-A)^{-k}x\|;\ 0 \le k/\lambda \le \tau,\ \lambda > \beta,\ k \ge 1\} \le K_\tau\|x\|_n$$

for $x \in D(A^n)$.

(iv) *A is closed and* $\varrho(A) \ne \emptyset$. *The problem* (ACP;x) *has a unique solution for each* $x \in D(A^{n+1})$.

Proof. If A is the generator of an n-times integrated semigroup $\{U(t); t \ge 0\}$ on X, then it is also the generator of the locally Lipschitz continuous $(n+1)$-times integrated semigroup $\{V(t); t \ge 0\}$ on X defined by $V(t)x = \int_0^t U(s)x\,ds$ for $x \in X$ and $t \ge 0$. We therefore deduce from the Main Theorem that (i) implies (ii). The implication (ii)⇒(iii) is obvious. It was proved by Oharu [11] that (iii) implies (iv). The implication (iv)⇒(i) was shown in [9, Theorem 3.3]. ∎

We next deduce the Arendt theorem from the Main Theorem (although Arendt's original proof is quite elegant).

COROLLARY 2. *Let n be a nonnegative integer. Then A is the generator of an exponentially Lipschitz continuous $(n+1)$-times integrated semigroup on X if and only if it is a closed linear operator in X and there exist $M > 0$ and $a \ge 0$ such that $\varrho(A) \supset (a,\infty)$ and*

$$\|(1/(k-1)!)(d/d\lambda)^{k-1}((\lambda I-A)^{-1}/\lambda^n)\| \le M(\lambda-a)^{-k} \tag{3.3}$$

for $\lambda > a$ and $k \ge 1$.

Proof. The necessity part is straightforward. We prove the sufficiency part using the Main Theorem. Since $A(\lambda I-A)^{-1} = \lambda(\lambda I-A)^{-1} - I$ for $\lambda > a$, it is shown inductively that

$$A^n(\lambda I-A)^{-1}x = \lambda^n(\lambda I-A)^{-1}x - \sum_{l=0}^{n-1}\lambda^l A^{n-1-l}x$$

for $x \in D(A^n)$ and $\lambda > a$. Dividing this equality by λ^n and differentiating the resulting equality $k-1$ times, we find

$$\begin{aligned}\|(\lambda I-A)^{-k}x\| \le{}& \frac{1}{(k-1)!}\,\|(d/d\lambda)^{k-1}((\lambda I-A)^{-1}/\lambda^n)A^nx\| \\ &+ \lambda^{-k}\sum_{l=0}^{n-1}\binom{n-l+k-2}{n-l-1}\lambda^{-(n-l-1)}\|A^{n-l-1}x\|\end{aligned}$$

for $x \in D(A^n)$, $\lambda > a$ and $k \geq 1$. By (3.3) the first term on the right-hand side is estimated by $M(\lambda - a)^{-k}\|A^n x\|$ for $\lambda > a$. Since $(1-t)^{-k} = \sum_{p=0}^{\infty} \binom{k+p-1}{p} t^p$ for $|t| < 1$ and $k \geq 0$, the second term is bounded by $\lambda^{-k}(1-1/\lambda)^{-k} \max_{0\leq p\leq n-1} \|A^p x\|$ for $\lambda > \max(a,1)$. Let $\beta = \max(a,1)$ and $K = \max(M,1)$. Then we have $(\beta,\infty) \subset \varrho(A)$ and $\|(\lambda I - A)^{-k}x\| \leq K(\lambda-\beta)^{-k}\|x\|_n$ for $x \in D(A^n)$, $\lambda > \beta$ and $k \geq 1$.

By an argument similar to that in [8, Section 4] there exists a norm $N(\cdot)$ on the Banach space $D(A^n)$ equipped with the norm $\|\cdot\|_n$ such that $\|x\| \leq N(x) \leq K\|x\|_n$ for $x \in D(A^n)$ and $N((\lambda I - A)^{-1}x) \leq (\lambda - \beta)^{-1}N(x)$ for $x \in D(A^n)$ and $\lambda > \beta$. This fact shows that

$$\Big\|\prod_{l=1}^{k}(I - h_l A)^{-1}x\Big\| \leq K\prod_{l=1}^{k}(1-h_l\beta)^{-1}\|x\|_n$$

for $x \in D(A^n)$ and every finite sequence $\{h_l\}_{l=1}^k$ of positive numbers with $h_l\beta < 1$ for $1 \leq l \leq k$. Since $(1-t)^{-1} \leq \exp(2t)$ for $0 \leq t \leq 1/2$, we see that condition (A2) is satisfied with $M_\tau = K\exp(2\beta\tau)$ and $\omega = 2\beta$. By the Main Theorem together with (3.2), A is the generator of an $(n+1)$-times integrated semigroup $\{U(t);\, t \geq 0\}$ on X and $\|U(t) - U(s)\| \leq \max(M_{2\tau},1)Le^{bt}(t-s)$ for $0 \leq s \leq t \leq \tau$ and $\tau > 0$. This means that A is the generator of an exponentially Lipschitz continuous $(n+1)$-times integrated semigroup on X. ∎

Example. *Let $X = l^\infty$. Let (a_k) be the sequence in $\mathbb{C}$ defined by $a_k = k + ie^{k^2}$ for $k \geq 1$, and define a linear operator A in X by $D(A) = \{x = (x_k) \in X;\, (a_k x_k) \in X\}$ and $Ax = (a_k x_k)$ for $x = (x_k) \in D(A)$. Then*:

(i) *$D(A)$ is not dense in X.*

(ii) *A is not the generator of any exponentially Lipschitz continuous $(n+1)$-times integrated semigroup on X, for any nonnegative integer n.*

(iii) *A is the generator of the locally Lipschitz continuous twice integrated semigroup $\{U(t);\, t \geq 0\}$ on X defined by*

$$U(t)x = \Big(\int_0^t (t-s)\exp(a_k s)x_k\,ds\Big)$$

for $x = (x_k) \in X$ and $t \geq 0$.

Proof. Since $\lim_{k\to\infty}|a_k| = \infty$ we have $\lim_{k\to\infty} x_k = 0$ for $x = (x_k) \in D(A)$, which implies (i). To prove (ii), assume to the contrary that A is the generator of an exponentially Lipschitz continuous $(n+1)$-times integrated semigroup $\{S(t);\, t \geq 0\}$ on X for some nonnegative integer n. By (2.2) we have $S(t)x = (\int_0^t((t-s)^n/n!)\exp(a_k s)x_k\,ds)$ for $x = (x_k) \in X$ and

$t \geq 0$, since the kth component $f_k(t)$ of $S(t)x$ must satisfy the equation $f_k(t) = \int_0^t a_k f_k(s)\,ds + (t^{n+1}/(n+1)!)x_k$ for $t \geq 0$.

Let $l \geq 1$. Then we have

$$|a_k|^{-l}|\exp(a_k t)| = \exp(k(t-lk))(1+k^2e^{-2k^2})^{-l/2}$$

and $\sup_{k\geq 1}\exp(k(t-lk)) = \exp(t^2/4l)$ for $t \geq 2l$. Since

$$\int_0^t \frac{(t-s)^{l-1}}{(l-1)!}\exp(a_k s)\,ds = (a_k)^{-l}\exp(a_k t) - \sum_{p=1}^{l}(a_k)^{-p}\,\frac{t^{l-p}}{(l-p)!}$$

and $|\sum_{p=1}^{l}(a_k)^{-p}t^{l-p}/(l-p)!| \leq e^t$ for $t \geq 0$ and $k \geq 1$, there exist $C_l \geq c_l > 0$ such that

$$c_l\exp(t^2/4l) - e^t \leq \sup_{k\geq 1}\left|\int_0^t \frac{(t-s)^{l-1}}{(l-1)!}\exp(a_k s)\,ds\right| \leq C_l\exp(t^2/4l) + e^t \tag{3.4}$$

for $t \geq 2l$, where the second inequality is true for all $t \geq 0$. By (3.4) with $l = n+1$ we see that $\|S(t)\|$ $(= \sup_{k\geq 1}|\int_0^t((t-s)^n/n!)\exp(a_k s)\,ds|)$ is not exponentially bounded, which contradicts the fact that $\{S(t);\, t \geq 0\}$ is exponentially Lipschitz continuous.

Finally, we prove (iii). We use the inequality (3.4) with $l = 1$ to obtain $\|U(t) - U(s)\| \leq (C_1e^{t^2/4} + e^t)(t-s)$ for $t \geq s \geq 0$, which implies that $\{U(t);\, t \geq 0\}$ is a locally Lipschitz continuous family in $B(X)$. The functional equation (I2) with $n = 2$ is clearly satisfied. If B is the generator of $\{U(t);\, t \geq 0\}$ then it is obvious that $A \subset B$. Since $\varrho(A) \supset \mathbb{R}$, the intersection of $\varrho(A)$ and $\varrho(B)$ is nonempty. The above two facts together imply $A = B$. ∎

Remark 3.1. In [4], the relationship between integrated semigroups and regularized semigroups was investigated. In this direction, it is seen from the above proof that the following result holds: Let A be a closed linear operator in X with nonempty resolvent set $\varrho(A)$. Let n be a nonnegative integer and $c \in \varrho(A)$. Then the following statements are mutually equivalent:

(i) A is the generator of a locally Lipschitz continuous $(n+1)$-times integrated semigroup on X.

(ii) A is the generator of a locally Lipschitz continuous C-regularized semigroup on X with $C = (cI - A)^{-(n+1)}$.

(iii) The resolvent set of A contains (ω, ∞) for some $\omega \geq 0$. For each $\tau > 0$ there exists $M_\tau > 0$ such that $\|\prod_{l=1}^{k}(I - h_lA)^{-1}x\| \leq M_\tau\|x\|_n$ for $x \in D(A^n)$ and every finite sequence $\{h_l\}_{l=1}^{k}$ of positive numbers such that $h_l\omega < 1$ for $1 \leq l \leq k$ and $\sum_{l=1}^{k} h_l \leq \tau$.

Remark 3.2. In [9], the generators of integrated semigroups were characterized in terms of the associated abstract Cauchy problems. See also [3] and [4].

Remark 3.3. Even for any local $(n+1)$-times integrated semigroup $\{U(t);\, t \in [0,T)\}$ which is locally Lipschitz continuous, the definition (1.1) of generators makes sense. However, we do not know whether the non-densely defined generators satisfy (2.1) and (2.2) for $t \in [0,T)$. Notice that a complex characterization of another type of "generators" was given in [2]. The problem of real characterization of the non-densely defined generators of such local integrated semigroups remains open except for our case of $T=\infty$, although a Hille–Yosida type theorem was found in [15, Theorem 4.2]. (See also [6].)

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