On the weak decomposition property ($\delta_w$)

by

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Abstract. We study a new class of bounded linear operators which strictly contains the class of bounded linear operators with the decomposition property ($\delta$) or the weak spectral decomposition property (weak-SDP). We treat general local spectral properties for operators in this class and compare them with the case of operators with ($\delta$).

1. Introduction

1.1. Definitions. Throughout this paper, $X$ is a Banach space and $\mathcal{L}(X)$ denotes the space of all bounded linear operators on $X$. For $T \in \mathcal{L}(X)$, let $N(T)$, $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_{com}(T)$ and $\sigma_{s}(T)$ denote the null space, the spectrum, the approximate point spectrum, the compression spectrum and the surjectivity spectrum of $T$, respectively.

Let $D(\lambda, r)$ be the open disc centred at $\lambda \in \mathbb{C}$ and with radius $r \geq 0$; the corresponding closed disc will be denoted by $\overline{D}(\lambda, r)$. For a closed subset $F$ in $\mathbb{C}$, the associated glocal spectral analytic space $\mathcal{X}_T(F)$ is the vector space of elements $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ such that $(T - \mu)f(\mu) = x$ for $\mu \in \mathbb{C} \setminus F$. The local resolvent set $\mathcal{R}_T(x)$ of $T$ at $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exists an analytic $X$-valued function $f$ on some open neighbourhood $U$ of $\lambda$ such that $(T - \mu)f(\mu) = x$ for all $\mu \in U$. The local spectrum of $T$ at $x$ is $\sigma_T(x) = \mathbb{C} \setminus g_T(x)$ (see [13]). Note that $\sigma_T(x)$ is a closed subspace of $\sigma(T)$ and it may be empty.

We say that $T$ has the single-valued extension property (SVEP) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$, the only analytic solution of the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. In this case, $\sigma_T(x) = \emptyset$ if and only if $x = 0$, and we have $\mathcal{X}_T(F) = X_T(F)$, where $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$. The operator $T$ is said to satisfy the Dunford condition (C) if $X_T(F)$ is closed for all closed subsets $F$ in $\mathbb{C}$; and $T$ has the Bishop property ($\beta$) if for every open subset $U$ and for any sequence $(f_n)_n$ of analytic $X$-valued functions on $U$...
with \((T - \lambda)f_n(\lambda) \to 0\) as \(n \to \infty\) uniformly on compact subsets of \(U\), we have \(f_n(\lambda) \to 0\) as \(n \to \infty\) uniformly on compact subsets of \(U\). For more details, see [3, 13].

An operator \(T \in \mathcal{L}(X)\) is said to be **decomposable** provided that for every finite open cover \(\{U_1, \ldots, U_n\}\) of \(\mathbb{C}\), there exist closed \(T\)-invariant subspaces \(X_1, \ldots, X_n\) of \(X\) such that

\[
\sigma(T|X_i) \subseteq U_i \quad \text{for} \quad i = 1, \ldots, n, \quad X_1 + \cdots + X_n = X.
\]

The class of decomposable operators contains all normal operators and more generally all spectral operators. Operators with totally disconnected spectrum are decomposable by the Riesz functional calculus. In particular, compact and algebraic operators are decomposable.

A weaker version of decomposable operators is given by operators that have the weak spectral decomposition property. Namely, \(T\) is said to have the **weak spectral decomposition property** (weak-SDP) if, for every finite open cover \(\{U_1, \ldots, U_n\}\) of \(\mathbb{C}\), there exist closed \(T\)-invariant subspaces \(X_1, \ldots, X_n\) of \(X\) such that

\[
\sigma(T|X_i) \subseteq U_i \quad \text{for} \quad i = 1, \ldots, n, \quad X_1 + \cdots + X_n = X.
\]

E. Albrecht [1] gives an example that shows that the class of bounded linear operators with weak-SDP contains strictly the class of decomposable operators.

An operator \(T\) has the **decomposition property** \((\delta)\) if for every finite open cover \(\{U_1, \ldots, U_n\}\) of \(\mathbb{C}\), we have

\[
\mathcal{X}_T(\overline{U}_1) + \cdots + \mathcal{X}_T(\overline{U}_n) = X.
\]

Properties \((\beta)\) and \((\delta)\) are known to be dual to each other in the sense that \(T\) has \((\delta)\) if and only if \(T^*\) satisfies \((\beta)\). It is also known that \((\beta)\) characterizes operators with decomposable extensions, and in particular isometries and subnormal operators have \((\beta)\) (see [2]). Property \((\beta)\) is hence conserved by restrictions while \((\delta)\) is inherited by quotient operators. See also [13] for more details.

**Definition 1.1.** Let \(T \in \mathcal{L}(X)\). We say that \(T\) has the **weak decomposition property** \((\delta_w)\) at \(\lambda \in \mathbb{C}\) if there exists \(r(\lambda) > 0\) such that for every \(0 \leq r \leq r(\lambda)\) and every finite open cover \(\{U_1, \ldots, U_n\}\) of \(\mathbb{C}\) with \(\sigma(T) \setminus D(\lambda, r) \subseteq U_1, \ldots, U_n\),

\[
\mathcal{X}_T(\overline{U}_1) + \cdots + \mathcal{X}_T(\overline{U}_n) \quad \text{is dense in} \quad X.
\]

We will say that \(T\) has \((\delta_w)\) if it has \((\delta_w)\) at every \(\lambda \in \mathbb{C}\).

The decomposition property \((\delta)\) and the Dunford condition \((C)\) characterize the decomposability of bounded linear operators. Indeed, \(T \in \mathcal{L}(X)\) is decomposable if and only if it has both \((C)\) and \((\delta)\).
Operators with \((\delta_w)\) and (C) are called quasi-decomposable and have been treated in the literature (see [1, 9]). It is clear that quasi-decomposable operators satisfy weak-SDP.

In this paper we investigate the properties of operators with \((\delta_w)\). Our main objective is to compare them systematically with operators having \((\delta)\). Examples showing that the class of operators with \((\delta_w)\) is different from known classes are given at the end of this section. In Section 2 we show that the adjoints of operators with \((\delta_w)\) have the single-valued extension property but may fail to satisfy the Dunford condition (C).

We also link the \((\delta_w)\) property with generalized derivations to obtain results on the stability of this property under some transformations.

The localizable spectrum and the support points set are introduced in Section 3. A description of these spectral sets is given for operators with \((\delta_w)\).

Section 4 is devoted to property \((\delta_w)\) for multipliers.

1.2. Examples. From the definition, it follows that \((\delta)\) and weak-SDP each imply \((\delta_w)\). The question of whether \((\delta_w)\) implies \((\delta)\) or weak-SDP arises naturally.

We now exhibit operators satisfying \((\delta_w)\) without satisfying \((\delta)\) or weak-SDP. This shows that the class of operators considered here is strictly larger than the class of operators with \((\delta)\) or with weak-SDP.

A first example uses the classical shift on the Hardy space.

**Example 1.1.** Let \(B\) be the backward unilateral shift on \(X_1 = l^2(\mathbb{N})\). The operator \(B\) has the decomposition property \((\delta)\) and does not have weak-SDP. Indeed, since \(B^*\) is an isometry and isometries have the Bishop property \((\beta)\), it follows that \(B\) has \((\delta)\). According to Beurling’s characterization of the subspaces invariant under \(B^*\) (see [8]), it is easy to see that \(B\) does not have weak-SDP. In particular, \(B\) satisfies \((\delta_w)\) but not weak-SDP.

An example with \((\delta_w)\) but without \((\delta)\) is provided by the operator \(R_0 \in \mathcal{L}(X_2)\), for some Banach space, of Albrecht [1], since weak-SDP implies \((\delta_w)\).

Now set \(R = 3I + R_0\) and take \(T = B \oplus R \in \mathcal{L}(X_1 \oplus X_2)\). It is not difficult to see that for every closed subset \(F\) in \(\mathbb{C}\),

\[ (X_1 \oplus X_2)_T(F) = X_1B(F) \oplus X_2R(F). \]

Thus \(T\) has \((\delta_w)\). If \(T\) had weak-SDP, then for an open cover \(\{U, V\}\) of \(\mathbb{C}\) there would exist closed \(T\)-invariant subspaces \(Y\) and \(Z\) of \(X_1 \oplus X_2\) such that \(\sigma(T|Y) \subseteq U\), \(\sigma(T|Z) \subseteq V\) and \(Y + Z\) is dense in \(X_1 \oplus X_2\). If \(Y\) and \(Z\) are trivial then \(Y = Y_1 \oplus Y_2\) and \(Z = Z_1 \oplus Z_2\), where \(Y_1\) and \(Z_1\) (resp. \(Y_2\) and \(Z_2\)) are closed \(B\)-invariant subspaces (resp. \(R\)-invariant subspaces) of \(X_1\) (resp. \(X_2\)). Then \(\sigma(B|Y_1) \subseteq U\), \(\sigma(B|Z_1) \subseteq V\) and \(Y_1 + Z_1\) is dense in \(X_1\). Now suppose that \(Y\) or \(Z\) is not trivial. Since \(\sigma(R_0) = \overline{D}(0, 1)\) we have \(\eta(\sigma(B)) \cap \eta(\sigma(R)) = \emptyset\), where \(\eta(\cdot)\) is the polynomially convex hull. Then it
follows from [19] that \( Y = Y_1 \oplus Y_2 \) and \( Z = Z_1 \oplus Z_2 \). Hence \( B \) has weak-SDP, which is not the case. Thus \( T \) has \((\delta_w)\) but neither \((\delta)\) nor weak-SDP.

**Example 1.2.** Let \( T \) be the unilateral weighted shift on \( X = l^2(\mathbb{N}) \) defined by

\[
Te_n = \begin{cases} 
0 & \text{if } n = p! \text{ for some } p \in \mathbb{N}, \\
n & \text{otherwise}.
\end{cases}
\]

The adjoint operator of \( T \) is given by

\[
T^*e_n = \begin{cases} 
0 & \text{if } n = 0 \text{ or } n = p! + 1 \text{ for some } p \in \mathbb{N}, \\
n & \text{otherwise}.
\end{cases}
\]

Since \( \sigma_p(T) = \sigma_p(T^*) = \{0\} \), \((\text{SVEP})\) holds for \( T \) and \( T^* \). It is also clear that \( T^n e_n = 0 \) for all \( n \geq 1 \) and hence \( \sigma_T(e_n) = \{0\} \). It follows that \( \mathcal{X}_T(\{0\}) \subseteq X_T(\{0\}) \) is dense in \( X \). Thus \( T \) has \((\delta_w)\). Since \( \sigma(T) = \overline{D}(0, 1) = \sigma(T^*) \), \( T \) does not have the Dunford property \((C)\). Indeed, otherwise

\[
\{0\} = \sigma(T|X_T(\{0\})) = \sigma(T|X) = \overline{D}(0, 1).
\]

The same argument shows that \( T^* \) has \((\delta_w)\) and fails \((C)\). So \( T \) and \( T^* \) do not have \((\delta)\). To see that \( T \) does not have weak-SDP, suppose that there exist nontrivial closed \( T \)-invariant subspaces \( X_1, X_2 \) of \( X \) such that

\[
\sigma(T|X_1) \subseteq D(0, 1/2),
\]

\[
\sigma(T|X_2) \subseteq \mathbb{C} \setminus \overline{D}(0, 1/4) \quad \text{and} \quad X_1 + X_2 \text{ is dense in } X.
\]

In particular \( 0 \notin \sigma_T(x) \) for all \( x \in X_2 \setminus \{0\} \). But \( \bigcap_{n \geq 0} T^n(X) = \{0\} \) implies that \( 0 \notin \sigma_T(x) \) for every nonzero \( x \in X \). Contradiction.

**2. Properties of operators with \((\delta_w)\).** In this section we generalize some known results for operators with \((\delta)\) to the class of operators with \((\delta_w)\).

**Theorem 2.1.** If \( T \in \mathcal{L}(X) \) has \((\delta_w)\) at \( \lambda \), then \( T^* \) has \((\text{SVEP})\) at \( \lambda \).

**Proof.** Let \( U \subset D(\lambda, \varepsilon) \) be an open set and \( \varphi(\cdot) \) be an analytic \( X^* \)-valued function on \( U \) such that \( (T^* - \mu)\varphi(\mu) = 0 \) for all \( \mu \in U \). Choose \( \varepsilon < r(\lambda) \) such that \( U \cap D(\lambda, \varepsilon/4) \neq \emptyset \) and \( U \cap (\mathbb{C} \setminus D(\lambda, \varepsilon/2)) \neq \emptyset \). As \( T \) has \((\delta_w)\) at \( \lambda \), it follows that

\[
\mathcal{X}_T(\overline{D}(\lambda, \varepsilon/2)) + \mathcal{X}_T(\mathbb{C} \setminus D(\lambda, \varepsilon/4)) \text{ is dense in } X.
\]

For each \( x \in \mathcal{X}_T(\overline{D}(\lambda, \varepsilon/2)) + \mathcal{X}_T(\mathbb{C} \setminus D(\lambda, \varepsilon/4)) \) there are \( x_1 \in \mathcal{X}_T(\overline{D}(\lambda, \varepsilon/2)) \) and \( x_2 \in \mathcal{X}_T(\mathbb{C} \setminus D(\lambda, \varepsilon/4)) \) such that \( x = x_1 + x_2 \). Let \( f_1(\mu) \) and \( f_2(\mu) \) be analytic \( X \)-valued functions on \( \mathbb{C} \setminus \overline{D}(\lambda, \varepsilon/2) \) and \( D(\lambda, \varepsilon/4) \) respectively such that \( x_i = (T - \mu) f_i(\mu) \) for \( i = 1, 2 \).

Now for \( \mu \in U \cap D(\lambda, \varepsilon/4) \), we have

\[
\langle \varphi(\mu), x_2 \rangle = \langle \varphi(\mu), (T - \mu) f_2(\mu) \rangle = \langle (T^* - \mu) \varphi(\mu), f_2(\mu) \rangle = 0.
\]
Since $\mu \to \langle \varphi(\mu), x_2 \rangle$ is an analytic function on $U$ which vanishes on $U \cap D(\lambda, \varepsilon/4)$, it is identically null, and hence $\langle \varphi(\mu), x_2 \rangle = 0$ for all $\mu \in U$. Similarly, $\langle \varphi(\mu), x_1 \rangle = 0$ on $U$. Then $\langle \varphi(\mu), x \rangle = 0$ for all $x \in X_T(D(\lambda, \varepsilon/2)) + X_T(\mathbb{C} \setminus D(\lambda, \varepsilon/4))$. It follows from (5) that $\varphi \equiv 0$. 

The following corollaries are immediate.

**Corollary 2.1.** If $T \in \mathcal{L}(X)$ has $(\delta_w)$, then $T^*$ has (SVEP), in particular

$$\sigma(T) = \sigma_{ap}(T).$$

Recall that $T \in \mathcal{L}(X)$ is said to be semi-Fredholm (resp. Fredholm) if $T(X)$ is closed and either $N(T)$ or $X/T(X)$ is finite-dimensional (resp. both are). The semi-Fredholm spectrum is defined by $\sigma_{SF}(T) := \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not semi-Fredholm on } X \}$.

**Corollary 2.2.** If $T \in \mathcal{L}(X)$ has $(\delta_w)$ and (SVEP), then every point of $\sigma(T) \setminus \sigma_{SF}(T)$ is an isolated point.

**Proof.** It follows from Corollary 2.1 that $T^*$ has (SVEP); then the result follows by [16, Corollary 1.8].

**Remark 2.1.** 1) Corollary 2.1 generalizes [13, Proposition 1.4] concerning the case when $T$ has $(\delta)$ or weak-SDP.

2) It is well known that $T$ having $(\delta)$ implies that $T^*$ satisfies the Dunford condition (C) and hence has (SVEP). As shown by Example 1.2, if $T$ has $(\delta_w)$ then $T^*$ does not necessarily have (C).

3) Corollary 2.2 together with [11, Lemma 1] yields the following result:
If $T$ has $(\delta_w)$, then $\sigma_{SF}(T)$ consists of all cluster points of $\sigma(T)$ and the isolated points $\lambda \in \sigma(T)$ for which $X_T(\{ \lambda \})$ is infinite-dimensional.

This extends Corollary 1 of K. B. Laursen [11].

Just as for operators with $(\delta)$, property $(\delta_w)$ is inherited by quotients and some limits, and is preserved by functional calculus as shown by the next propositions.

**Proposition 2.1.** Let $T \in \mathcal{L}(X)$ have $(\delta_w)$ and let $S \in \mathcal{L}(Y)$. If $RT = SR$ for some $R \in \mathcal{L}(X,Y)$ with dense range, then $S$ has $(\delta_w)$.

**Proof.** Note that $R(X_T(F)) \subseteq Y_S(F)$ for every closed subset $F$ of $\mathbb{C}$. Let $\{U_1, \ldots, U_n\}$ be an open cover of $\mathbb{C}$. Since $T$ has $(\delta_w)$, we have

$$Y = R(X) \subseteq R(X_T(\overline{U}_1) + \cdots + X_T(\overline{U}_n))$$

$$\subseteq R(X_T(\overline{U}_1)) + \cdots + R(X_T(\overline{U}_n)) \subseteq Y_S(\overline{U}_1) + \cdots + Y_S(\overline{U}_n).$$

For $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, let $\delta_{S,T}$ be the generalized derivation induced by $T$ and $S$ defined on $\mathcal{L}(X,Y)$ by $\delta_{S,T}(R) = SR - RT$ for all $R \in \mathcal{L}(X,Y)$. 

Next we give an asymptotic version of Proposition 2.1.

**Theorem 2.2.** Let $T_k \in \mathcal{L}(X)$ be a sequence of operators with $(\delta_w)$ and let $S \in \mathcal{L}(Y)$. Suppose that there exist $R_k \in \mathcal{L}(X,Y)$ with dense range such that $r_{S,T_k}(R_k) \to 0$ as $k \to \infty$. Then $S$ has $(\delta_w)$. 

**Proof.** Let $\{U_1, \ldots, U_n\}$ be an open cover of $\mathbb{C}$. Choose an open cover $\{V_1, \ldots, V_n\}$ of $\mathbb{C}$ such that $V_i \subseteq \bar{V}_i \subseteq U_i$ for $i = 1, \ldots, n$. Since $r_k := r_{S,T_k}(R_k) \to 0$ as $k \to \infty$ and the sets $(\bar{V}_i + \overline{D}(0, r_k)) \cap \sigma(S)$ are compact, there exists $k_0 \geq 0$ such that 

$$(\bar{V}_i + \overline{D}(0, r_{k_0})) \cap \sigma(S) \subseteq U_i \quad \text{for } i = 1, \ldots, n.$$ 

Hence by [13, Proposition 3.4.2], for $i = 1, \ldots, n,$ 

$$(\overline{X}_i)(\bar{V}_i) \subseteq \mathcal{Y}_S(\overline{V}_i + \overline{D}(0, r_{k_0})) = \mathcal{Y}_S((\overline{V}_i + \overline{D}(0, r_{k_0})) \cap \sigma(S)) \subseteq \mathcal{Y}_S(\overline{U}_i).$$

Now we proceed as in the proof of Proposition 2.1. ■

The following corollary is immediate by setting $R_n = (T - \lambda)^{-1}$ for some $\lambda \notin \sigma(T)$ in Theorem 2.2.

**Corollary 2.3.** Let $T_n$ be operators with $(\delta_w)$ norm converging to $T$. If $T$ commutes with $T_n$ for all $n \geq 0$, then $T$ has $(\delta_w)$.

**Proposition 2.2.** Let $T$ have $(\delta_w)$ and let $f : U \to X$ be an analytic function on an open neighbourhood $U$ of $\sigma(T)$. Then $f(T)$ has $(\delta_w)$.

**Proof.** We recall that $\overline{X}_{f(T)}(F) = \overline{X}_{T}(f^{-1}(F))$ for all closed subsets $f$ of $\mathbb{C}$ (see [13, Theorem 3.3.6]). Let $\{U_1, \ldots, U_n\}$ be an open cover of $\mathbb{C}$. Since $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$ is an open cover of $\sigma(T)$ and $T$ has $(\delta_w)$, it follows that

$$X = \overline{X}_{T}(f^{-1}(U_1)) + \cdots + \overline{X}_{T}(f^{-1}(U_n))$$

$$\subseteq \overline{X}_{T}(f^{-1}(U_1)) + \cdots + \overline{X}_{T}(f^{-1}(U_n))$$

$$= \overline{X}_{f(T)}(U_1) + \cdots + \overline{X}_{f(T)}(U_n).$$

Hence $f(T)$ has $(\delta_w)$. ■

**Proposition 2.3.** Let $T \in \mathcal{L}(X)$ have $(\delta_w)$ and let $S \in \mathcal{L}(Y)$ satisfy the Dunford condition (C). Then $\delta_{S,T}$ has (SVEP).

**Proof.** Let $R : U \to \mathcal{L}(X,Y)$ be an analytic function on an open connected set $U$ of $\mathbb{C}$ such that $(\delta_{S,T} - \mu)R(\mu) = 0$ for all $\mu \in U$. If $\mu \in U$ and $x \in X$, then $(\delta_{S,T} - \mu)R(\mu)x = 0$ implies $SR(\mu)x = R(\mu)(T + \mu)x$. Hence $R(\mu)x \in \mathcal{Y}_S(\sigma_T(x) + \mu).$ Now let $D_1, D_2 \subseteq U$ be two closed discs with nonempty interiors and $\text{dist}(D_1, D_2) > \varepsilon$. Then $R(\mu)x \in \mathcal{Y}_S(\sigma_T(x) + D_i)$ for all $\mu \in D_i$. Since $S$ satisfies the Dunford condition (C), $\mathcal{Y}_S(\sigma_T(x) + D_i)$ is a closed subspace and by the identity theorem for analytic functions, $R(\mu)x \in \mathcal{Y}_S(\sigma_T(x) + D_i)$ for all $\mu \in U$. Then if $\text{diam}(\sigma_T(x)) \leq \varepsilon$, we have
The localizable spectrum for operators with \((\delta_w)\). For a bounded operator \(T\), the localizable spectrum \(\sigma_{\text{loc}}(T)\) of \(T\) is the set of all points \(\lambda \in \sigma(T)\) such that

\[
X_T(\overline{D}(\lambda, r)) \neq \{0\} \quad \text{for every } r > 0.
\]

We list some elementary observations related to the localizable spectrum:

(a) If \(T\) does not have (SVEP), then \(\sigma_{\text{loc}}(T) = \sigma(T)\). Indeed, \(X_T(\emptyset)\) is nontrivial and contained in \(X_T(D(\lambda, r))\) for all \(\lambda\) and \(r\).

(b) \(\sigma_p(T) \subseteq \sigma_{\text{loc}}(T)\), because \(\sigma_T(x) \subseteq \{\lambda\}\) for any eigenvector associated with the eigenvalue \(\lambda\).

(c) The localizable spectrum may be strictly contained in the spectrum; for example, if \(T\) is the unilateral forward unweighted shift on the Hardy space, then \(\sigma_{\text{loc}}(T) = \emptyset\), while \(\sigma(T)\) is the closed unit disk. On the other hand, if \(T\) satisfies \((\delta)\) or weak-SDP, then \(\sigma_{\text{loc}}(T) = \sigma(T)\). See [7, 18].

If \(T\) is decomposable, then \(T\) has \((\delta)\) and \((\delta^*)\). Thus \(\sigma_{\text{loc}}(T) = \sigma_{\text{loc}}(T^*)\). This is still valid if \((\delta)\) is replaced by \((\delta_w)\).

**Proposition 3.1.** If \(T \in \mathcal{L}(X)\) has \((\delta_w)\), then

\[
\sigma_{\text{com}}(T) \subseteq \sigma_{\text{loc}}(T^*) \subseteq \sigma_{\text{loc}}(T).
\]

In particular if both \(T\) and \(T^*\) have \((\delta_w)\), then \(\sigma_{\text{loc}}(T) = \sigma_{\text{loc}}(T^*)\).

**Proof.** The first inclusion is trivial since \(\sigma_{\text{com}}(T) = \sigma_p(T^*)\). Suppose \(\lambda \notin \sigma_{\text{loc}}(T)\) and let \(r > 0\) be such that \(X_T(D(\lambda, r)) = \{0\}\). It follows that \(X_T(D(\lambda, r)) = \{0\}\) and because \(T\) satisfies \((\delta_w)\), we see that \(X_T(\mathbb{C} \setminus D(\lambda, r/2))\) is dense. Now, from [13, Proposition 2.5.1] we have the inclusion

\[
X_{T^*}(D(\lambda, r/4)) \subseteq (X_T(\mathbb{C} \setminus D(\lambda, r/2)))'.
\]

Thus \(X_{T^*}(D(\lambda, r/4)) = \{0\}\), and (SVEP) for \(T^*\) leads to \(X_{T^*}(D(\lambda, r/4)) = \{0\}\). The proof is complete. ■

In contrast with decomposable operators, Example 1.2 provides an operator \(T\) with \((\delta_w)\) and \((\delta^*_w)\) such that \(\sigma_{\text{loc}}(T) \neq \sigma(T)\).

In the proposition above, the inclusion \(\sigma_{\text{com}}(T) \subseteq \sigma_{\text{loc}}(T)\) may be strict. Let \(T\) be the operator given in Example 1.2. Then \(\sigma_{\text{com}}(T) = \{0\}\). Indeed, \(\{0\} \subseteq \sigma_{\text{com}}(T) \subseteq \sigma_{\text{loc}}(T^*) \subseteq \sigma_{\text{loc}}(T)\). On the other hand, since \(\cap_{n \geq 0} T^n(H) = \{0\}\), we obtain \(0 \in \sigma_T(x)\) for every nonzero \(x\), and therefore
we conclude that $\sigma_{\text{loc}}(T) = \{0\}$. Thus

$$\sigma_{\text{com}}(T) = \sigma_{\text{loc}}(T^*) = \sigma_{\text{loc}}(T) = \{0\}.$$

If $T$ is any normal operator without eigenvectors, then $\sigma_{\text{com}}(T) = \emptyset$ while the equality $\sigma_{\text{loc}}(T) = \sigma(T)$ always holds.

**Proposition 3.2.** Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, and let $R \in \mathcal{L}(X,Y)$ be injective such that $RT = SR$. If $T$ has $(\delta_w)$, then

$$\sigma_{\text{com}}(T) \subseteq \sigma_{\text{loc}}(S).$$

**Proof.** Since $RT = SR$, we obtain $R(X_T(F)) \subseteq Y_S(F)$ for every closed set $F$, and as $R$ is injective, it follows that $\sigma_{\text{loc}}(T) \subseteq \sigma_{\text{loc}}(S)$. Proposition 3.1 allows us to conclude. $\blacksquare$

**Remark 3.1.** The inclusion (7) may be strict. To see this, let $T = S$ be a normal operator without eigenvectors and $R$ the identity map. Then $T$ has $(\delta_w)$ and $\sigma_{\text{com}}(T) = \emptyset \subset \sigma_{\text{loc}}(S) = \sigma(S)$.

A notion closely related to the localizable spectrum is provided by the support points set introduced in [17] by taking, in (6), the glocal spectral analytic space instead of the analytic spectral space. More precisely, according to [17], $\lambda \in \mathbb{C}$ is a support point for $T$ if

$$X_T(\mathcal{D}(\lambda, r)) \neq \{0\} \quad \text{for every } r > 0.$$  

The set of all support points for $T$ is denoted $\text{spt}(T)$. It is trivial from (6) and (8) that $\sigma_p(T) \subseteq \text{spt}(T) \subseteq \sigma_{\text{loc}}(T)$ and that the last inclusion is an equality if and only if $T$ has (SVEP). It is also not difficult to see that $\text{spt}(T) = \sigma(T)$ if $T$ is decomposable, or more generally if $T$ has weak-SDP or $(\delta)$. The latter equality fails to be true in general for operators with $(\delta_w)$ as shown by Example 1.2.

The analytic core $K(T)$ associated with $T$ is the (not necessarily closed) invariant subspace of $T$ that consists of elements $x \in X$ for which there exists $c > 0$ and a sequence $x_n \in X$ such that $x_0 = x$, $\|x_n\| \leq c^n\|x\|$ and $Tx_{n+1} = x_n$. The quasi-nilpotent part of $T$ is $X_0(T) := \{x \in X : \lim_{n \to \infty} \|T^n x\|^{1/n} = 0\}$. The analytic core and the quasi-nilpotent part of an operator have been extensively studied by M. Mbekhta in [14–16]. In particular

$$K(T) = X_T(\mathbb{C} \setminus \{0\}) \quad \text{and} \quad X_0(T) = X_T(\{0\}).$$

The question when the analytic core is closed has interested some mathematicians and is central in [17], where the special case of operators with $(\delta)$ has been developed. In the rest of this section we discuss the same phenomena for operators with $(\delta_w)$. 
For operators with (SVEP), it is proved in [17, Lemma 4] that $K(T)$ is not closed when 0 is a cluster point of $\text{spt}(T)$. For operators with $(\delta_w)$, we have

**Theorem 3.1.** Let $T$ be noninvertible and suppose $T$ has $(\delta_w)$. If $K(T)$ is closed, then $0 \in \text{spt}(T)$.

**Proof.** Let $F_n = D(0, 1/n)$ and $G_n = \mathbb{C} \setminus F_{2n}$. Then $\mathcal{X}_T(F_n) + \mathcal{X}_T(G_n)$ is dense and $\mathcal{X}_T(G_n) \subseteq K(T)$. In particular, if $\mathcal{X}_T(F_n) = \{0\}$ for some $n$, then $\mathcal{X}_T(G_n) \subseteq K(T)$ is dense, and since $K(T)$ is closed, it follows that $T$ is onto. Now since $0 \in \sigma(T)$, we have $0 \in \sigma_p(T) \subseteq \text{spt}(T)$.

Set $\sigma_{\text{HK}}(T) = \{\lambda \in \mathbb{C} : K(T - \lambda) \text{ is not closed}\}$. We have

**Corollary 3.1.** If $T$ has $(\delta_w)$, then $\sigma(T) \setminus \sigma_{\text{HK}}(T) \subseteq \text{spt}(T)$.

In [17, Corollary 8], it is shown that an operator $T$ is quasinilpotent if and only if $T$ has $(\delta)$ and $K(T) = \{0\}$. We note that the assumption $(\delta)$ cannot be relaxed to $(\delta_w)$ as shown by Example 1.2.

**4. Multipliers and $(\delta_w)$.** We devote this section to showing that for multipliers on semisimple Banach algebras, the notions of decomposability and $(\delta_w)$ coincide.

Let $A$ be a semisimple commutative Banach algebra. $\Sigma(A)$ denotes the spectrum of $A$, that is, the set of nontrivial multiplicative linear functionals on $A$. For each $a \in A$, let $\hat{a}$ denote the Gelfand transform given by $\hat{a}(\chi) = \chi(a)$ for all $\chi \in \Sigma(A)$. The *Gelfand topology* is the coarsest topology on $\Sigma(A)$ for which all the Gelfand transforms $\hat{a}$ are continuous.

For $B \subseteq A$ and $E \subseteq \Sigma(A)$, we define $h(B) = \{\Psi \in \Sigma(A) : \Psi(b) = 0 \text{ for all } b \in B\}$ and $k(E) = \{a \in A : \varphi(a) = 0 \text{ for all } \varphi \in E\}$. We say that $E$ is closed for the *hull-kernel topology* if $E = hk(E)$. The last topology is always coarser than the Gelfand topology, and they coincide exactly when $A$ is a regular algebra. For further information, see [4, 19]. For $a \in A$ let $T_a$ denote the corresponding multiplication operator given by $T_a(x) = ax$ for all $x \in A$.

A mapping $T : A \to A$ is called a *multiplier* if $T(xy) = xT(y)$ for all $x, y \in A$. By semisimplicity of $A$, every multiplier is a bounded linear operator on $A$. Moreover, $M(A)$, the set of multipliers on $A$, is a semisimple commutative unital subalgebra of $\mathcal{L}(A)$. The spectrum $\Sigma(M(A))$ may be represented as the disjoint union of $\Sigma(A)$ and $H(A)$, where $H(A) = \{\chi \in \Sigma(M(A)) : \chi(a) = 0, \forall a \in A\}$ and $\Sigma(A)$ is canonically embedded in $\Sigma(M(A))$. For $T \in M(A)$, $a, b \in A$ and $\chi \in \Sigma(A)$ we have $\chi(a)\chi(T(b)) = \chi(b)\chi(T(a))$. Hence $\chi(T(a))/\chi(a) = \chi(T(b))/\chi(b)$. This allows us to define $\tilde{T}(\chi) := \chi(T) = \chi(T(a))/\chi(a)$ for some $a \in A$ such that $\chi(a) \neq 0$. See for instance [10] and [13].
It is shown in [6, Proposition 1] that weak-SDP multipliers have hull-kernel continuous transform on \( \Sigma(A) \). This result is extended as follows:

**Proposition 4.1.** Let \( A \) be a semisimple commutative Banach algebra. If \( T \in M(A) \) has \((\delta_w)\), then \( \hat{T}|\Sigma(A) \) is hull-kernel continuous on \( \Sigma(A) \).

**Proof.** Suppose that \( \hat{T} \) is not hull-kernel continuous on \( \Sigma(A) \). There exists a closed subset \( F \) of \( \mathbb{C} \) such that \( E = \{ \chi \in \Sigma(A) : \chi(T) \in F \} \subset \text{hk}(E) \). Let \( \chi \in \text{hk}(E) \setminus E \). Then \( \chi(T) = \lambda \notin F \) and hence there exist open subsets \( U, V \) of \( \mathbb{C} \) such that \( \lambda \in U, F \subset V \) and \( U \cap V = \emptyset \).

Since \( T \) has \((\delta_w)\), it follows that \( \mathcal{A}_T(C \setminus U) + \mathcal{A}_T(C \setminus V) \) is dense in \( A \).

For each \( x \in \mathcal{A}_T(C \setminus U) \), there exists \( y \in A \) such that \( x = (T - \lambda)y \). Then \( \chi(x) = 0 \). Also for each \( x \in \mathcal{A}_T(C \setminus V) \), \( \psi(x) = 0 \) for all \( \psi \in E \) and hence \( \chi(x) = 0 \). It follows that \( \chi \equiv 0 \) on \( A \). Contradiction. \( \blacksquare \)

We deduce the following corollary in the spirit of [13, Theorem 4.4.5]. The proof is a simple adaptation and is omitted.

**Corollary 4.1.** Let \( A \) be a semisimple commutative Banach algebra and \( a \in A \). Then the following statements are equivalent:

(i) \( T_a \) is decomposable.

(ii) \( T_a \) has \((\delta_w)\).

(iii) The Gelfand transform \( \hat{a} \) is hull-kernel continuous on \( \Sigma(A) \).

Let \( T \) be multiplier on \( A \). It is not hard to see that \( \hat{T}(\Sigma(A)) \subset \sigma(T) = \hat{T}(\Sigma(M(A))) \). We say that \( T \) has natural spectrum if

\[
\sigma(T) = \overline{\hat{T}(\Sigma(A))}.
\]

Multipliers with weak-SDP or \((\delta)\) are known to have natural spectrum (see [6, Proposition 1] and [13, Proposition 4.6.3] respectively). This may fail to be true for an operator with \((\delta_w)\). We have

**Proposition 4.2.** If \( T \) is a multiplier with \((\delta_w)\), then

\[
\sigma_{\text{com}}(T) = \overline{\hat{T}(\Sigma(A))}.
\]

**Proof.** Let \( S \) be multiplication by \( \hat{T} \) on the Banach space \( Y \) of all continuous bounded \( \mathbb{C} \)-valued functions on \( \Sigma(A) \) equipped with the sup-norm, and \( \hat{R} : A \to Y \) the Gelfand transform. It follows from Proposition 3.2 that \( \sigma_{\text{com}}(T) \subseteq \sigma(S) \).

Let \( \lambda \notin \hat{T}(\Sigma(A)) \). If \( (S - \lambda)f = 0 \) then \( (\hat{T}(\chi) - \lambda)f(\chi) = 0 \) for all \( \chi \in \Sigma(A) \). Thus \( f(\chi) = 0 \) and hence \( S - \lambda \) is injective. Now let \( g \in Y \). The
mapping
\[ f(\chi) = \frac{1}{T(\chi) - \lambda} g(\chi) \]
is continuous and bounded. Moreover, it satisfies \((S - \lambda)f = g\). Thus \(S - \lambda\) is surjective and then \(\sigma_{\text{com}}(T) \subseteq \sigma(S) \subseteq \hat{T}(\Sigma(A))\).

On the other hand, \(\hat{T}(\Sigma(A)) \subseteq \sigma_{\text{com}}(T)\). Indeed, let \(\lambda \notin \sigma_{\text{com}}(T)\). If \(\lambda \in \hat{T}(\Sigma(A))\), then there exists \(\chi \in \Sigma(A)\) such that \(\hat{T}(\chi) = \lambda\). Hence \(\chi((T - \lambda)a) = 0\) for all \(a \in A\). Since \((T - \lambda)A\) is dense in \(A\), it follows that \(\chi\) vanishes on \(A\).

If \(A\) is a regular semisimple commutative Tauberian Banach algebra, then it follows from the proof of [6, Proposition 5] that every multiplier \(T\) on \(A\) has \((\delta_w)\). By Corollary 2.1, \(\sigma(T) = \sigma_{\text{ap}}(T)\). Thus in Proposition 4.8.6 of [13] the assumption that \(\Sigma(A)\) is discrete is not necessary.

Let \(G\) be a locally compact abelian group, \(\Gamma\) its dual group, \(L^1(G)\) the space of \(\mathbb{C}\)-valued functions on \(G\) integrable with respect to Haar measure and \(M(G)\) the Banach algebra of regular complex Borel measures on \(G\). We recall that \(L^1(G)\) is a regular semisimple commutative Tauberian Banach algebra. Then we have the following proposition.

**Proposition 4.3.** Let \(G\) be a locally compact abelian group, \(\mu \in M(G)\) and \(X = L^1(G)\). Then every convolution operator \(T_\mu : X \to X\), \(T_\mu(k) = \mu \ast k\), has \((\delta_w)\) and
\[ \sigma_{\text{com}}(T_\mu) = \hat{\mu}(\Gamma). \]

Note that when \(G\) is nondiscrete there exists a measure \(\mu \in M(G)\) such that the convolution operator \(T_\mu\) does not have a natural spectrum (see for instance [5, Corollary 3] or [13, pp. 370]). In that example \(\hat{\mu}(T_\mu) \subset \sigma(T_\mu)\). Hence this also gives another example of an operator with \((\delta_w)\) but with neither weak-SDP nor \((\delta)\).

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