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Explicit formulas for optimal rearrangement-invariant norms in Sobolev imbedding inequalities

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Abstract. We study imbeddings of the Sobolev space

$$W^{m,\varrho}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ with } \varrho(\partial^\alpha u/\partial x^\alpha) < \infty \text{ when } |\alpha| \le m\},$$

in which Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, ϱ is a rearrangement-invariant (r.i.) norm and $1 \le m \le n-1$. For such a space we have shown there exist r.i. norms, τ_ϱ and σ_ϱ, that are optimal with respect to the inclusions

$$W^{m,\varrho}(\Omega) \subset W^{m,\tau_\varrho}(\Omega) \subset L_{\sigma_\varrho}(\Omega).$$

General formulas for τ_ϱ and σ_ϱ are obtained using the $\mathcal{K}$-method of interpolation. These lead to explicit expressions when ϱ is a Lorentz Gamma norm or an Orlicz norm.

1. Introduction. Suppose Ω is a bounded domain in $\mathbb{R}^n$ which has a Lipschitz boundary. Let $\partial^\alpha/\partial x^\alpha := \partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$ be a differential operator of order $|\alpha| := \alpha_1+\cdots+\alpha_n$. Denote by $|D^m u|$ the Euclidean length of the vector $D^m u := \{\partial^\alpha u/\partial x^\alpha\}_{0\le|\alpha|\le m}$, $1 \le m \le n-1$, of all derivatives of u of order m or less, whenever such derivatives exist in the weak sense. In this paper we study Sobolev imbedding inequalities of the form

$$\sigma(u) \le C\varrho(|D^m u|), \tag{1.1}$$

where ϱ and σ are rearrangement-invariant (r.i.) norms (such as those of Lebesgue, Orlicz and Lorentz) and u belongs to the r.i. Sobolev space

$$W^{m,\varrho}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ with } \varrho(|D^m u|) < \infty\}.$$

The focus is on cases in which ϱ and σ are optimal, namely, σ cannot be replaced by an essentially larger norm and ϱ cannot be replaced by an

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essentially smaller one. The goal is to give concrete expressions for these optimal ϱ and σ.

One starts with an r.i. Sobolev space, $W^{m,\varrho}(\Omega)$. Then, as shown in [KP2], one can construct (at least implicitly) r.i. norms, τ_ϱ and σ_ϱ, optimal with respect to the inclusions

$$W^{m,\varrho}(\Omega) \subset W^{m,\tau_\varrho}(\Omega) \subset L_{\sigma_\varrho}(\Omega), \tag{1.2}$$

where $L_{\sigma_\varrho}(\Omega) := \{f : \Omega \to \mathbb{R} : \sigma_\varrho(f) < \infty\}$. The norm τ_ϱ is called the *optimal hull norm* of ϱ.

The constructions are spoken of as implicit since the formulas are given in terms of the Köthe duals, τ'_ϱ and σ'_ϱ, of τ_ϱ and σ_ϱ. (For the precise definition of a Köthe dual see Section 2.) Thus,

$$\tau'_\varrho(g) := \bar{\varrho}'\Big(t^{m/n-1} \sup_{0<s<t} s^{-m/n} \int_0^s g^*(y)\,dy\Big), \tag{1.3}$$

$$\sigma'_\varrho(g) := \bar{\varrho}'\Big(t^{m/n-1} \int_0^t g^*(s)\,ds\Big), \quad g \in \mathfrak{M}_+(\Omega). \tag{1.4}$$

Here, $\mathfrak{M}_+(\Omega)$ is the class of nonnegative measurable real-valued functions on Ω, $\bar{\varrho}$ is an r.i. norm on $\mathfrak{M}_+(\mathbb{R}_+)$ related to ϱ, and g^* denotes the decreasing rearrangement of g on $\mathbb{R}_+$, given by

$$g^*(t) := \inf\{s \in \mathbb{R}_+ : |\{x \in \Omega : g(x) > s\}| \le t\}, \quad t \in \mathbb{R}_+.$$

As usual, $\mathbb{R}_+ := (0,\infty)$.

The expression for $\sigma'_\varrho(g)$ turns out to be unsatisfactory in that the function

$$t \mapsto t^{m/n-1} \int_0^t g^*(s)\,ds, \quad t \in \mathbb{R}_+,$$

need not be decreasing. This complicates the construction of explicit formulas for σ_ϱ. (However, see [EKP, Section 4] and [KP2, Section 4].)

Theorem 3.1 in Section 3 shows

$$\sigma_\varrho(f) \approx \mu'_\varrho(t^{-m/n} f^*(t)), \quad f \in \mathfrak{M}_+(\Omega), \tag{1.5}$$

where

$$\mu_\varrho(g) := \bar{\varrho}'\Big(t^{m/n-1} \int_0^t g^*(s) s^{-m/n}\,ds\Big), \quad g \in \mathfrak{M}_+(\Omega), \tag{1.6}$$

thereby eliminating the problem, since the function

$$t \mapsto t^{m/n-1} \int_0^t g^*(s) s^{-m/n}\,ds, \quad t \in \mathbb{R}_+$$

is decreasing, being a weighted average of a decreasing function.

(As usual, the equivalence sign signifies that the quantities are within constant multiples of one another, the constants being independent of the functions involved.)

The main task of this paper, then, is to compute the Köthe duals of τ'_ϱ and μ_ϱ. Our approach is to use the Brudnyi–Krugljak duality theory for the $\mathcal{K}$-method of interpolation, as elaborated for r.i. spaces in [KMS] and summarized in Section 3. The analysis is given in Section 4, leading to the (general) Theorem 4.2. For now, we briefly indicate how the $\mathcal{K}$-method comes in.

Let X_1 and X_2 be Banach spaces, compatible in the sense that they are imbedded in a common Hausdorff topological vector space $\mathcal{H}$. Suppose $x \in X_1 + X_2$ and $t \in \mathbb{R}_+$. The *Peetre K-functional* is defined by

$$K(t, x; X_1, X_2) := \inf_{x = x_1 + x_2} [\|x_1\|_{X_1} + t\|x_2\|_{X_2}], \qquad t > 0.$$

It is an increasing concave function of t on $\mathbb{R}_+$, so that

$$k(t, x; X_1, X_2) := \frac{d}{dt} K(t, x; X_1, X_2)$$

is decreasing on $\mathbb{R}_+$.

Given an r.i. norm ϱ on $\mathfrak{M}_+(\mathbb{R}_+)$ for which $\varrho\big(\frac{1}{1+t}\big) < \infty$, the space X, with $\|x\|_X$ defined at $x \in X_1 + X_2$ by

$$\|x\|_X := \varrho(t^{-1} K(t, x; X_1, X_2)),$$

satisfies

$$X_1 \cap X_2 \subset X \subset X_1 + X_2;$$

moreover, for any linear operator T defined on $X_1 + X_2$,

$$T : X_i \to X_i, \ i = 1, 2, \quad \text{implies} \quad T : X \to X.$$

We say the space X is generated by the $\mathcal{K}$-*method of interpolation*.

The asserted connection of the duality theory for the $\mathcal{K}$-method with our task is through certain reformulations of (1.3) and (1.5), namely

$$(1.7) \qquad \begin{aligned} \tau'_\varrho(g) &\approx \bar{\varrho}'(t^{m/n-1} K(t^{1-m/n}, g; L_{n/(n-m),\infty}(I_\Omega), L_\infty(I_\Omega))), \\ \mu_\varrho(g) &\approx \bar{\varrho}'(t^{m/n-1} K(t^{1-m/n}, g; L_{n/(n-m),1}(I_\Omega), L_\infty(I_\Omega))), \end{aligned}$$

in which $I_\Omega = (0, |\Omega|)$, $L_\infty(I_\Omega)$ is the usual Lebesgue space of essentially bounded functions on I_Ω, $L_{n/(n-m),\infty}(I_\Omega)$ has the norm

$$\varrho_{n/(n-m),\infty}(f) := \sup_{0<t<|\Omega|} t^{-m/n} \int_0^t f^*(s)\, ds$$

and $L_{n/(n-m),1}(I_\Omega)$ has the norm

$$\varrho_{n/(n-m),1}(f) := \int_0^{|\Omega|} f^*(t) t^{-m/n}\, dt, \qquad f \in \mathfrak{M}_+(I_\Omega).$$

Theorem 4.2 is applied to obtain explicit expressions equivalent to τ_ϱ and σ_ϱ for two classes of r.i. Sobolev spaces. The application to Lorentz–Sobolev spaces in Section 5 is more or less direct. It is the application to Orlicz–Sobolev spaces in Section 6 which requires a good deal of preliminary work. Indeed, the desire to obtain a result such as Theorem 6.3 was the motivating force behind the whole project. A. Cianchi [C] obtains a description of σ_ϱ different from the one in Theorem 6.3 by the use of techniques specific to the Orlicz context.

The final section presents two concrete examples.

2. Rearrangement-invariant norms. Suppose Ω is a domain in $\mathbb{R}^n$. Let $\mathfrak{M}(\Omega)$ be the class of real-valued, measurable functions on Ω and $\mathfrak{M}_+(\Omega)$ the class of nonnegative functions in $\mathfrak{M}(\Omega)$. Given $f \in \mathfrak{M}(\Omega)$, we define its decreasing rearrangement, f^*, on $\mathbb{R}_+$ by

$$f^*(t) := \inf\{\lambda > 0 : \mu_f(\lambda) \le t\}, \qquad t \in \mathbb{R}_+,$$

where

$$\mu_f(\lambda) := |\{x \in \Omega : |f(x)| > \lambda\}|.$$

Definition 2.1. A *rearrangement-invariant* (*r.i.*) *Banach function norm* ϱ on $\mathfrak{M}_+(\Omega)$ satisfies the following seven axioms:

(A.1) $\varrho(f) \ge 0$ with $\varrho(f) = 0$ if and only if $f = 0$ a.e. on Ω;
(A.2) $\varrho(cf) = c\varrho(f)$, $c \ge 0$;
(A.3) $\varrho(f + g) \le \varrho(f) + \varrho(g)$;
(A.4) $f_n \nearrow f$ implies $\varrho(f_n) \nearrow \varrho(f)$;
(A.5) $\varrho(\chi_E) < \infty$ for measurable $E \subset \Omega$, $|E| < \infty$;
(A.6) $\int_E f(x)\,dx \le C_E \varrho(f)$, $E \subset \Omega$, $|E| < \infty$, $C_E > 0$ independent of f;
(A.7) $\varrho(f) = \varrho(g)$ for all $f, g \in \mathfrak{M}_+(\Omega)$ such that $f^* = g^*$.

A fundamental result of Luxemburg [BS, Chapter 2, Theorem 4.10] asserts that to every r.i. norm ϱ on $\mathfrak{M}_+(\Omega)$ there corresponds an r.i. norm $\overline{\varrho}$ on $\mathfrak{M}_+(\mathbb{R}_+)$ such that

$$\varrho(f) = \overline{\varrho}(f^*), \qquad f \in \mathfrak{M}_+(\Omega). \tag{2.1}$$

The *Köthe dual* of an r.i. norm ϱ is another such norm, ϱ', with

$$\varrho'(g) := \sup_{\varrho(h) \le 1} \int_\Omega g(x)h(x)\,dx, \qquad g, h \in \mathfrak{M}_+(\Omega).$$

It obeys the Principle of Duality, namely,

$$\varrho'' := (\varrho')' = \varrho.$$

Moreover, the Hölder inequality

$$\int_\Omega f(x)g(x)\,dx \le \varrho(f)\varrho'(g)$$

holds for all $f, g \in \mathfrak{M}_+(\Omega)$; this inequality is saturated in the sense that, given $f \in \mathfrak{M}_+(\Omega)$ and $\varepsilon > 0$, there exists $g_0 \in \mathfrak{M}_+(\Omega)$ such that $\varrho'(g_0) \le 1$ and

$$\int_\Omega f(x) g_0(x)\, dx \ge (1-\varepsilon)\varrho(f)\varrho'(g_0).$$

We remark that, given (2.1), one has

$$\varrho'(g) = \bar{\varrho}'(g^*), \qquad g \in \mathfrak{M}_+(\Omega).$$

The first example of what we now call an r.i. norm was the Lebesgue norm ϱ_p, $1 \le p \le \infty$, with

$$\varrho_p(f) := \begin{cases} \Big(\int_\Omega f(x)^p\, dx\Big)^{1/p} = \Big(\int_{I_\Omega} f^*(t)^p\, dt\Big)^{1/p} & \text{when } p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} f(x) = f^*(0+) & \text{when } p = \infty,\ f \in \mathfrak{M}_+(\Omega). \end{cases}$$

It follows from the classical Hölder inequality and its converse that $\varrho_p' = \varrho_{p'}$, where $p = p/(p-1)$ (with the usual modification when $p = 1$).

Generalizations of the Lebesgue space $L_p(\Omega)$ will be studied in Sections 5 and 6 below.

Finally, we state a basic result concerning the k-functionals defined in the Introduction. It follows easily from a special case of [BS, Chapter 5, Theorem 1.19, pp. 305–306].

THEOREM 2.2. *Fix $b > 0$ and let ϱ_0, ϱ_1, μ_0, μ_1 and λ be r.i. norms on $\mathfrak{M}_+(I_b)$, $I_b = (0, b)$. Set*

$$\begin{aligned} \tau(f) &:= \lambda(k(t, f; L_{\varrho_0}(I_b), L_{\varrho_1}(I_b))), \\ \sigma(f) &:= \lambda(k(t, f; L_{\mu_0}(I_b), L_{\mu_1}(I_b))), \\ L_\tau(I_b) &:= \{f \in \mathfrak{M}(I_b) : \tau(|f|) < \infty\}, \\ L_\sigma(I_b) &:= \{f \in \mathfrak{M}(I_b) : \sigma(|f|) < \infty\}. \end{aligned}$$

Then $L_\tau(I_b)$ and $L_\sigma(I_b)$ are r.i. spaces with norms $\|f\|_\tau = \tau(|f|)$ and $\|f\|_\sigma = \sigma(|f|)$, respectively. Moreover, if T is any linear operator satisfying

$$T : L_{\varrho_0}(I_b) \to L_{\mu_0}(I_b) \quad \textit{and} \quad T : L_{\varrho_1}(I_b) \to L_{\mu_1}(I_b),$$

then $T : L_\tau(I_b) \to L_\sigma(I_b)$.

3. The norm of the optimal imbedding space for $W^{m,\varrho}(\Omega)$. The purpose of this section is to verify the assertion about σ_ϱ in (1.5). Since $W^{m,\varrho_{n/m,1}}(\Omega) \hookrightarrow L_\infty(\Omega)$ when Ω is bounded, we may assume

$$L_\varrho(\Omega) \supsetneq L_{n/m,1}(\Omega). \tag{3.1}$$

THEOREM 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Suppose ϱ is an r.i. norm on $\mathfrak{M}_+(\Omega)$ satisfying* (3.1). *Set*

$$\mu_\varrho(g) := \overline{\varrho}'\Big(t^{m/n-1}\int_0^t g^*(s)s^{-m/n}\,ds\Big), \qquad g \in \mathfrak{M}_+(I_\Omega),$$

in which $\overline{\varrho}$ is an r.i. norm on $\mathfrak{M}_+(\mathbb{R}_+)$, indeed on $\mathfrak{M}_+(I_\Omega)$, such that (2.1) *holds. Then, for the norm, σ_ϱ, of the optimal imbedding space for $W^{m,\varrho}(\Omega)$ in* (1.2) *we have*

$$\sigma_\varrho(f) \approx \nu(f) := \mu_\varrho'(t^{-m/n}f^*(t)), \qquad f \in \mathfrak{M}_+(\Omega). \tag{3.2}$$

Proof. The functional μ_ϱ is readily seen to be an r.i. norm. Now, by the definition of μ_ϱ, $L_{\mu_\varrho}(I_\Omega)$ is the largest r.i. space *from* which the mapping

$$g \mapsto t^{m/n-1}\int_0^t g^*(s)s^{-m/n}\,ds, \qquad t \in \mathbb{R}_+$$

is bounded into $L_{\overline{\varrho}'}(I_\Omega)$, whence $L_{\mu_\varrho'}(I_\Omega)$ is the smallest r.i. space *into* which the associate mapping

$$f \mapsto t^{-m/n}\int_t^{|\Omega|} f(s)s^{m/n-1}\,ds, \qquad t \in \mathbb{R}_+,$$

is bounded from $L_{\overline{\varrho}}(I_\Omega)$. So,

$$\nu(\chi_\Omega) = \mu_\varrho'(t^{-m/n}\chi_{I_\Omega}) \approx \mu_\varrho'\Big(t^{-m/n}\int_t^{|\Omega|}\chi_{I_\Omega}(s)s^{m/n-1}\,ds\Big) \le C\overline{\varrho}(\chi_{I_\Omega}) < \infty.$$

It is a routine exercise to verify ν satisfies the other axioms for an r.i. norm.

In view of [KP1, Theorem A], $W^{m,\varrho}(\Omega) \hookrightarrow L_\nu(\Omega)$ if and only if

$$\overline{\nu}\Big(\int_t^{|\Omega|} f(s)s^{m/n-1}\,ds\Big) \le C\overline{\varrho}(f), \qquad f \in \mathfrak{M}_+(I_\Omega).$$

But

$$\begin{aligned}\overline{\nu}\Big(\int_t^{|\Omega|} f(s)s^{m/n-1}\,ds\Big) &= \mu_\varrho'\Big(t^{-m/n}\int_t^{|\Omega|} f(s)s^{m/n-1}\,ds\Big)\\ &\le C\overline{\varrho}(f), \qquad f \in \mathfrak{M}_+(I_\Omega),\end{aligned}$$

so $\nu \le C\sigma_\varrho$. Again, by the above-observed optimality of $L_{\mu_\varrho'}(I_\Omega)$, we see that, in order to show $\sigma_\varrho \le C\nu$, it suffices to prove that

$$\tau\Big(\int_t^{|\Omega|} f(s)s^{m/n-1}\,ds\Big) \le C\overline{\varrho}(f), \qquad f \in \mathfrak{M}_+(I_\Omega),$$

(where we have $\tau = \overline{\sigma}_\varrho$ in mind) implies

$$\tau(f) \approx \lambda(t^{-m/n}f^*(t)), \qquad f \in \mathfrak{M}_+(I_\Omega),$$

for some r.i. norm λ on $\mathfrak{M}_+(I_\Omega)$. Indeed, one would then have

$$\lambda\Big(t^{-m/n}\int_t^{|\Omega|} f(s)s^{m/n-1}\,ds\Big) \approx \tau\Big(\int_t^{|\Omega|} f(s)s^{m/n-1}\,ds\Big) \le C\overline{\varrho}(f), \quad f\in\mathfrak{M}_+(I_\Omega),$$

hence $\lambda \le C\mu_\varrho'$ and so $\sigma_\varrho \le C\nu$.

To find such a λ for a given τ, set

$$\kappa(f) := \tau(f^*(t^{1-m/n})), \quad f\in\mathfrak{M}_+(I_\Omega);$$

of course, this is equivalent to

$$\tau(f) = \kappa(f^*(t^{n/(n-m)})), \quad f\in\mathfrak{M}_+(I_\Omega).$$

Since $W^{m,\varrho_1}(\Omega) \hookrightarrow L_{n/(n-m),1}(\Omega)$, we may suppose $\varrho_{n/(n-m),1} \le \tau$. Then κ is clearly seen to be an r.i. norm. In fact, only the verification of (A.6) requires comment: we have, for $f\in\mathfrak{M}_+(I_\Omega)$,

$$\int_0^{|\Omega|} f^*(t)\,dt \le \varrho_{n/(n-m),1}(f^*(t^{1-m/n})) \le C\tau(f^*(t^{1-m/n})) = \kappa(f).$$

By Holmstedt's formula [H],

$$K(t,h^*;L_1(I_\Omega),L_{n/m,\infty}(I_\Omega)) \approx \int_0^{t^{n/(n-m)}} h^*(s)\,ds + t\sup_{t^{n/(n-m)}<s<|\Omega|} s^{m/n}h^*(s)$$

for $h\in\mathfrak{M}_+(I_\Omega)$, so

$$\begin{aligned}(3.3)\quad & K(t,s^{-m/n}f^*(s);L_1(I_\Omega),L_{n/m,\infty}(I_\Omega))\\ &\approx \int_0^{t^{n/(n-m)}} s^{-m/n}f^*(s)\,ds + tf^*(t^{n/(n-m)})\\ &\approx \int_0^{t^{n/(n-m)}} s^{-m/n}f^*(s)\,ds \approx \int_0^t f^*(s^{n/(n-m)})\,ds\end{aligned}$$

for $f\in\mathfrak{M}_+(I_\Omega)$ and $t\in I_\Omega$. We conclude that

$$\int_0^t k(s,y^{-m/n}f^*(y);L_1(I_\Omega),L_{n/m,\infty}(I_\Omega))\,ds \approx \int_0^t f^*(s^{n/(n-m)})\,ds$$

for $f\in\mathfrak{M}_+(I_\Omega)$ and $t\in I_\Omega$, and hence, from the Hardy–Littlewood–Pólya Principle [BS, Chapter 2, Theorem 4.6, p. 61],

$$\kappa(k(t,s^{-m/n}f^*(s);L_1(I_\Omega),L_{n/m,\infty}(I_\Omega))) \approx \kappa(f^*(s^{n/(n-m)})), \quad f\in\mathfrak{M}_+(I_\Omega).$$

Therefore,

$$\tau(f) \approx \lambda(t^{-m/n}f^*(t)), \quad f\in\mathfrak{M}_+(I_\Omega),$$

where

$$\lambda(g) := \kappa(k(t,g^*; L_1(I_\Omega), L_{n/m,\infty}(I_\Omega))), \qquad g \in \mathfrak{M}_+(I_\Omega),$$

is an r.i. norm, according to Theorem 2.2. ■

4. The Brudnyi–Krugljak duality theory for the $\mathcal{K}$-method. The $\mathcal{K}$-method of interpolation was outlined for an arbitrary pair of compatible spaces X_1 and X_2 in the Introduction. Theorem C of [KMS] gives the consequences of the Brudnyi–Krugljak duality theory for the $\mathcal{K}$-method when X_1 and X_2 are r.i. spaces. A special case of that result is

THEOREM 4.1. *Let $I_b = (0,b)$, $b > 0$, and suppose ϱ_1 and ϱ_2 are r.i. norms on $\mathfrak{M}_+(I_b)$. Assume, further, that*

$$L_{\varrho_1'}(I_b) \cap L_{\varrho_2'}(I_b) \quad \textit{is dense in } L_{\varrho_2'}(I_b)$$

and

$$\varrho_2'(\chi_{(0,a)}) \downarrow 0 \qquad \textit{as } a \downarrow 0,\ 0 < a < b.$$

Then, for any r.i. norm ϱ on $\mathfrak{M}_+(\mathbb{R}_+)$ such that $\varrho\big(\frac{1}{1+t}\big) < \infty$, the functional

$$\sigma(f) := \varrho(t^{-1}K(t,f; L_{\varrho_1}(I_b), L_{\varrho_2}(I_b))), \qquad f \in \mathfrak{M}_+(\mathbb{R}_+),$$

is a norm. Moreover, if, in addition,

$$\varrho'\bigg(\frac{1}{1+t}\bigg) < \infty \quad \textit{and} \quad \varrho(\chi_{(0,a)}) \downarrow 0 \quad \textit{as } a \downarrow 0,$$

then

$$\sigma'(g) \approx \varrho'_\Gamma(k(t,g; L_{\varrho_2'}(I_b), L_{\varrho_1'}(I_b)), \qquad g \in \mathfrak{M}_+(\mathbb{R}_+);$$

here, ϱ_Γ is the r.i. norm defined by

$$\varrho_\Gamma(f) := \varrho\Big(t^{-1}\int_0^t f^*(s)\,ds\Big), \qquad f \in \mathfrak{M}_+(\mathbb{R}_+).$$

The implications of Theorem 4.1 for determining τ_ϱ and σ_ϱ are given in

THEOREM 4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, having a Lipschitz boundary. Suppose ϱ is an r.i. norm on $\mathfrak{M}_+(\Omega)$ defined in terms of an r.i. norm $\bar\varrho$ on $\mathfrak{M}_+(\mathbb{R}_+)$ by*

$$\varrho(f) = \bar\varrho(f^*), \qquad f \in \mathfrak{M}_+(\Omega).$$

Assume, in addition, that

$$\bar\varrho\bigg(\frac{1}{1+t}\bigg) < \infty, \quad \bar\varrho'\bigg(\frac{1}{1+t}\bigg) < \infty, \quad \bar\varrho(\chi_{(0,a)}) \downarrow 0 \quad \textit{as } a \downarrow 0.$$

Then the optimal hull norm τ_ϱ of ϱ satisfies

$$\tau_\varrho(f) \approx \lambda'_\Gamma(k(t,f^*; L_1(I_\Omega), L_{n/m,1}(I_\Omega))), \qquad f \in \mathfrak{M}_+(\Omega),$$

where

$$\lambda(h) := \bar{\varrho}'(h^*(t^{1-m/n})), \quad h \in \mathfrak{M}_+(\mathbb{R}_+).$$

Again, for the norm σ_ϱ of the smallest r.i. space of functions on Ω into which $W^{m,\varrho}(\Omega)$ imbeds, we have

$$\sigma_\varrho(f) \approx \lambda'_\Gamma(k(t, s^{-m/n} f^*(s); L_1(I_\Omega), L_{n/m,\infty}(I_\Omega))), \quad f \in \mathfrak{M}_+(\Omega).$$

Proof. According to [KP2, Theorem B] and Holmstedt's formula,

$$\begin{aligned}
\tau'_\varrho(g) &\approx \bar{\varrho}'\Big(t^{m/n-1} \sup_{0<s<t} s^{-m/n} \int_0^s g^*(y)\,dy\Big) \\
&\approx \bar{\varrho}'(t^{m/n-1} \sup_{0<s<t} s^{1-m/n} g^*(s)) \\
&\approx \bar{\varrho}'(t^{m/n-1} K(t^{1-m/n}, g^*; L_{n/(n-m),\infty}(I_\Omega), L_\infty(I_\Omega))) \\
&\approx \lambda(t^{-1} K(t, g^*; L_{n/(n-m),\infty}(I_\Omega), L_\infty(I_\Omega))), \quad g \in \mathfrak{M}_+(\Omega).
\end{aligned}$$

Thus, by Theorem 4.1,

$$\tau_\varrho(f) \approx \lambda'_\Gamma(k(t, f^*; L^1(I_\Omega), L_{n/m,1}(I_\Omega))), \quad f \in \mathfrak{M}_+(\Omega).$$

Similarly, in view of Theorem 3.1,

$$\sigma_\varrho(f) \approx \nu(f) := \mu'_\varrho(t^{-m/n} f^*(t)), \quad f \in \mathfrak{M}_+(\Omega),$$

where

$$\begin{aligned}
\mu_\varrho(g) &= \bar{\varrho}'\Big(t^{m/n-1} \int_0^t g^*(s) s^{-m/n}\,ds\Big) \\
&\approx \bar{\varrho}'(t^{m/n-1} K(t^{1-m/n}, g^*; L_{n/(n-m),1}(I_\Omega), L_\infty(I_\Omega))) \\
&\approx \lambda(t^{-1} K(t, g^*; L_{n/(n-m),1}(I_\Omega), L_\infty(I_\Omega))), \quad g \in \mathfrak{M}_+(I_\Omega).
\end{aligned}$$

Hence, Theorem 4.1 yields

$$\sigma_\varrho(f) \approx \lambda'_\Gamma(k(t, s^{-m/n} f^*(s); L_1(I_\Omega), L_{n/m,\infty}(I_\Omega))), \quad f \in \mathfrak{M}_+(\Omega). \ \blacksquare$$

REMARK 4.3. It is clear that to compute τ_ϱ and σ_ϱ we need to know the Köthe dual of

$$\lambda_\Gamma(h) = \bar{\varrho}'\Big(t^{m/n-1} \int_0^{t^{1-m/n}} h^*(s)\,ds\Big), \quad h \in \mathfrak{M}_+(\mathbb{R}_+).$$

We look at this question in the next two sections, in which ϱ is first a Lorentz Gamma norm and then an Orlicz norm.

5. Lorentz Gamma norms

DEFINITION 5.1. For a nonnegative, locally integrable (weight) function Φ on $\mathbb{R}_+$ and an index $p \in [1, \infty)$, the *Lorentz Gamma norm* $\varrho_{p,\Phi}$ is defined by

$$\varrho_{p,\Phi}(f) := \Big(\int_0^{|\Omega|}\Big(t^{-1}\int_0^t f^*(s)\,ds\Big)^p \Phi(t)\,dt\Big)^{1/p}, \qquad f \in \mathfrak{M}_+(\Omega).$$

The *Lorentz Gamma space* $\Gamma_{p,\Phi} = \Gamma_{p,\Phi}(\Omega)$ is then

$$\Gamma_{p,\Phi} := \{f \in \mathfrak{M}(\Omega) : \varrho_{p,\Phi}(f) < \infty\},$$

with $\|f\|_{\Gamma_{p,\Phi}} := \varrho_{p,\Phi}(f)$.

We will require $\int_0^1 \Phi(t)\,dt < \infty$ and $\int_0^{|\Omega|} t^{-p}\Phi(t)\,dt = \infty$ to guarantee that, respectively, $\Gamma_{p,\Phi} \neq \{0\}$ and $\Gamma_{p,\Phi} \neq L_1(\Omega)$, the usual Lebesgue space of functions integrable on Ω.

The Köthe dual, $\varrho'_{p,\Phi}$, of $\varrho_{p,\Phi}$ satisfies

$$\varrho'_{p,\Phi}(g) \approx \varrho_{p',\Psi}(g), \qquad g \in \mathfrak{M}_+(\Omega),$$

where, at $t \in \mathbb{R}_+$,

$$\Psi(t) := \begin{cases} \dfrac{t^{p'+p-1}(\int_0^t \Phi(s)\,ds)(\int_t^{|\Omega|} s^{-p}\Phi(s)\,ds)}{(\int_0^t \Phi(s)\,ds + t^p \int_t^{|\Omega|} s^{-p}\Phi(s)\,ds)^{p'+1}} & \text{when } 1 < p < \infty,\\[2ex] \dfrac{t}{\int_0^t \Phi(s)\,ds + t\int_t^{|\Omega|} s^{-1}\Phi(s)\,ds} & \text{when } p = 1. \end{cases}$$

See [GP, Theorem 6.2] for the case $|\Omega| = \infty$.

Of special interest are the original Lorentz norms $\varrho_{p,q} = \varrho_{p,\Phi}$, in which

$$\Phi(t) = t^{q/p-1} \qquad \text{when } 1 < p < \infty,\ 1 \le q < \infty.$$

One also defines

$$\varrho_{p,\infty}(f) := \sup_{0<t<|\Omega|} t^{1/p-1}\int_0^t f^*(s)\,ds, \qquad f \in \mathfrak{M}_+(\Omega),\ 1 < p < \infty.$$

The corresponding Lorentz spaces are denoted by $L_{p,q}(\Omega)$ and one has, by Hardy's inequality,

$$\|f\|_{L_{p,q}(\Omega)} \approx \begin{cases} \Big(\int_0^{|\Omega|} (t^{1/p-1/q} f^*(t))^q\,dt\Big)^{1/q} & \text{when } 1 \le q < \infty,\\ \sup_{0<t<|\Omega|} t^{1/p} f^*(t) & \text{when } q = \infty. \end{cases}$$

Fix an index p, $1 < p < \infty$, and let Φ be a weight on I_b for some $b > 0$. To apply the Brudnyi–Krugljak theory we extend Φ to all of $\mathbb{R}_+$ by requiring

$$\Phi(t) = t^{\alpha}, \qquad t > b, \qquad \frac{m}{n}p - 1 < \alpha < p - 1. \tag{5.1}$$

An expression equivalent to the Köthe dual of the λ_Γ defined in terms of

$$\varrho(f) = \varrho_{p,\Phi}(f) = \Big(\int_{\mathbb{R}_+} \Big(t^{-1} \int_0^t f^*(s)\, ds \Big)^p \Phi(t)\, dt \Big)^{1/p}$$

is given in

THEOREM 5.2. *Fix an index p, $1 < p < \infty$, and let Φ be a weight on I_b, $b > 0$, satisfying*

$$\int_0^b \Phi(t)\, dt < \infty \quad \text{and} \quad \int_0^b t^{-p} \Phi(t)\, dt = \infty. \tag{5.2}$$

Extend Φ to all of $\mathbb{R}_+$ as in (5.1). Then the norm $\varrho_{p,\Phi}$ on $\mathfrak{M}_+(\mathbb{R}_+)$ has Köthe dual norm

$$\varrho'_{p,\Phi}(g) \approx \varrho_{p',\Psi}(g), \qquad g \in \mathfrak{M}_+(\mathbb{R}_+),$$

with

$$\Psi(t) := \begin{cases} \dfrac{t^{p'+p-1} \int_0^t \Phi(s)\, ds \int_t^b s^{-p} \Phi(s)\, ds}{(\int_0^t \Phi(s)\, ds + t^p \int_t^b s^{-p} \Phi(s)\, ds)^{p'+1}}, & 0 < t < b/2, \\ t^{\alpha(1-p')}, & t \geq b/2. \end{cases} \tag{5.3}$$

If, further,

$$\int_0^b t^{(m/n-1)p'} \Psi(t)\, dt = \infty, \tag{5.4}$$

then the r.i. norm

$$\lambda_\Gamma(g) := \varrho'_{p,\Phi}\Big(t^{m/n-1} \int_0^{t^{1-m/n}} g^*(s)\, ds \Big), \qquad g \in \mathfrak{M}_+(\mathbb{R}_+),$$

has the Köthe dual norm

$$\lambda'_\Gamma(f) \approx \varrho_{p,\widetilde{\Phi}}(f), \qquad f \in \mathfrak{M}_+(\mathbb{R}_+), \tag{5.5}$$

where

$$\widetilde{\Phi}(t) := \begin{cases} \dfrac{t^{p'+p-1} \int_0^t \widetilde{\Psi}(s)\, ds \int_t^b s^{-p'} \widetilde{\Psi}(s)\, ds}{(\int_0^t \widetilde{\Psi}(s)\, ds + t^{p'} \int_t^b s^{-p'} \widetilde{\Psi}(s)\, ds)^{p+1}}, & 0 < t < b/2, \\ t^{\frac{m(1-p)+\alpha n}{n-m}}, & t \geq b/2, \end{cases} \tag{5.6}$$

in which

$$\widetilde{\Psi}(t) := t^{m/(n-m)} \Psi(t^{n/(n-m)}), \qquad t \in \mathbb{R}_+.$$

Proof. The conditions (5.1) and (5.2) guarantee that Φ satisfies the hypotheses of Theorem 6.2 of [GP] and, as well, that $\int_0^\infty \Phi(t)\, dt = \infty$. Hence,

$$\varrho'_{p,\Phi}(g) \approx \varrho_{p',\Psi}(g), \qquad g \in \mathfrak{M}_+(\mathbb{R}_+),$$

since

$$\Psi(t) \approx \frac{t^{p'+p-1}\int_0^t \Phi(s)\,ds \int_t^\infty s^{-p}\Phi(s)\,ds}{(\int_0^t \Phi(s)\,ds + t^p\int_t^\infty s^{-p}\Phi(s)\,ds)^{p'+1}}, \qquad t \in \mathbb{R}_+.$$

Similarly, (5.3) and (5.4) ensure that

$$\int_0^1 \widetilde{\Psi}(t)\,dt < \infty \quad \text{and} \quad \int_0^1 t^{-p'}\widetilde{\Psi}(t)\,dt = \int_1^\infty \widetilde{\Psi}(t)\,dt = \infty,$$

which means

$$\varrho'_{p',\widetilde{\Psi}}(f) \approx \varrho_{p,\widetilde{\Phi}}(f), \qquad f \in \mathfrak{M}_+(\mathbb{R}_+),$$

as

$$\widetilde{\Phi}(t) \approx \frac{t^{p'+p-1}\int_0^t \widetilde{\Psi}(s)\,ds \int_t^\infty s^{-p'}\widetilde{\Psi}(s)\,ds}{(\int_0^t \widetilde{\Psi}(s)\,ds + t^{p'}\int_t^\infty s^{-p'}\widetilde{\Psi}(s)\,ds)^{p+1}}, \qquad t \in \mathbb{R}_+.$$

It only remains to verify that

$$\lambda'_\Gamma(f) \approx \varrho_{p,\widetilde{\Phi}}(f), \qquad f \in \mathfrak{M}_+(\mathbb{R}_+),$$

or, equivalently,

$$\lambda_\Gamma(g) \approx \Big(\int_{\mathbb{R}_+} \Big(t^{-1}\int_0^t g^*(s)\,ds\Big)^{p'} \widetilde{\Psi}(t)\,dt\Big)^{1/p'}, \qquad g \in \mathfrak{M}_+(\mathbb{R}_+).$$

But,

$$\begin{aligned}
\lambda_\Gamma(g) &= \varrho'_{p,\Phi}\Big(t^{m/n-1}\int_0^{t^{1-m/n}} g^*(s)\,ds\Big) \approx \varrho_{p',\Psi}\Big(t^{m/n-1}\int_0^{t^{1-m/n}} g^*(s)\,ds\Big)\\
&\approx \Big(\int_{\mathbb{R}_+}\Big(t^{-1}\int_0^t s^{m/n-1}\int_0^{s^{1-m/n}} g^*(y)\,dy\,ds\Big)^{p'}\Psi(t)\,dt\Big)^{1/p'}\\
&\approx \Big(\int_{\mathbb{R}_+}\Big(t^{m/n-1}\int_0^t g^*(s)\,ds\Big)^{p'}\Psi(t)\,dt\Big)^{1/p'}\\
&\approx \Big(\int_{\mathbb{R}_+}\Big(t^{-1}\int_0^t g^*(s)\,ds\Big)^{p'}\widetilde{\Psi}(t)\,dt\Big)^{1/p'}.
\end{aligned}$$

The result now follows from Theorem 2.2. ∎

Given (5.5) we can now prove

THEOREM 5.3. *Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, having a Lipschitz boundary. Fix an index p, $1 < p < \infty$. Let Φ be a weight on $\mathbb{R}_+$ satisfying* (5.1) *and* (5.2). *Assume, further, Φ is such that* (5.4) *holds for the Ψ given by* (5.3). *Define $\widetilde{\Phi}$ as in* (5.6). *Then $f \in \mathfrak{M}_+(\Omega)$ belongs to the* *optimal hull of* $\Gamma_{p,\Phi}(\Omega)$ *if and only if*

$$\begin{aligned}&\int\limits_{I_\Omega}\Big(t^{-1}\int\limits_0^{t^{n/(n-m)}} f^*(s)\,ds\Big)^p\widetilde{\Phi}(t)\,dt<\infty,\\ &\int\limits_{I_\Omega}\Big(\int\limits_{t^{n/(n-m)}}^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big)^p\widetilde{\Phi}(t)\,dt<\infty.\end{aligned}\tag{5.7}$$

Again, $f\in\mathfrak{M}_+(\Omega)$ *belongs to the optimal r.i. imbedding range of* $W^{m,\varrho_{p,\Phi}}(\Omega)$ *if and only if*

$$\int\limits_{I_\Omega}\Big(t^{-1}\int\limits_0^t f^*(s^{n/(n-m)})\,ds\Big)^p\widetilde{\Phi}(t)\,dt<\infty.\tag{5.8}$$

Proof. Theorem 4.2, (5.5) and the Holmstedt formulas imply

$$\begin{aligned}\tau_\varrho(f)&\approx\lambda'_\Gamma(k(t,f;L_1(I_\Omega),L_{n/m,1}(I_\Omega)))\\ &\approx\varrho_{p,\tilde{\Phi}}(k(t,f;L_1(I_\Omega),L_{n/m,1}(I_\Omega)))\\ &=\Big(\int\limits_{\mathbb{R}_+}\Big(t^{-1}\int\limits_0^t k(s,f;L_1(I_\Omega),L_{n/m,1}(I_\Omega))\,ds\Big)^p\widetilde{\Phi}(t)\,dt\Big)^{1/p}\\ &=\Big(\int\limits_{\mathbb{R}_+}(t^{-1}K(t,f;L_1(I_\Omega),L_{n/m,1}(I_\Omega)))^p\widetilde{\Phi}(t)\,dt\Big)^{1/p}\\ &\approx\Big(\int\limits_0^{|\Omega|}\Big(t^{-1}\int\limits_0^{t^{n/(n-m)}} f^*(s)\,ds+\int\limits_{t^{n/(n-m)}}^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big)^p\widetilde{\Phi}(t)\,dt\Big)^{1/p}\\ &\quad+\Big(\int\limits_0^{|\Omega|} f^*(s)\,ds\Big)\Big(\int\limits_{|\Omega|^{1-m/n}}^\infty t^{-p}\widetilde{\Phi}(t)\,dt\Big)^{1/p},\end{aligned}$$

from which the criterion (5.7) follows, in view of (5.1).

Similarly, the fact that

$$\begin{aligned}\sigma_\varrho(f)&\approx\lambda'_\Gamma(k(t,s^{-m/n}f^*(s);L_1(I_\Omega),L_{n/m,\infty}(I_\Omega)))\\ &=\Big(\int\limits_{\mathbb{R}_+}(t^{-1}K(t,s^{-m/n}f^*(s);L_1(I_\Omega),L_{n/m,\infty}(I_\Omega)))^p\widetilde{\Phi}(t)\,dt\Big)^{1/p}\\ &\approx\Big(\int\limits_0^{|\Omega|^{1-m/n}}\Big(t^{-1}\int\limits_0^t f^*(s^{n/(n-m)})\,ds\Big)^p\widetilde{\Phi}(t)\,dt\Big)^{1/p}\\ &\quad+\Big(\int\limits_0^{|\Omega|^{1-m/n}} f^*(s^{n/(n-m)})\,ds\Big)\Big(\int\limits_{|\Omega|^{1-m/n}}^\infty t^{-p}\widetilde{\Phi}(t)\,dt\Big)^{1/p}\end{aligned}$$

yields the criterion (5.8). ∎

6. Orlicz norms. A generalization of the Lebesgue spaces, due to Orlicz, is defined in terms of certain *ϕ-functions*. A ϕ-function is an increasing continuous function from $\mathbb{R}_+$ onto itself. We associate to such a function the gauge functional

$$\varrho_\phi(f) := \inf\Big\{\lambda > 0 : \int_\Omega \phi(|f(x)|/\lambda)\,dx \le 1\Big\}, \qquad f \in \mathfrak{M}(\Omega). \tag{6.1}$$

When a ϕ-function is convex it is referred to as a *Young function*. If A is a Young function one can show that the functional, now ϱ_A, in (6.1) is, in fact, an r.i. norm, called the *gauge norm*. See [BS, Chapter 4, Section 8] for details, and, as well, [RR] for the interesting history of such norms.

The Köthe dual of ϱ_A is equivalent to the gauge norm $\varrho_{\widetilde{A}}$, where, if $A(t) = \int_0^t a(s)\,ds$,

$$\widetilde{A}(t) := \int_0^t a^{-1}(s)\,ds, \qquad t \in \mathbb{R}_+;$$

indeed,

$$\varrho_{\widetilde{A}}(g) \le \varrho_A'(g) \le 2\varrho_{\widetilde{A}}(g), \qquad g \in \mathfrak{M}_+(\Omega).$$

The Orlicz spaces determined by ϱ_A and $\varrho_{\widetilde{A}}$ are denoted by $L_A(\Omega)$ and $L_{\widetilde{A}}(\Omega)$, respectively.

Let A be a Young function and assume that for fixed $b > 0$ and $1 < q < n/m$,

$$A(t) = t^q, \qquad t \in I_b. \tag{6.2}$$

Suppose $\widetilde{A}$ is the Young function complementary to A and set

$$\lambda_\Gamma(g) := \varrho_{\widetilde{A}}\Big(t^{m/n-1}\int_0^{t^{1-m/n}} g^*(s)\,ds\Big), \qquad g \in \mathfrak{M}_+(I_b).$$

Our principal task in this section is to prove

THEOREM 6.1. *Let A be a Young function and fix $b > 0$. Assume A is given by* (6.2) *when $t \in I_b$ and suppose*

$$t^{m/n-1} \notin L_{\widetilde{A}}(\mathbb{R}_+). \tag{6.3}$$

Define B through the equation

$$B(\gamma(t)) := \Big(\frac{m}{n} - 1\Big)\widetilde{A}(t^{m/n-1})\frac{\gamma(t)}{t\gamma'(t)}, \tag{6.4}$$

in which

$$\gamma(t) := t^{-m/n}\int_t^\infty \widetilde{A}(s^{m/n-1})\,ds, \qquad t \in \mathbb{R}_+.$$

Then B is a Young function and

$$\lambda_\Gamma'(f) \approx \varrho_B(t^{-m/n}f^*(t^{1-m/n})), \qquad f \in \mathfrak{M}_+(I_b). \tag{6.5}$$

The first step towards proving Theorem 6.1 is taken in

PROPOSITION 6.2. *Suppose A is a Young function satisfying* (6.2) *and define B as in* (6.4). *Then, B is a Young function such that*

$$Q_{n/m} : L_A(\mathbb{R}_+) \to L_B(\mathbb{R}_+), \tag{6.6}$$

where

$$(Q_{n/m}f)(t) := t^{-m/n} \int_t^\infty f(s)s^{m/n-1}\,ds, \qquad f \in \mathfrak{M}_+(\mathbb{R}_+),\ t \in \mathbb{R}_+.$$

Proof. The methods of [BK] ensure that in order for there to be a constant $K > 0$, independent of $f \in \mathfrak{M}_+(\mathbb{R}_+)$, with

$$\int_{\mathbb{R}_+} \widetilde{A}\Big(t^{m/n-1}\int_0^t f(s)s^{-m/n}\,ds\Big)\,dt \le \int_{\mathbb{R}_+} \widetilde{B}(Kf(s))\,ds, \tag{6.7}$$

it is necessary and sufficient that

$$\int_0^t B\bigg(\frac{\alpha(\lambda,t)}{Cs^{m/n}}\bigg)\,ds \le \alpha(\lambda,t) < \infty, \tag{6.8}$$

where B is a Young function, $C > 0$ is independent of $\lambda, t \in \mathbb{R}_+$ and

$$\alpha(\lambda,t) := \int_t^\infty \widetilde{A}(\lambda s^{m/n-1})\,ds.$$

Further, (6.7) implies

$$Q'_{n/m} : L_{\widetilde{B}}(\mathbb{R}_+) \to L_{\widetilde{A}}(\mathbb{R}_+)$$

and hence (6.6). Thus, we will be done if we can prove that the B defined by (6.4) is a Young function for which (6.8) holds.

If, in the equation

$$\int_0^t B\bigg(\frac{\alpha(t)}{s^{m/n}}\bigg)\,ds = \alpha(t), \qquad \alpha(t) := \alpha(1,t),\ t \in \mathbb{R}_+, \tag{6.9}$$

we set $y = \alpha(t)/s^{m/n}$, it becomes

$$\int_{\gamma(t)}^\infty \frac{B(y)}{y^{n/m+1}}\,dy = \frac{m}{n}\alpha(t)^{1-n/m}, \tag{6.10}$$

which, on differentiation, yields (6.4).

Since $\gamma(t)$ is decreasing and convex, with $\gamma(0) = \infty$ and $\gamma(\infty) = 0$, we will have proved that B is a Young function if we can show $B(\gamma(t))$ is also decreasing and convex. But,

$$B(\gamma(t)) = \frac{(1-m/n)\tilde{A}(t^{m/n-1})\frac{\alpha(t)}{t}}{\frac{m}{n}\frac{\alpha(t)}{t} + \tilde{A}(t^{m/n-1})}, \qquad t \in \mathbb{R}_+. \tag{6.11}$$

Moreover, both $\tilde{A}(t^{m/n-1})$ and $\alpha(t)/t$ are decreasing and convex, which means $\frac{1}{\tilde{A}(t^{m/n-1})} + \frac{n}{m}\frac{t}{\alpha(t)}$ is increasing and concave, so $B(\gamma(t))$, a constant multiple of the reciprocal of the latter, is indeed decreasing and convex.

Now, (6.8) holds for $\lambda = 1$ by the definition of B. It only remains to observe that (6.8), with $\lambda = 1$ and t replaced by $t/\lambda^{n/(n-m)}$, reduces to the general (6.8) after some simple changes of variable. ■

Proof of Theorem 6.1. In view of (6.3) and the freedom of choice concerning $\widetilde{A}$ on I_b, one readily shows that, for a near 0,

$$\begin{aligned} \varrho_B(t^{-m/n}\chi_{(0,a)}(t)) &\ge \frac{a^{-m/n}}{G^{-1}(1/a)}, \\ \varrho_{\widetilde{A}}(t^{m/n-1}\chi_{(a,b)}(t)) &\ge \frac{a^{m/n-1}}{H^{-1}(1/a)}, \end{aligned} \tag{6.12}$$

where

$$G(t) := t^{n/m}\int_t^\infty \frac{B(s)}{s^{n/m+1}}\,ds, \qquad H(t) := t^{n/(n-m)}\int_0^t \frac{\widetilde{A}(s)}{s^{n/(n-m)+1}}\,ds, \qquad t \in \mathbb{R}_+.$$

Then,

$$\frac{G(t)}{t} \le G'(t) \le \frac{n}{m}\frac{G(t)}{t}, \qquad t \in \mathbb{R}_+,$$

the second inequality being obvious, while the first is equivalent to

$$t^{n/m-1}\int_t^\infty \frac{B(s)}{s^{n/m+1}}\,ds \ge \frac{m}{n-m}\frac{B(t)}{t},$$

and, indeed,

$$t^{n/m-1}\int_t^\infty \frac{B(s)}{s^{n/m+1}}\,ds \ge t^{n/m-1}\frac{B(t)}{t}\int_t^\infty s^{-n/m}\,ds = \frac{m}{n-m}\frac{B(t)}{t}. \tag{6.13}$$

Again,

$$\begin{aligned} H'(t) &= \frac{n}{n-m}t^{n/(n-m)-1}\int_0^t \frac{\widetilde{A}(s)}{s^{n/(n-m)+1}}\,ds + \frac{\widetilde{A}(t)}{t} \\ &\ge \frac{n}{n-m}\frac{H(t)}{t}, \qquad t \in \mathbb{R}_+. \end{aligned}$$

Consider, next,

$$\Phi_B(t) := \frac{n}{m}t^{-m/n}G(t), \qquad \Psi_A(t) := n/(n-m)t^{-n/(n-m)}H(t), \qquad t \in \mathbb{R}_+.$$

Observe that, by (6.10),

$$\Phi_B(\gamma(t)) = \alpha(t)^{1-n/m}$$

and, by a simple change of variable,

$$\Psi_A(t) = \alpha(t^{-n/(n-m)}), \qquad t \in \mathbb{R}_+.$$

So,

$$\Phi_B(t^{-m/n}\Psi_A(t^{m/n-1})) = \Psi_A(t^{m/n-1})^{1-n/m}$$

or

$$(t^{m/(n-m)}\Psi_A(t))^{n/m-1}\Phi_B(t^{m/(n-m)}\Psi_A(t)) = t,$$

that is,

$$\begin{aligned} G'(H'(t)) &\ge \frac{G(H'(t))}{H'(t)} \ge \frac{G(t^{n/(n-m)}\Psi_A(t))}{t^{n/(n-m)}\Psi_A(t)} \\ &= (t^{m/(n-m)}\Psi_A(t))^{n/m-1}\Phi_B(t^{m/(n-m)}\Psi_A(t)) = t. \end{aligned} \tag{6.14}$$

(The second inequality in (6.14) used the facts that $G(t)/t$ increases (in view of (6.13)) and $H'(t) \ge n/(n-m)\frac{H(t)}{t} = t^{m/(n-m)}\Psi_A(t)$.) We have

$$G'(t) \ge (H')^{-1}(t), \qquad G(t) \ge \widetilde{H}(t), \qquad G^{-1}(t) \le \widetilde{H}^{-1}(t),$$

and, finally,

$$\frac{1}{\widetilde{H}^{-1}(t)} \le \frac{1}{G^{-1}(t)}, \qquad t \in \mathbb{R}_+. \tag{6.15}$$

From (6.12), (6.15) and [RR, p. 12],

$$\begin{aligned} \varrho_B(t^{-m/n}\chi_{(0,a)}(t))\varrho_{\widetilde{A}}(t^{m/n-1}\chi_{(a,b)}(t)) &\ge \frac{a^{-m/n}}{G^{-1}(1/a)}\,\frac{a^{m/n-1}}{2H^{-1}(1/a)} \\ &\ge \frac{1/a}{2H^{-1}(1/a)\widetilde{H}^{-1}(1/a)} \ge \frac{1}{4}, \end{aligned}$$

for a near 0.

Then [BK, Section 4] implies

$$\int_0^b f^*(t)g^*(t)\,dt \le C\varrho_B(t^{-m/n}f^*(t))\varrho_{\widetilde{A}}\Big(t^{m/n-1}\int_0^t g^*(s)\,ds\Big), \qquad f,g \in \mathfrak{M}_+(I_b).$$

Replacing $f^*(t)$ by $f^*(t^{1-m/n})$ and $g^*(t)$ by $g^*(t^{1-m/n})t^{-m/n}$ and changing variables, we obtain

$$\int_0^b f^*(t)g^*(t)\,dt \le C\varrho_B(t^{-m/n}f^*(t^{1-m/n}))\lambda_\Gamma(g).$$

Taking the supremum over all g with $\lambda_\Gamma(g) \le 1$ yields

$$\lambda'_\Gamma(f) \le C\varrho_B(t^{-m/n}f^*(t^{1-m/n})).$$

We now prove the reverse estimate. Given $f \in \mathfrak{M}_+(I_b)$, there exists $k_0 \in \mathfrak{M}_+(I_b)$ such that $\varrho_A(k_0) \le 1$ and

$$\int_0^b t^{m/n-1} \int_0^{t^{1-m/n}} f^*(s)\, ds\, k_0(t)\, dt \ge 1/2 \varrho_{\widetilde{A}}\Big(t^{m/n-1} \int_0^{t^{1-m/n}} f^*(s)\, ds\Big).$$

Set

$$g_0(t) = g_0^*(t) := \int_{t^{n/(n-m)}}^b k_0(s) s^{m/n-1}\, ds, \qquad t \in I_b.$$

Then,

$$\begin{aligned}\int_0^b f^*(s) g_0^*(s)\, ds &= \int_0^b f^*(s) \int_{s^{n/(n-m)}}^b k_0(t) t^{m/n-1}\, dt\, ds \\ &= \int_0^b t^{m/n-1} \int_0^{t^{1-m/n}} f^*(s)\, ds\, k_0(t)\, dt \\ &\ge \frac{1}{2} \varrho_{\widetilde{A}}\Big(t^{m/n-1} \int_0^{t^{1-m/n}} f^*(s)\, ds\Big).\end{aligned}$$

Moreover, in view of Proposition 6.2,

$$\varrho_B(t^{-m/n} g_0^*(t^{1-m/n})) \le C,$$

that is,

$$\varrho_B\Big(t^{-m/n} \int_t^b k_0(s) s^{m/n-1}\, ds\Big) \le C \varrho_A(k_0) \le C.$$

We conclude that $\nu'(f) \ge c\lambda_\Gamma(f)$, or

$$\lambda'_\Gamma(h) \ge c\nu(h),$$

where

$$\nu(h) := \varrho_B(t^{-m/n} h^*(t^{1-m/n})), \qquad h \in \mathfrak{M}_+(I_b).$$

This completes the proof. ■

With (6.5) in hand we can now prove

THEOREM 6.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$, having a Lipschitz boundary. Consider a Young function A given by* (6.2) *when $t \in I_\Omega$ and suppose its complementary function $\widetilde{A}$ satisfies*

$$t^{m/n-1} \notin L_{\widetilde{A}}(I_\Omega).$$

Define the new Young function B as in (6.4) *and, in terms of it, the abso-*

lutely continuous ϕ-function $C(t) := \int_0^t c(s)\,ds$, with

$$c(s) := s\frac{d}{ds}\left(\frac{B(s)}{s}\right) = b(s) - \frac{B(s)}{s}, \qquad s \in \mathbb{R}_+.$$

Then $f \in \mathfrak{M}_+(\Omega)$ is in the optimal hull $L_{\tau_A}(\Omega)$ of $L_A(\Omega)$ if and only if $f \in L_B(\Omega)$ and

$$\int_{I_\Omega} C\Big(\lambda_f^{-1} t^{-m/n} \int_t^{|\Omega|} f^*(s) s^{m/n-1}\,ds\Big)\,dt < \infty \tag{6.16}$$

for some $\lambda_f \in \mathbb{R}_+$.

Again, $f \in \mathfrak{M}_+(\Omega)$ belongs to the optimal r.i. imbedding range $L_{\sigma_A}(\Omega)$ of $W^{m,A}(\Omega)$ if and only if

$$\int_{I_\Omega} C\Big(\lambda_f^{-1} t^{-1} \int_0^t f^*(s) s^{-m/n}\,ds\Big)\,dt < \infty \tag{6.17}$$

for some $\lambda_f \in \mathbb{R}_+$.

Proof. We begin by observing that, according to [GK, Theorem 2],

$$\varrho_B(f) \approx \varrho_{\Gamma_C}(f) := \inf\Big\{\lambda > 0 : \int_{\mathbb{R}_+} C\Big(\lambda^{-1} t^{-1} \int_0^t f^*(s)\,ds\Big)\,dt \le 1\Big\} \tag{6.18}$$

for $f \in \mathfrak{M}_+(\mathbb{R}_+)$.

Now, Theorem 4.2 and (6.5) imply

$$\begin{aligned}\tau_\varrho(f) &\approx \lambda'_\Gamma(k(t, f^*; L_1(I_\Omega), L_{n/m,1}(I_\Omega)))\\ &\approx \varrho_B(t^{-m/n} k(t^{1-m/n}, f^*; L_1(I_\Omega), L_{n/m,1}(I_\Omega))), \qquad f \in \mathfrak{M}_+(\Omega).\end{aligned}$$

So, from (6.18), $\tau_\varrho(f) < \infty$ if and only if, for some $\lambda_f \in \mathbb{R}_+$,

$$\infty > \int_{\mathbb{R}_+} C\Big(\lambda_f^{-1} t^{-1} \int_0^t s^{-m/n} k(s^{1-m/n}, f^*; L_1(I_\Omega), L_{n/m,1}(I_\Omega))\,ds\Big)\,dt,$$

or, setting $y = s^{1-m/n}$,

$$\begin{aligned}\infty &> \int_{\mathbb{R}_+} C\Big(\lambda_f^{-1} t^{-1} \int_0^{t^{1-m/n}} k(y, f^*; L_1(I_\Omega), L_{n/m,1}(I_\Omega))\,dy\Big)\,dt\\ &= \int_{\mathbb{R}_+} C(\lambda_f^{-1} t^{-1} K(t^{1-m/n}, f^*; L_1(I_\Omega), L_{n/m,1}(I_\Omega)))\,dt.\end{aligned}$$

The Holmstedt formula and the linearity of the class

$$\Gamma_C := \{f \in \mathfrak{M}_+(\mathbb{R}_+) : \varrho_{\Gamma_C}(f) < \infty\}$$

show the latter holds if and only if, for some $\lambda_f \in \mathbb{R}_+$, one has (6.16),

$$\int_{I_\Omega} C\Big(\lambda_f^{-1} t^{-1} \int_0^t f^*(s)\,ds\Big)\,dt < \infty \tag{6.19}$$

and

$$\int_{|\Omega|}^{\infty} C\Big(\lambda_f^{-1} \int_0^{|\Omega|} f^*(s)\,ds t^{-1}\Big)\,dt < \infty. \tag{6.20}$$

But, in view of (6.18), the condition (6.19) is the assertion that $f \in L_B(\Omega)$, while (6.20) is always true, since $C(t) = t^q$, $q > 1$, near 0.

Similarly,

$$\begin{aligned}\sigma_\varrho(f) &\approx \lambda'_\Gamma(k(t, s^{-m/n} f^*(s); L_1(I_\Omega), L_{n/m,\infty}(I_\Omega)))\\ &\approx \varrho_B(t^{-m/n} k(t^{1-m/n}, s^{-m/n} f^*(s); L_1(I_\Omega), L_{n/m,\infty}(I_\Omega))).\end{aligned}$$

Hence, by (6.18), $\sigma_\varrho(f) < \infty$ if and only if, for some $\lambda_f \in \mathbb{R}_+$,

$$\begin{aligned}\infty &> \int_{\mathbb{R}_+} C\Big(\lambda_f^{-1} t^{-1} \int_0^t s^{-m/n} k(s^{1-m/n}, y^{-m/n} f^*(y); L_1(I_\Omega), L_{n/m,\infty}(I_\Omega))\,ds\Big)\,dt\\ &\approx \int_{\mathbb{R}_+} C\Big(\lambda_f^{-1} t^{-1} \int_0^{t^{1-m/n}} k(s, y^{-m/n} f^*(y); L_1(I_\Omega), L_{n/m,\infty}(I_\Omega))\,ds\Big)\,dt\\ &= \int_{\mathbb{R}_+} C(\lambda_f^{-1} t^{-1} K(t^{1-m/n}, s^{-m/n} f^*(s); L_1(I_\Omega), L_{n/m,\infty}(I_\Omega)))\,dt\\ &\approx \int_{\mathbb{R}_+} C\Big(\lambda_f^{-1} t^{-1} \int_0^t s^{-m/n} f^*(s)\,ds\Big)\,dt, \qquad \text{by (3.3). } \blacksquare\end{aligned}$$

7. Examples. We illustrate Theorem 6.3 in two important cases.

EXAMPLE 7.1. The case $p = n/m$. Here,

$$A(t) := \begin{cases} t^q, & 1 < q < n/m,\ 0 < t < 1,\\ t^{n/m}, & t \ge 1,\end{cases}$$

and

$$\gamma^{-1}(t) \approx \begin{cases} t^{-\frac{n}{n-m}(q-1)}, & 0 < t < 2,\\ t^{-n/m}(\ln t)^{n/m}, & t \ge 2.\end{cases}$$

It follows that

$$B(t) \approx C(t) \approx \begin{cases} t^q, & 0 < t < 2,\\ t^{n/m}(\ln t)^{-n/m}, & t > 2.\end{cases} \tag{7.1}$$

We conclude that $f \in L_{\tau_A}(\Omega)$ if and only if

(7.2)
$$\int_0^{|\Omega|}\Big(t^{-m/n}\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big)^{n/m}\ln_+^{-n/m}\Big(t^{-m/n}\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big)\,dt<\infty,$$

since (7.1) and (7.2) imply $f \in L_B(\Omega)$.

Again, $f \in L_{\sigma_A}(\Omega)$ if and only if

$$\int_0^{|\Omega|}\Big(t^{-1}\int_0^t f^*(s)s^{-m/n}\,ds\Big)^{n/m}\ln_+^{-n/m}\Big(t^{-1}\int_0^t f^*(s)s^{-m/n}\,ds\Big)\,dt < \infty.$$

But,

$$\varrho_B\Big(t^{-1}\int_0^t g^*(s)\,ds\Big) \approx \varrho_B(g^*), \qquad g \in \mathfrak{M}_+(\mathbb{R}_+),$$

so $f \in L_{\sigma_A}(\Omega)$ if and only if

$$\int_0^{|\Omega|} (t^{-m/n}f^*(t))^{n/m}\ln_+^{-n/m}(t^{-m/n}f^*(t))\,dt < \infty.$$

Therefore, since $L_{\sigma_A}(\Omega) \subset L_{n/(n-m),\infty}(\Omega)$,

$$\sigma_A(f) \approx \Big(\int_0^{|\Omega|} f^*(t)^{n/m}\ln^{-n/m}\Big(\frac{|\Omega|}{t}\Big)\frac{dt}{t}\Big)^{m/n}. \tag{7.3}$$

For the history of this norm, see the introductory section in [EKP].

Example 7.2. Consider the Young function

$$A_\beta(t) := \int_0^t \ln^\beta(1+s)\,ds \approx t\ln^\beta(1+t), \qquad 0<\beta<1,\ t \in \mathbb{R}_+.$$

Since the operator $Q_{n/m}$ maps $L_{A_\beta}(\mathbb{R}_+)$ to itself we may take $B(t) = A_\beta(t)$.

As shown in [GK, Example 4.1], C satisfies

$$C(t) \approx \int_0^t \ln^{\beta-1}(1+s)\,ds \approx t\ln^{\beta-1}(1+t) \qquad \text{for } t>1;$$

in particular, C is *not* a Young function.

We find that $f \in L_{\tau_{A_\beta}}(\Omega)$ if and only if

$$\int_0^{|\Omega|} f^*(t)\ln^\beta(1+f^*(t))\,dt < \infty$$

and

$$\int_0^{|\Omega|} \Big(t^{-m/n} \int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big) \ln^{\beta-1}\Big(1+t^{-m/n}\int_t^{|\Omega|} f^*(s)s^{m/n-1}\,ds\Big)\,dt<\infty.$$

Again, $f\in L_{\sigma_{A_\beta}}(\Omega)$ if and only if

$$\int_0^{|\Omega|} \Big(t^{-1}\int_0^t f^*(s)s^{-m/n}\,ds\Big) \ln^{\beta-1}\Big(1+t^{-1}\int_0^t f^*(s)s^{-m/n}\,ds\Big)\,dt<\infty$$

or

$$\int_0^{|\Omega|} f^*(t)t^{-m/n}\ln^{\beta}(1+f^*(t)t^{-m/n})\,dt<\infty.$$

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