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Mappings on some reflexive algebras characterized by action on zero products or Jordan zero products

by

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Abstract. Let $\mathcal{L}$ be a subspace lattice on a Banach space X and let $\delta : \operatorname{Alg}\mathcal{L} \to B(X)$ be a linear mapping. If $\bigvee\{L \in \mathcal{L} : L_- \not\supseteq L\} = X$ or $\bigwedge\{L_- : L \in \mathcal{L}, L_- \not\supseteq L\} = (0)$, we show that the following three conditions are equivalent: (1) $\delta(AB) = \delta(A)B + A\delta(B)$ whenever $AB = 0$; (2) $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB + BA = 0$; (3) δ is a generalized derivation and $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$. If $\bigvee\{L \in \mathcal{L} : L_- \not\supseteq L\} = X$ or $\bigwedge\{L_- : L \in \mathcal{L}, L_- \not\supseteq L\} = (0)$ and δ satisfies $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ whenever $AB = 0$, we show that δ is a generalized derivation and $\delta(I)A \in (\operatorname{Alg}\mathcal{L})'$ for every $A \in \operatorname{Alg}\mathcal{L}$. We also prove that if $\bigvee\{L \in \mathcal{L} : L_- \not\supseteq L\} = X$ and $\bigwedge\{L_- : L \in \mathcal{L}, L_- \not\supseteq L\} = (0)$, then δ is a local generalized derivation if and only if δ is a generalized derivation.

1. Introduction. Throughout this paper, X denotes a Banach space over the real or complex field $\mathbb{F}$ and X^* is the topological dual of X. When X is a Hilbert space, we relabel it as H. We denote by $B(X)$ the set of all bounded linear operators on X. For $A \in B(X)$, we denote by A^* the adjoint of A. A *subspace* of X means a norm closed linear manifold. For a subset $L \subseteq X$, denote by $L^\perp$ the annihilator of L, that is, $L^\perp = \{f \in X^* : f(x) = 0$ for all $x \in L\}$. By a *subspace lattice* on X, we mean a collection $\mathcal{L}$ of subspaces of X with (0) and X in $\mathcal{L}$ such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\bigwedge M_r$ and $\bigvee M_r$ belong to $\mathcal{L}$, where $\bigwedge M_r$ denotes the intersection of $\{M_r\}$, and $\bigvee M_r$ denotes the closed linear span of $\{M_r\}$. We use $\operatorname{Alg}\mathcal{L}$ to denote the algebra of operators in $B(X)$ that leave members of $\mathcal{L}$ invariant.

Let $x \in X$ and $f \in X^*$ be non-zero. The rank-one operator $x \otimes f$ is defined by $y \mapsto f(y)x$ for $y \in X$. If $\mathcal{L}$ is a subspace lattice on X and $E \in \mathcal{L}$, we define

$$E_- = \bigvee\{F \in \mathcal{L} : F \not\supseteq E\}, \quad E_+ = \bigwedge\{F \in \mathcal{L} : F \not\subseteq E\}$$

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and

$$\mathcal{J}_{\mathcal{L}} = \{L \in \mathcal{L} : L \neq (0) \text{ and } L_- \neq X\}, \quad \mathcal{P}_{\mathcal{L}} = \{L \in \mathcal{L} : L_- \not\supseteq L\}.$$

It is obvious that $\mathcal{P}_{\mathcal{L}} \subseteq \mathcal{J}_{\mathcal{L}}$. It is well known [14] that a rank one operator $x \otimes f$ is in $\operatorname{Alg}\mathcal{L}$ if and only if there exists a $K \in \mathcal{J}_{\mathcal{L}}$ such that $x \in K$ and $f \in K_-^{\perp}$. A subspace lattice $\mathcal{L}$ is called a *completely distributive lattice* if $L = \bigvee\{E \in \mathcal{L} : E_- \not\supseteq L\}$ for every $L \in \mathcal{L}$ (see [14]); $\mathcal{L}$ is called a $\mathcal{J}$*-subspace lattice* if $L \wedge L_- = (0)$ for every $L \in \mathcal{J}_{\mathcal{L}}$, $X = \bigvee\{L : L \in \mathcal{J}_{\mathcal{L}}\}$ and $\bigwedge\{L_- : L \in \mathcal{J}_{\mathcal{L}}\} = (0)$ (see [15]). A totally ordered subspace lattice $\mathcal{N}$ is called a *nest*. Recall that a nest $\mathcal{N}$ is called *discrete* if $N_- \neq N$ for every non-trivial subspace N in $\mathcal{N}$.

We say that $\mathcal{L}$ is a $\mathcal{P}$*-subspace lattice* on X if $\bigvee\{L : L \in \mathcal{P}_{\mathcal{L}}\} = X$ or $\bigwedge\{L_- : L \in \mathcal{P}_{\mathcal{L}}\} = (0)$. It is obvious that this class of subspace lattices contains $\mathcal{J}$-subspace lattices, discrete nests and subspace lattices with $X_- \neq X$ or $(0)_+ \neq (0)$. The following example is also a $\mathcal{P}$-subspace lattice. As usual, in a Hilbert space, we disregard the distinction between a subspace and the orthogonal projection onto it.

EXAMPLE 1.1. Let $\{e_n : n \in \mathbb{Z}^+\}$ be an orthonormal basis of a complex Hilbert space H, $P_n = \operatorname{span}\{e_i : i = 1, \dots, n\}$, $\xi = \sum_{n=1}^{\infty}(1/n)e_n$ and $P_\xi = \mathbb{C}\xi$. It follows from [20, Theorem 2.11] and [7, Lemma 3.2] that $\mathcal{L} = \{0, I, P_n, P_\xi, P_\xi \vee P_n : n = 1, 2, \dots\}$ is a reflexive $\mathcal{P}$-subspace lattice.

A subspace lattice on a Hilbert space H is called a *commutative subspace lattice* (or *CSL* for short) if it consists of mutually commuting projections. In this paper, we assume that H is a complex separable Hilbert space.

Let δ be a linear mapping from a unital algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$. Recall that δ is a *derivation* (respectively, *generalized derivation*) if $\delta(AB) = \delta(A)B + A\delta(B)$ (respectively, $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$) for all A, B in $\mathcal{A}$. We say that δ is *derivable* at $Z \in \mathcal{A}$ if $\delta(AB) = \delta(A)B + A\delta(B)$ for any $A, B \in \mathcal{A}$ with $AB = Z$; δ is *Jordan derivable* at $Z \in \mathcal{A}$ if $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for any $A, B \in \mathcal{A}$ with $AB + BA = Z$. If $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for any $A, B \in \mathcal{A}$ with $AB = 0$, we say that δ has the *WJD* (weak Jordan derivation) property.

In recent years, there have been a number of papers on the study of conditions under which derivations and Jordan derivations of operator algebras can be completely determined by the action on some subsets of operator algebras (for example, see [1, 3, 8, 9, 21]). For instance, Zhao and Zhu [21] show that every linear mapping δ from a triangular algebra $\mathcal{T}$ into itself with the WJD property is a derivation. In [8], Jiao and Hou prove that every additive mapping δ derivable or Jordan derivable at zero on some nest algebras has the form $\delta(A) = \tau(A) + cA$ for some additive derivation τ and some scalar $c \in \mathbb{F}$.

The purpose of this paper is to consider some mappings which behave like derivations on $\mathcal{P}$-subspace lattice algebras and completely distributive commutative subspace lattice (CDCSL) algebras.

In Section 2, we show that every linear (respectively, bounded linear) mapping δ on a $\mathcal{P}$-subspace lattice (respectively, CDCSL) algebra which is Jordan derivable at zero point is a generalized derivation and $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$.

In Section 3, for a $\mathcal{P}$-subspace lattice algebra $\operatorname{Alg}\mathcal{L}$, we prove that δ has the WJD property if and only if δ is a generalized derivation and $\delta(I)A \in (\operatorname{Alg}\mathcal{L})'$ for every $A \in \operatorname{Alg}\mathcal{L}$.

In Section 4, we investigate mappings derivable at zero and some linear mappings which behave like left (respectively, right) multipliers, isomorphisms or local generalized derivations on $\mathcal{P}$-subspace lattice algebras. One of the main results of the section is that if $\bigvee\{L \in \mathcal{L} : L_- \not\supseteq L\} = X$ and $\bigwedge\{L_- : L \in \mathcal{L},\, L_- \not\supseteq L\} = (0)$, then δ is a local generalized derivation from $\operatorname{Alg}\mathcal{L}$ into $B(X)$ if and only if δ is a generalized derivation.

The following proposition will be used in our proofs.

Proposition 1.2 ([19, Proposition 1.1]). *Let E and F be non-zero subspaces of X and X^*, respectively. Let $\Phi : E \times F \to B(X)$ be a bilinear mapping such that $\Phi(x,f)\ker(f) \subseteq \mathbb{F}x$ for all $x \in E$ and $f \in F$. Then there exist two linear mappings $T : E \to X$ and $S : F \to X^*$ such that $\Phi(x,f) = Tx \otimes f + x \otimes Sf$ for all $x \in E$ and $f \in F$.*

2. Mappings Jordan derivable at zero. The following lemma is included in the proof of [8, Theorem 3.1]. We leave the proof to the reader.

Lemma 2.1. *If δ is Jordan derivable at zero from a unital algebra $\mathcal{A}$ into a unital $\mathcal{A}$ bimodule, then for any idempotents P and Q in $\mathcal{A}$,*

(i) $\delta(I)P = P\delta(I)$;
(ii) $\delta(P) = \delta(P)P + P\delta(P) - P\delta(I)$;
(iii) $\delta(PQ+QP) = \delta(P)Q + P\delta(Q) + \delta(Q)P + Q\delta(P) - \delta(I)(PQ+QP)$.

For a subspace lattice $\mathcal{L}$ and a subspace $E \in \mathcal{P}_{\mathcal{L}}$, we denote by $\mathcal{T}_E$ the ideal $\operatorname{span}\{x \otimes f : x \in E,\, f \in E_-^{\perp}\}$ of $\operatorname{Alg}\mathcal{L}$.

Lemma 2.2. *If $\mathcal{L}$ is a subspace lattice on X and E is in $\mathcal{P}_{\mathcal{L}}$, then for every x in E and every f in $E_-^{\perp}$, $x \otimes f$ is a linear combination of idempotents in $\mathcal{T}_E$.*

Proof. Suppose $f(x) \neq 0$; then $x \otimes f = f(x)\big(\frac{1}{f(x)}x \otimes f\big)$, where $\frac{1}{f(x)}x \otimes f$ is an idempotent in $\mathcal{T}_E$.

Suppose $f(x) = 0$. Since $E \in \mathcal{P}_{\mathcal{L}}$, there exist $z \in E$ and $g \in E_-^{\perp}$ such that $g(z) = 1$.

Case 1. If $g(x)=\mu\neq 0$, then $x\otimes f=x\otimes\big(\frac{1}{\mu}g+f\big)-x\otimes\frac{1}{\mu}g$, where $x\otimes\big(\frac{1}{\mu}g+f\big)$ and $x\otimes\frac{1}{\mu}g$ are idempotents in $\mathcal{T}_E$.

Case 2. If $f(z)=\lambda\neq 0$, then $x\otimes f=\big(x+\frac{1}{\lambda}z\big)\otimes f-\frac{1}{\lambda}z\otimes f$, where $\big(x+\frac{1}{\lambda}z\big)\otimes f$ and $\frac{1}{\lambda}z\otimes f$ are idempotents in $\mathcal{T}_E$.

Case 3. If $f(z)=g(x)=0$, then $x\otimes f=\frac{1}{4}((z+x)\otimes(g+f)+(z-x)\otimes(g-f)-(z+x)\otimes(g-f)-(z-x)\otimes(g+f))$, where $(z+x)\otimes(g+f)$, $(z-x)\otimes(g-f)$, $(z+x)\otimes(g-f)$ and $(z-x)\otimes(g+f)$ are idempotents in $\mathcal{T}_E$. The proof is complete. ∎

Lemma 2.3. *Let $\mathcal{L}$ be a subspace lattice on X, E be in $P_{\mathcal{L}}$ and δ be a linear mapping from* $\operatorname{Alg}\mathcal{L}$ *into $B(X)$. If δ is Jordan derivable at zero, then for every idempotent P in $\mathcal{T}_E$ and every A in* $\operatorname{Alg}\mathcal{L}$*,*

(i) $\delta(AP+PA)=\delta(A)P+A\delta(P)+\delta(P)A+P\delta(A)-\delta(I)(AP+PA)$;
(ii) $\delta(PAP)=\delta(P)AP+P\delta(A)P+PA\delta(P)-2\delta(I)PAP$.

Proof. (i) For every idempotent $P\in\mathcal{T}_E$ and every $A\in\operatorname{Alg}\mathcal{L}$, since $P^{\perp}AP^{\perp}P+PP^{\perp}AP^{\perp}=0$, by assumption we have

$$\delta(P^{\perp}AP^{\perp})P+P^{\perp}AP^{\perp}\delta(P)+\delta(P)P^{\perp}AP^{\perp}+P\delta(P^{\perp}AP^{\perp})=0.$$

Since $A-P^{\perp}AP^{\perp}=PA+P^{\perp}AP\in\mathcal{T}_E$, it follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned}\delta(AP+PA)&=\delta((A-P^{\perp}AP^{\perp})P+P(A-P^{\perp}AP^{\perp}))\\&=\delta(A-P^{\perp}AP^{\perp})P+(A-P^{\perp}AP^{\perp})\delta(P)\\&\quad+\delta(P)(A-P^{\perp}AP^{\perp})+P\delta(A-P^{\perp}AP^{\perp})\\&\quad-\delta(I)(AP+PA)\\&=\delta(A)P+A\delta(P)+\delta(P)A+P\delta(A)-\delta(I)(AP+PA)\\&\quad-\delta(P^{\perp}AP^{\perp})P-P^{\perp}AP^{\perp}\delta(P)-\delta(P)P^{\perp}AP^{\perp}\\&\quad-P\delta(P^{\perp}AP^{\perp})\\&=\delta(A)P+A\delta(P)+\delta(P)A+P\delta(A)-\delta(I)(AP+PA).\end{aligned}$$

(ii) The substitution of $AP+PA$ for A in (i) gives (ii). ∎

One of the main results of this section is the following theorem.

Theorem 2.4. *Let $\mathcal{L}$ be a subspace lattice on X such that $\bigvee\{L:L\in\mathcal{P}_{\mathcal{L}}\}=X$ and δ be a linear mapping from* $\operatorname{Alg}\mathcal{L}$ *into $B(X)$. Then δ is Jordan derivable at zero if and only if δ is a generalized derivation and $\delta(I)\in(\operatorname{Alg}\mathcal{L})'$, where $(\operatorname{Alg}\mathcal{L})'$ is the commutant of* $\operatorname{Alg}\mathcal{L}$ *in $B(X)$. In particular, if $\delta(I)=0$, then δ is Jordan derivable at zero if and only if δ is a derivation.*

Proof. The sufficiency is obvious, so we only need to prove the necessity. Let $E\in\mathcal{P}_{\mathcal{L}}$, $z\in E$ and $g\in E_{-}^{\perp}$ with $g(z)=1$. We divide the proof into several claims.

Claim 1. $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$.

For every $x \in E$, $f \in E_-^\perp$ and $T \in \operatorname{Alg}\mathcal{L}$, by Lemmas 2.1 and 2.2, we have $\delta(I)Tx \otimes f = Tx \otimes f\delta(I) = T\delta(I)x \otimes f$. That is, $\delta(I)Tx = T\delta(I)x$ for every $x \in E$. Since $\bigvee\{E : E \in \mathcal{P}_\mathcal{L}\} = X$, it follows that $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$.

Now define $\tau(A) = \delta(A) - \delta(I)A$ for $A \in \operatorname{Alg}\mathcal{L}$. It is easy to see that τ is Jordan derivable at zero and $\tau(I) = 0$.

Claim 2. *$\tau(x \otimes f)\ker(f) \subseteq \mathbb{F}x$ for every $x \in E$ and $f \in E_-^\perp$.*

Case 1. If $f(x) = \mu \neq 0$, then by Lemma 2.1, we have

$$\tau\left(\frac{1}{\mu}x \otimes f\right) = \tau\left(\frac{1}{\mu}x \otimes f\right)\left(\frac{1}{\mu}x \otimes f\right) + \left(\frac{1}{\mu}x \otimes f\right)\tau\left(\frac{1}{\mu}x \otimes f\right).$$

Thus $\tau(x \otimes f)\ker(f) \subseteq \mathbb{F}x$.

Case 2. If $f(x) = 0$ and $f(z) \neq 0$, then by Case 1, for every $y \in \ker(f)$,

$$\tau((z+x) \otimes f)y = \lambda_1(z+x),$$
$$\tau((z-x) \otimes f)y = \lambda_2(z-x),$$
$$\tau(z \otimes f)y = \lambda_3 z,$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$. By the above equations, it follows that

$$2\lambda_3 z = (\lambda_1 + \lambda_2)z + (\lambda_1 - \lambda_2)x,$$

and the independence of z and x implies $\lambda_1 = \lambda_2 = \lambda_3$. Hence

$$\tau(x \otimes f)y = \tau((z+x) \otimes f)y - \tau(z \otimes f)y = \lambda_1 x.$$

This means $\tau(x \otimes f)\ker(f) \subseteq \mathbb{F}x$.

Case 3. Suppose that $f(x) = 0$ and $f(z) = 0$. Since $z \otimes (g+f)$ and $z \otimes (g-f)$ are idempotents in $\mathcal{T}_E$, it follows from Lemma 2.3 that

$$\begin{aligned}
&\tau((z \otimes (g+f))(x \otimes g)(z \otimes (g+f))) \\
&\quad = \tau(z \otimes (g+f))(x \otimes g)(z \otimes (g+f)) + (z \otimes (g+f))\tau(x \otimes g)(z \otimes (g+f)) \\
&\qquad + (z \otimes (g+f))(x \otimes g)\tau(z \otimes (g+f)), \\
&\tau((z \otimes (g-f))(x \otimes g)(z \otimes (g-f))) \\
&\quad = \tau(z \otimes (g-f))(x \otimes g)(z \otimes (g-f)) + (z \otimes (g-f))\tau(x \otimes g)(z \otimes (g-f)) \\
&\qquad + (z \otimes (g-f))(x \otimes g)\tau(z \otimes (g-f)),
\end{aligned}$$

and

$$\begin{aligned}
\tau((z \otimes g)(x \otimes g)(z \otimes g)) = {}& \tau(z \otimes g)(x \otimes g)(z \otimes g) + (z \otimes g)\tau(x \otimes g)(z \otimes g) \\
&+ (z \otimes g)(x \otimes g)\tau(z \otimes g).
\end{aligned}$$

From the above three equations, we have

$$\begin{aligned}0 &= \tau((z\otimes f)(x\otimes g)(z\otimes f))\\ &= \tau(z\otimes f)(x\otimes g)(z\otimes f) + (z\otimes f)\tau(x\otimes g)(z\otimes f)\\ &\quad + (z\otimes f)(x\otimes g)\tau(z\otimes f)\\ &= \tau(z\otimes f)(x\otimes f) + (z\otimes f)\tau(x\otimes g)(z\otimes f).\end{aligned}$$

Thus

$$\tau(z\otimes f)x = -f(\tau(x\otimes g)z)z. \tag{2.1}$$

Hence by (2.1), and Lemmas 2.2 and 2.3, it follows that

$$\begin{aligned}\tau(x\otimes f) &= \tau((z\otimes f)(x\otimes g) + (x\otimes g)(z\otimes f))\\ &= -f(\tau(x\otimes g)z)z\otimes g + (z\otimes f)\tau(x\otimes g)\\ &\quad + \tau(x\otimes g)(z\otimes f) + (x\otimes g)\tau(z\otimes f).\end{aligned}$$

Let y be in $\ker(f)$. Applying the above equations to y gives

$$\tau(x\otimes f)y = -g(y)f(\tau(x\otimes g)z)z + f(\tau(x\otimes g)y)z + g(\tau(z\otimes f)y)x. \tag{2.2}$$

Notice that (2.2) is valid for every $z\in E$ satisfying $g(z)=1$ and $f(z)=0$. If $g(x)=\mu\neq 0$, replacing z by $(1/\mu)x$ in (2.2), we have $\tau(x\otimes f)y\in\mathbb{F}x$. If $g(x)=0$, by the proof of [18, Lemma 2.3], we have $g(y)f(\tau(x\otimes g)z) - f(\tau(x\otimes g)y)=0$, whence $\tau(x\otimes f)y = g(\tau(z\otimes f)y)x\in\mathbb{F}x$.

Claim 3. *τ is a derivation.*

By Claim 2 and Proposition 1.2, there exist linear mappings $T: E\to X$ and $S: E_-^\perp\to X^*$ such that

$$\tau(x\otimes f) = Tx\otimes f + x\otimes Sf \tag{2.3}$$

for every $x\in E$ and $f\in E_-^\perp$. It follows from Lemmas 2.2 and 2.3 that for every $A\in\operatorname{Alg}\mathcal{L}$,

$$\begin{aligned}\tau(Ax\otimes g + x\otimes gA) &= \tau(A)x\otimes g + A\tau(x\otimes g) + \tau(x\otimes g)A\\ &\quad + x\otimes g\tau(A).\end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we have

$$\begin{aligned}&TAx\otimes g + Ax\otimes Sg + Tx\otimes A^*g + x\otimes SA^*g\\ &= \tau(A)x\otimes g + ATx\otimes g + Ax\otimes Sg + Tx\otimes A^*g + x\otimes A^*Sg + x\otimes\tau(A)^*g.\end{aligned}$$

That is,

$$(\tau(A) + AT - TA)x\otimes g = x\otimes(SA^* - \tau(A)^* - A^*S)g.$$

Thus there exists a linear mapping $\lambda:\operatorname{Alg}\mathcal{L}\to\mathbb{F}$ such that

$$\tau(A)x = (TA - AT)x + \lambda(A)x \tag{2.5}$$

for all $A \in \operatorname{Alg}\mathcal{L}$ and $x \in E$. Hence by (2.5), for all A, B in $\operatorname{Alg}\mathcal{L}$ and x in E,

$$\tau(AB)x = (\tau(A)B + A\tau(B))x + \lambda(AB)x - \lambda(A)Bx - \lambda(B)Ax. \tag{2.6}$$

In the following, we show $\lambda(A) = 0$ for every $A \in \operatorname{Alg}\mathcal{L}$. Putting $A = B = z \otimes g$ and $x = z$ in (2.6) gives $\lambda(z \otimes g) = g(\tau(z \otimes g)z)$, and Lemma 2.1(ii) implies $g(\tau(z \otimes g)z) = 0$. Hence

$$\lambda(z \otimes g) = 0. \tag{2.7}$$

Notice that (2.7) is valid for all z in E and g in $E_-^\perp$ satisfying $g(z) = 1$. Now fix $z \in E$ and $g \in E_-^\perp$ such that $g(z) = 1$. Thus for all $f \in E_-^\perp$, if $f(z) = \mu \neq 0$, then $\lambda(z \otimes f) = \mu\lambda(z \otimes (1/\mu)f) = 0$; if $f(z) = 0$, then $\lambda(z \otimes f) = \lambda(z \otimes (g + f)) - \lambda(z \otimes g) = 0$. Hence $\lambda(z \otimes f) = 0$ for every $f \in E_-^\perp$. Similarly, we have $\lambda(x \otimes g) = 0$ for every $x \in E$. Now for every $A \in \operatorname{Alg}\mathcal{L}$, by (2.6), we have

$$\tau(Az \otimes g)z = \tau(A)z + A\tau(z \otimes g)z - \lambda(A)z, \tag{2.8}$$

$$\tau(z \otimes gA)z = \tau(z \otimes g)Az + g(\tau(A)z)z - \lambda(A)z. \tag{2.9}$$

By Lemma 2.3(i), we have

$$\begin{aligned} \tau(Az \otimes g + z \otimes gA)z = \tau(A)z &+ A\tau(z \otimes g)z \\ &+ \tau(z \otimes g)Az + g(\tau(A)z)z. \end{aligned} \tag{2.10}$$

Combining (2.8)–(2.10) gives $\lambda(A) = 0$ for every $A \in \operatorname{Alg}\mathcal{L}$. Then by (2.6), we obtain

$$\tau(AB)x = (\tau(A)B + A\tau(B))x$$

for all $A, B \in \operatorname{Alg}\mathcal{L}$ and $x \in E$. Since $\bigvee\{L : L \in \mathcal{P}_\mathcal{L}\} = X$, it follows that τ is a derivation. From $\delta(A) = \tau(A) + \delta(I)A$, it is easy to show that δ is a generalized derivation. ■

Applying the ideas in the proof of Theorem 2.4, we can obtain the following result.

THEOREM 2.5. *Let $\mathcal{L}$ be a subspace lattice on X such that $\bigwedge\{L_- : L \in \mathcal{P}_\mathcal{L}\} = (0)$ and δ be a linear mapping from $\operatorname{Alg}\mathcal{L}$ into $B(X)$. Then δ is Jordan derivable at zero if and only if δ is a generalized derivation and $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$. In particular, if $\delta(I) = 0$, then δ is Jordan derivable at zero if and only if δ is a derivation.*

Proof. We only prove the necessity. Let $x \mapsto \hat{x}$ be the canonical mapping from X into X^{**}. Then $(x \otimes f)^* = f \otimes \hat{x}$ for all $x \in X$ and $f \in X^*$. The hypothesis $\bigwedge\{L_- : L \in \mathcal{P}_\mathcal{L}\} = (0)$ implies that $\bigvee\{L_-^\perp : L \in \mathcal{P}_\mathcal{L}\} = X^*$. With a proof similar to that of Theorem 2.4, we have $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$. Let $\tau(A) = \delta(A) - \delta(I)A$ for $A \in \operatorname{Alg}\mathcal{L}$. Then τ is Jordan derivable at zero and $\tau(I) = 0$. In the following, we show τ is a derivation. Let $E \in \mathcal{P}_\mathcal{L}$. We choose

$z \in E$ and $g \in E_-^\perp$ such that $g(z) = 1$. One can easily verify that for all $x \in E$ and $f \in E_-^\perp$, $\tau(x \otimes f)^* \ker(\hat{x}) \subseteq \mathbb{F}f$. Let $\Phi(f, \hat{x}) = \tau(x \otimes f)^*$ for all $x \in E$ and $f \in E_-^\perp$. Then Φ is a bilinear mapping from $E_-^\perp \times \hat{E}$ into $B(X^*)$, where $\hat{E} = \{\hat{x} : x \in E\}$. Hence there exist linear mappings $T : E_-^\perp \to X^*$ and $S : \hat{E} \to X^{**}$ such that

$$\tau(x \otimes f)^* = \Phi(f, \hat{x}) = Tf \otimes \hat{x} + f \otimes S\hat{x}$$

for all $x \in E$ and $f \in E_-^\perp$. Hence for $A \in \operatorname{Alg}\mathcal{L}$ and $f \in E_-^\perp$, we have

$$(\tau(A)^* + A^*T - TA^*)f \otimes \hat{z} = f \otimes (S\widehat{Az} - \widehat{\delta(A)z} - A^{**}S\hat{z}).$$

It follows that $\tau(A)^*f = (TA^* - A^*T)f + \lambda(A)f$, where $\lambda : \operatorname{Alg}\mathcal{L} \to \mathbb{F}$ is a linear mapping. Hence for all $A, B \in \operatorname{Alg}\mathcal{L}$ and $f \in E_-^\perp$,

$$\tau(AB)^*f = (B^*\tau(A)^* + \tau(B)^*A^*)f - \lambda(A)B^*f - \lambda(B)A^*f + \lambda(AB)f.$$

With a proof similar to that of Theorem 2.4, we can show that $\lambda(A) = 0$ for every $A \in \operatorname{Alg}\mathcal{L}$. Since $\bigvee\{L_-^\perp : L \in \mathcal{P}_\mathcal{L}\} = X^*$, it follows that τ is a derivation. Hence δ is a generalized derivation. ■

Next we investigate the bounded linear mappings which are Jordan derivable at zero on CDCSL algebras. Recall that a CSL algebra $\operatorname{Alg}\mathcal{L}$ is irreducible if and only if $(\operatorname{Alg}\mathcal{L})' = \mathbb{C}I$, which is equivalent to the condition that $\mathcal{L} \cap \mathcal{L}^\perp = \{0, I\}$, where $\mathcal{L}^\perp = \{E^\perp : E \in \mathcal{L}\}$.

Lemma 2.6 ([5]). *Let $\operatorname{Alg}\mathcal{L}$ be a CDCSL algebra on H. Then there exists a countable set $\{P_n : n \in \Lambda\}$ of mutually orthogonal projections in $\mathcal{L} \cap \mathcal{L}^\perp$ such that $\bigvee_n P_n = I$ and each $(\operatorname{Alg}\mathcal{L})P_n$ is an irreducible CDCSL algebra on P_nH; moreover, $\operatorname{Alg}\mathcal{L}$ can be written as a direct sum $\operatorname{Alg}\mathcal{L} = \sum_n \bigoplus (\operatorname{Alg}\mathcal{L})P_n$.*

Lemma 2.7 ([16]). *Let $\operatorname{Alg}\mathcal{L}$ be a non-trivially irreducible CDCSL algebra on H. Then there exists a non-trivial projection P in $\mathcal{L}$ such that $P(\operatorname{Alg}\mathcal{L})P^\perp$ is faithful, that is, for $T, S \in \operatorname{Alg}\mathcal{L}$, $TP(\operatorname{Alg}\mathcal{L})P^\perp = \{0\}$ implies $TP = 0$, and $P(\operatorname{Alg}\mathcal{L})P^\perp S = \{0\}$ implies $P^\perp S = 0$.*

Lemma 2.8. *Let $\operatorname{Alg}\mathcal{L}$ be an irreducible CDCSL algebra on H and let $\delta : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{L}$ be a bounded linear mapping with $\delta(I) = 0$. If δ is Jordan derivable at zero, then δ is a derivation.*

Proof. Suppose that $\mathcal{L}$ is trivial. Then $\operatorname{Alg}\mathcal{L} = B(H)$ is a von Neumann algebra. It follows from [1, Theorem 3.2] that δ is a Jordan derivation. Since every von Neumann algebra is a semiprime ring, by [2, Theorem 1], δ is a derivation.

Suppose that $\mathcal{L}$ is non-trivial. Let P be the non-trivial projection in $\mathcal{L}$ provided by Lemma 2.7. Since $P(\operatorname{Alg}\mathcal{L})P^\perp$ is faithful, by [1, Theorem 2.1], δ is a Jordan derivation. Since every Jordan derivation on a CSL algebra is a derivation [17, Theorem 3.2], it follows that δ is a derivation. ■

THEOREM 2.9. *Let* $\operatorname{Alg}\mathcal{L}$ *be a CDCSL algebra on* H *and* δ *be a bounded linear mapping from* $\operatorname{Alg}\mathcal{L}$ *into itself. Then* δ *is Jordan derivable at zero if and only if* δ *is a generalized derivation and* $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$. *In particular, if* $\delta(I) = 0$, *then* δ *is Jordan derivable at zero if and only if* δ *is a derivation.*

Proof. We only prove the necessity. Since every rank one operator in $\operatorname{Alg}\mathcal{L}$ is a linear combination of idempotents in $\operatorname{Alg}\mathcal{L}$ [6, Lemma 2.3] and the rank one subalgebra of $\operatorname{Alg}\mathcal{L}$ is dense in $\operatorname{Alg}\mathcal{L}$ in the weak topology [10, Theorem 3], by Lemma 2.1(i), we have $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$. Let $\tau(A) = \delta(A) - \delta(I)A$ for $A \in \operatorname{Alg}\mathcal{L}$. Then τ is Jordan derivable at zero and $\tau(I) = 0$.

Let $\operatorname{Alg}\mathcal{L} = \sum_n \bigoplus (\operatorname{Alg}\mathcal{L})P_n$ be the irreducible decomposition of $\operatorname{Alg}\mathcal{L}$ as in Lemma 2.6. Let A be in $\operatorname{Alg}\mathcal{L}$ and fix an index n. Since $P_nAP_nP_n^\perp + P_n^\perp P_nAP_n = 0$, we have

$$\begin{aligned} 0 &= \tau(P_nAP_nP_n^\perp + P_n^\perp P_nAP_n) \\ &= \tau(P_nAP_n)P_n^\perp + P_nAP_n\tau(P_n^\perp) + \tau(P_n^\perp)P_nAP_n + P_n^\perp\tau(P_nAP_n), \end{aligned}$$

which yields $P_n^\perp\tau(P_nAP_n)P_n^\perp = 0$. Since $P_n \in \mathcal{L} \cap \mathcal{L}^\perp$, we have $\tau(AP_n) = \tau(AP_n)P_n$. In the same way, we obtain $\tau(AP_n^\perp) = \tau(AP_n^\perp)P_n^\perp$. Since

$$0 = \tau(I) = \tau(P_n + P_n^\perp) = \tau(P_n)P_n + \tau(P_n^\perp)P_n^\perp,$$

it follows that $\tau(P_n) = 0$. Now define a linear mapping $\tau_n : (\operatorname{Alg}\mathcal{L})P_n \to (\operatorname{Alg}\mathcal{L})P_n$ by

$$\tau_n(AP_n) = \tau(AP_n)P_n$$

for every $A \in \operatorname{Alg}\mathcal{L}$. It is easy to show that τ_n is bounded and Jordan derivable at zero. Since $(\operatorname{Alg}\mathcal{L})P_n$ is irreducible and $\tau_n(P_n) = \tau(P_n)P_n = 0$, by Lemma 2.8, τ_n is a derivation. Hence from $\tau(A)P_n = \tau(AP_n)P_n + \tau(AP_n^\perp)P_n = \tau_n(AP_n)$, we see that τ is a derivation. Thus δ is a generalized derivation. ∎

3. Mappings with the WJD property. Our first result in this section says that the set of all mappings Jordan derivable at zero from a $\mathcal{P}$-subspace lattice algebra into $B(X)$ is bigger than the set of all mappings with the WJD property. The following lemma is included in the proof of [4, Lemma 2.6].

LEMMA 3.1. *If* δ *is a linear mapping with the WJD property from a unital algebra* $\mathcal{A}$ *into a unital* $\mathcal{A}$*-bimodule, then for every idempotent* $P \in \mathcal{A}$ *and every* $A \in \mathcal{A}$,

(i) $\delta(I)P = P\delta(I)$ *and* $\delta(P) = \delta(P)P + P\delta(P) - \delta(I)P$;
(ii) $\delta(PA+AP) = \delta(P)A + P\delta(A) + \delta(A)P + A\delta(P) - \delta(I)PA - PA\delta(I)$;
(iii) $\delta(PA+AP) = \delta(P)A + P\delta(A) + \delta(A)P + A\delta(P) - \delta(I)AP - AP\delta(I)$;
(iv) $\delta(PAP) = \delta(P)AP + P\delta(A)P + PA\delta(P) - \delta(I)AP - \frac{1}{2}(PA + AP)\delta(I)$.

Theorem 3.2. *Let $\mathcal{L}$ be a subspace lattice on X such that $\bigvee\{L : L \in \mathcal{P}_{\mathcal{L}}\} = X$ and δ be a linear mapping from* $\operatorname{Alg}\mathcal{L}$ *into $B(X)$. Then δ has the WJD property if and only if δ is a generalized derivation and $\delta(I)A \in (\operatorname{Alg}\mathcal{L})'$ for every $A \in \operatorname{Alg}\mathcal{L}$. In particular, if $\delta(I) = 0$, then δ has the WJD property if and only if δ is a derivation.*

Proof. Since the sufficiency is evident, we will show the necessity. Suppose δ has the WJD property. We claim that $\delta(I)A \in (\operatorname{Alg}\mathcal{L})'$ for every $A \in \operatorname{Alg}\mathcal{L}$. By Lemma 3.1(i) and the proof of Claim 1 in Theorem 2.4, we have $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$. Hence by Lemma 3.1(ii) & (iii), we deduce that $\delta(I)AP = PA\delta(I)$ for every idempotent $P \in \operatorname{Alg}\mathcal{L}$ and every $A \in \operatorname{Alg}\mathcal{L}$. Hence for all $x \in E$, $f \in E_{-}^{\perp}$ and $T \in \operatorname{Alg}\mathcal{L}$, we have $\delta(I)ATx \otimes f = Tx \otimes fA\delta(I) = T\delta(I)Ax \otimes f$. Since $\bigvee\{L : L \in \mathcal{P}_{\mathcal{L}}\} = X$, it follows that $\delta(I)A \in (\operatorname{Alg}\mathcal{L})'$ for every $A \in \operatorname{Alg}\mathcal{L}$. Let $\tau(A) = \delta(A) - \delta(I)A$ for $A \in \operatorname{Alg}\mathcal{L}$. It is easy to show that τ has the WJD property and $\tau(I) = 0$. Similar to the proof of Theorem 2.4, we may show τ is a derivation and so δ is a generalized derivation. ∎

Similarly, we have the following theorem.

Theorem 3.3. *Let $\mathcal{L}$ be a subspace lattice on X such that $\bigwedge\{L_- : L \in \mathcal{P}_{\mathcal{L}}\} = (0)$ and δ be a linear mapping from* $\operatorname{Alg}\mathcal{L}$ *into $B(X)$. Then δ has the WJD property if and only if δ is a generalized derivation and $\delta(I)A \in (\operatorname{Alg}\mathcal{L})'$ for every $A \in \operatorname{Alg}\mathcal{L}$. In particular, if $\delta(I) = 0$, then δ has the WJD property if and only if δ is a derivation.*

Corollary 3.4. *Let $\mathcal{L}$ be as in Example* 1.1. *Then $\delta : \operatorname{Alg}\mathcal{L} \to B(H)$ has the WJD property if and only if δ is a derivation.*

Proof. By Theorem 3.2, we only need to show that if δ has the WJD property, then $\delta(I) = 0$. Let $n \geq 2$. By [7, Lemma 3.2], we have $(P_n)_- \not\geq P_n$. Hence there exist $z_n \in P_n$ and $g_n \in (P_n)_{-}^{\perp}$ such that $g_n(z_n) = 1$. Also, there exists $y_n \in P_n$ such that y_n and z_n are linearly independent. Since δ has the WJD property, we have $\delta(I)A \in \mathcal{A}'$ for every $A \in \mathcal{A}$, which implies that there exists some scalar λ_n such that $\delta(I)x = \lambda_n x$ for every $x \in P_n$ and $\delta(I)(z_n \otimes g_n)(y_n \otimes g_n) = \delta(I)(y_n \otimes g_n)(z_k \otimes g_n)$. That is, $\lambda_n g_n(y_n)z_n = \lambda_n y_n$. The independence of y_n and z_n gives $\lambda_n = 0$ and $\delta(I)x = 0$ for every $x \in P_n$. Since $\bigvee\{P_n \in \mathcal{L} : n = 2, 3, \dots\} = H$, it follows that $\delta(I) = 0$. The proof is complete. ∎

Corollary 3.5. *Let $\mathcal{L}$ be a subspace lattice on H with* $\dim H \geq 2$ *such that $\bigvee\{L : L \in \mathcal{P}_{\mathcal{L}}\} = H$ or $\bigwedge\{L_- : L \in \mathcal{P}_{\mathcal{L}}\} = (0)$. If $\mathcal{L}$ has a non-trivial comparable element, then $\delta : \operatorname{Alg}\mathcal{L} \to B(H)$ has the WJD property if and only if δ is a derivation.*

Proof. According to Theorem 3.2, we only need to show that if δ has the WJD property, then $\delta(I) = 0$. By [11, Proposition 2.9], we have

$(\mathrm{Alg}\,\mathcal{L})' = \mathbb{C}I$. Hence by Theorem 3.2, we have $\delta(I) = \lambda I$ and $\delta(I)A = \mu_A I$ for every $A \in \mathrm{Alg}\,\mathcal{L}$ (where $\lambda, \mu_A \in \mathbb{C}$). We claim that $\lambda = 0$. Suppose that $\lambda \neq 0$; then every operator in $\mathrm{Alg}\,\mathcal{L}$ is a scalar multiple of the identity I. That is, for every $A \in \mathrm{Alg}\,\mathcal{L}$, the range of A is H or 0. However, since $\mathrm{Alg}\,\mathcal{L}$ contains a rank one operator, this is impossible. Hence $\delta(I) = 0$. ∎

Using Corollary 3.5, we can easily show the following result.

Corollary 3.6. *Let $\mathcal{L}$ be a subspace lattice on H with $\dim H \geq 2$ such that $H_- \neq H$ or $(0)_+ \neq (0)$. Then $\delta : \mathrm{Alg}\,\mathcal{L} \to B(H)$ has the WJD property if and only if δ is a derivation.*

Remark. It follows from Theorems 2.4, 2.5, 3.2 and 3.3 that every linear mapping with the WJD property from a $\mathcal{P}$-subspace lattice algebra into $B(X)$ is Jordan derivable at zero. But the converse is not true. For example, let $\mathcal{T}_2(\mathbb{C})$ be the algebra of all 2×2 upper triangular matrices over the complex field $\mathbb{C}$. Define a linear mapping $\delta : \mathcal{T}_2(\mathbb{C}) \to \mathcal{T}_2(\mathbb{C})$ by

$$\delta\left(\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix}\right) = \begin{pmatrix} x_{11} & x_{11} - x_{22} + x_{12} \\ 0 & x_{22} \end{pmatrix}$$

for every $x_{ij} \in \mathbb{C}$ $(1 \leq i \leq j \leq 2)$. It is easy to show that δ is a generalized derivation and $\delta(I) = I \in (\mathcal{T}_2(\mathbb{C}))'$, that is, δ is Jordan derivable at zero. However, it follows from Corollary 3.6 that δ does not have the WJD property since $\delta(I) \neq 0$.

4. Mappings derivable at zero and local generalized derivations. Let $\mathcal{A}$ be a unital algebra, $\mathcal{M}$ be an $\mathcal{A}$-bimodule and $\mathcal{T}$ be an ideal of $\mathcal{A}$. We say that $\mathcal{T}$ is a *left* (respectively, *right*) *separating set* of $\mathcal{M}$ if for every m in $\mathcal{M}$, $m\mathcal{T} = \{0\}$ implies $m = 0$ (respectively, $\mathcal{T}m = \{0\}$ implies $m = 0$). $\mathcal{T}$ is called a *separating set* of $\mathcal{M}$ if $\mathcal{T}$ is a left separating set and a right separating set of $\mathcal{M}$. The following result is obvious.

Lemma 4.1. *Suppose that $\mathcal{L}$ is a subspace lattice on X such that $\bigvee\{L : L \in \mathcal{P}_\mathcal{L}\} = X$ (respectively, $\bigwedge\{L_- : L \in \mathcal{P}_\mathcal{L}\} = (0)$). Then the ideal $\mathcal{T} = \mathrm{span}\{x \otimes f : x \in E,\ f \in E_-^\perp,\ E \in \mathcal{P}_\mathcal{L}\}$ of $\mathrm{Alg}\,\mathcal{L}$ is a left (respectively, right) separating set of $B(X)$.*

By Lemmas 2.2 and 4.1, we have the following result.

Theorem 4.2. *Let $\mathcal{L}$ be a subspace lattice on X such that $\bigvee\{L : L \in \mathcal{P}_\mathcal{L}\} = X$ or $\bigwedge\{L_- : L \in \mathcal{P}_\mathcal{L}\} = (0)$ and δ be a linear mapping from $\mathrm{Alg}\,\mathcal{L}$ into $B(X)$. Then δ is derivable at zero if and only if δ is a generalized derivation and $\delta(I) \in (\mathrm{Alg}\,\mathcal{L})'$. In particular, if $\delta(I) = 0$, then δ is derivable at zero if and only if δ is a derivation.*

Proof. We will show that if $\bigvee\{L : L \in \mathcal{P}_{\mathcal{L}}\} = X$ and δ is derivable at zero, then δ is a generalized derivation and $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$. The proof for $\bigwedge\{L_- : L \in \mathcal{P}_{\mathcal{L}}\} = (0)$ is similar. By the proof of [9, Lemma 3], we can show that $\delta(AP) = \delta(A)P + A\delta(P) - A\delta(I)P$ and $\delta(I)P = P\delta(I)$ for every $A \in \operatorname{Alg}\mathcal{L}$ and every idempotent $P \in \operatorname{Alg}\mathcal{L}$. With a proof similar to the proof of Claim 1 in Theorem 2.4, we have $\delta(I) \in (\operatorname{Alg}\mathcal{L})'$. Now for all $A, B \in \operatorname{Alg}\mathcal{L}$ and $T \in \mathcal{T}$,

$$\delta(ABT) = \delta(AB)T + AB\delta(T) - AB\delta(I)T$$

and

$$\begin{aligned}\delta(ABT) &= \delta(A)BT + A\delta(BT) - A\delta(I)BT \\ &= \delta(A)BT + A\delta(B)T + AB\delta(T) - AB\delta(I)T - A\delta(I)BT.\end{aligned}$$

It follows that $\delta(AB)T = \delta(A)BT + A\delta(B)T - A\delta(I)BT$. Since $\mathcal{T}$ is a left separating set of $B(X)$, we obtain $\delta(AB) = \delta(A)B + A\delta(B)T - A\delta(I)B$ for all $A, B \in \operatorname{Alg}\mathcal{L}$. That is, δ is a generalized derivation. ∎

Recall that a linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is a *left* (respectively, *right*) *multiplier* if $\delta(AB) = \delta(A)B$ (respectively, $\delta(AB) = A\delta(B)$) for all $A, B \in \mathcal{A}$; δ is a *local generalized derivation* if for every $A \in \mathcal{A}$ there is a generalized derivation $\delta_A : \mathcal{A} \to \mathcal{M}$ (depending on A) such that $\delta(A) = \delta_A(A)$. In the following we give some applications of Lemmas 2.2 and 4.1. The proofs are similar to the proof of Theorem 4.2, and we leave them to the reader.

THEOREM 4.3. *Suppose that $\mathcal{L}$ is a subspace lattice on X such that $\bigvee\{L : L \in \mathcal{P}_{\mathcal{L}}\} = X$ (respectively, $\bigwedge\{L_- : L \in \mathcal{P}_{\mathcal{L}}\} = (0)$) and δ is a linear mapping from $\operatorname{Alg}\mathcal{L}$ into $B(X)$. Then δ has the following properties:*

- (i) *if $\delta(AB) = \delta(A)B$ (respectively, $\delta(AB) = A\delta(B)$) for any $A, B \in \operatorname{Alg}\mathcal{L}$ with $AB = 0$, then δ is a left (respectively, right) multiplier;*
- (ii) *if $\delta(AB) = \delta(A)B + \delta(B)A$ (respectively, $\delta(AB) = A\delta(B) + B\delta(A)$) for any $A, B \in \operatorname{Alg}\mathcal{L}$ with $AB = 0$ and $\delta(I) = 0$, then $\delta \equiv 0$;*
- (iii) *if $\delta(A^2) = 2\delta(A)A$ (respectively, $\delta(A^2) = 2A\delta(A)$) for all $A \in \operatorname{Alg}\mathcal{L}$, then $\delta \equiv 0$.*

Combining Theorem 4.3(i) and [12, Proposition 1.1], we have

COROLLARY 4.4. *Suppose that $\mathcal{L}$ is a subspace lattice on X such that $\bigvee\{L : L \in \mathcal{P}_{\mathcal{L}}\} = X$ and $\bigwedge\{L_- : L \in \mathcal{P}_{\mathcal{L}}\} = (0)$ and δ is a linear mapping from $\operatorname{Alg}\mathcal{L}$ into $B(X)$. Then the following are equivalent:*

- (i) *δ is a generalized derivation;*
- (ii) *δ is a local generalized derivation;*
- (iii) *$A\delta(B)C = 0$ whenever $A, B, C \in \operatorname{Alg}\mathcal{L}$ and $AB = BC = 0$.*

Combining Lemmas 2.2, 4.1 and [13, Theorem 2.8], we also have

THEOREM 4.5. *Let $\mathcal{L}$ be a subspace lattice on X such that $\bigvee\{L : L \in \mathcal{P}_{\mathcal{L}}\} = X$ and $\bigwedge\{L_- : L \in \mathcal{P}_{\mathcal{L}}\} = (0)$. If h is a bijective linear mapping from $\operatorname{Alg}\mathcal{L}$ onto a unital algebra satisfying $h(A)h(B)h(C) = 0$ for all $A, B, C \in \operatorname{Alg}\mathcal{L}$ with $AB = BC = 0$ and $h(I) = I$, then h is an isomorphism.*

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