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Metric version of flatness and Hahn–Banach type theorems for normed modules over sequence algebras

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Abstract. We introduce and study the metric or extreme versions of the notions of a flat and an injective normed module. The relevant definitions, in contrast with the standard known ones, take into account the exact value of the norm of the module. The main result gives a full characterization of extremely flat objects within a certain category of normed modules. As a corollary, some Hahn–Banach type theorems for normed modules are obtained.

Introduction: Formulation of the main results and comments. The concepts of a flat module and of an injective module are among the most important in algebra. In particular, they are two of the three pillars of the whole building of homological algebra; the remaining one is the notion of a projective module. The first functional-analytic versions of the three above mentioned notions appeared about 40 years ago (see [6] and the references therein). They were introduced in connection with the rise in functional analysis of interest in such topics as derivations of Banach algebras, their extensions and amenability. The relevant definitions were given in the framework of a certain kind of relative homology, adapted to the context of functional analysis. They were formulated in terms of the norm topology of the modules in question rather than the norm itself.

Quite recently, the birth of new areas of analysis, notably of the so-called quantum functional analysis (= operator space theory), has caused the introduction and study of metric or extreme versions of these notions (cf. [10]–[18]). The specific feature of the new versions is that they take into account the exact value of the norm.

Before proceeding to our main definitions, we recall and fix some terminology and notation.

Let A be a normed algebra, generally speaking, not unital. We denote by A-mod (respectively, mod-A) the category of all left (respectively, right)

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normed A-modules and their bounded morphisms. Throughout this paper, all normed algebras and modules are always supposed to be contractive; this means that the norms of all relevant inner and outer multiplications, as bilinear operators, are ≤ 1.

If $X \in A$-mod, we denote the closure of the linear span of the set $\{a \cdot x : a \in A, x \in X\}$ by X_{es} and call it the *essential part* of X. It is, of course, a submodule. X is called *essential* (or 'non-degenerate') if $X = X_{\mathrm{es}}$. The quotient normed A-module X/X_{es} is denoted by X_{an}; obviously it has zero outer multiplication. The *annihilator* of A in X is the closed left submodule $\{x : a \cdot x = 0 \text{ for all } a \in A\}$ in X, denoted by $\mathrm{Ann}(X)$. The quotient left normed A-module $X/\mathrm{Ann}(X)$ is called the *reduced module* of X and denoted by X^{red}.

As usual, we call a left A-module X *faithful* if $\mathrm{Ann}(X) = 0$. Of course, the reduced module of every module is faithful.

Let A be a normed algebra. We shall use the symbol $\otimes_A$ for the *non-completed* projective tensor product of A-modules and of their bounded morphisms. (See, e.g., [9] or, for the initial 'completed' version, the pioneering paper of Rieffel [12] or the textbooks [6, II.3], [7, VI.3.2].)

If A is commutative, then, as usual, we identify both types of modules and say just 'A-module'. Accordingly, we speak about projective tensor products of normed A-modules and of bounded morphisms of normed A-modules.

Recall that in this situation the tensor product of two modules, say X and Y, is itself a normed contractive A-module with the outer multiplication, well defined by $a \cdot (x \otimes_A y) := (a \cdot x) \otimes_A y$ (or $:= x \otimes_A (a \cdot y)$). Moreover, the tensor product of two bounded morphisms of normed A-modules is itself a bounded morphism of appropriate modules.

The identity operator on a linear space (or a module) Z will be denoted by $\mathbf{1}_Z$, or just $\mathbf{1}$, if there is no danger of misunderstanding.

Finally, let us distinguish a class, so far arbitrary, of right normed A-modules and denote it by $\mathcal{K}$. In the spirit of the old definitions of a flat or strictly flat Banach module ([6, VII.1], [7, VII.1.3]), we give the following

Definition I. A normed left A-module Z is called *extremely flat with respect to* (or *relative to*) *the class* $\mathcal{K}$ if, for every isometric morphism $\mathsf{i} : X \to Y$ of right modules belonging to $\mathcal{K}$, the operator $\mathsf{i} \otimes_A \mathbf{1}_Z : X \otimes_A Z \to Y \otimes_A Z$ is also isometric.

If we deal with just normed spaces, the well known theorem of Grothendieck [5, Thm. 1], adapted to non-complete spaces, gives the following description of extremely flat objects. *A normed space* ('unital normed $\mathbb{C}$-module') *is extremely flat with respect to the class of all normed spaces if and only if it is isometrically isomorphic to a dense subspace of* $L_1(\Omega, \mu)$ *for some measure space* (Ω, μ).

Definition II. A normed right A-module Z is called *extremely injective with respect to the class* $\mathcal{K}$ if, for every isometric morphism $\mathsf{i} : X \to Y$ of right modules belonging to $\mathcal{K}$, and for every bounded morphism $\varphi : X \to Z$ of right A-modules, there exists a bounded morphism $\psi : Y \to Z$ of right modules such that the diagram

$$\begin{tikzcd} X \arrow[r, "\mathsf{i}"] \arrow[d, "\varphi"'] & Y \arrow[dl, "\psi"] \\ Z & \end{tikzcd}$$

is commutative, and $\|\varphi\| = \|\psi\|$. In other words, every bounded morphism of right modules from X into Z can be extended, after the identification of X with a submodule of Y, to a morphism from Y to Z with the same norm.

Thus the assertion that a certain Z is extremely injective with respect to $\mathcal{K}$ can be considered as a 'Hahn–Banach type' theorem for given A and $\mathcal{K}$, with Z playing the role of $\mathbb{C}$.

If again $A := \mathbb{C}$, then the extremely injective objects are described by a theorem, connected with the names of Nachbin, Goodner, Hasumi and Kelley (see [2, p. 123] or [17, Thm. 25.5.1]), which can be easily adapted to the non-complete case. Namely, *a normed space is extremely injective with respect to the class of all normed spaces if and only if it is isometrically isomorphic to the space $C(\Omega)$, where Ω is an extremely disconnected compact space.*

Remark. The word 'extremely' in both definitions is chosen because isometric operators or morphisms are exactly the so-called extreme monomorphisms in some principal categories of spaces or modules in functional analysis (cf., e.g., [1, p. 4], [8, Ch. 0.5]).

The two notions introduced are closely connected. A link is provided by a proper functional-analytic version of the algebraic 'law of adjoint associativity', established by Rieffel [13]. With its help, we shall prove (see the beginning of Section 2) the following easy statement.

Proposition. *Let A, $\mathcal{K}$ and Z be as above. Then Z is extremely flat with respect to $\mathcal{K}$ if and only if its dual normed left A-module Z^* is extremely injective with respect to $\mathcal{K}$.*

The notions defined above were actually introduced in [10], but only for some special algebras and modules. The role of a base algebra was played by $\mathcal{B}(H)$, and $\mathcal{K}$ consisted of the so-called semi-Ruan $\mathcal{B}(H)$-modules. (Speaking informally, these are modules satisfying a proper one-sided version of Ruan axioms for an operator space; cf. [9]). It was shown that certain $\mathcal{B}(H)$-modules are extremely flat with respect to that $\mathcal{K}$, and certain Hahn–Banach type theorems for modules over $\mathcal{B}(H)$ were obtained as corollaries. These theorems, in their turn, led to a conceptually new proof of one of the basic theorems of operator space theory, the Arveson–Wittstock theorem on extensions of completely bounded operators (see, e.g., [4] or [9]).

Afterwards the results of [10] were generalized and considerably strengthened by Wittstock [18], who, in particular, replaced $\mathcal{B}(H)$ by an arbitrary properly infinite C^*-algebra and established that every semi-Ruan module is extremely flat with respect to the above-mentioned class. As an application, Wittstock presented a new transparent proof of the Arveson–Wittstock theorem in its more sophisticated version, that for operator modules.

After the cited papers it seemed natural to look for extremely flat modules over other classes of normed algebras and, accordingly, for related Hahn–Banach type theorems. In the present paper we consider another class of 'popular' algebras, which is opposite, in a sense, to those in [10], [18]. We mean commutative normed algebras, consisting moreover of sequences. These algebras apparently represent the next degree of complication after $\mathbb{C}$. Nevertheless we hope to show that even in this case, after the proper choice of $\mathcal{K}$, there is something to say.

Denote by $\mathbf{p}^n$ the sequence $(0,\dots,0,1,0,\dots)$ with 1 as the nth entry, and by c_{00} the linear space of finite sequences, i.e., $\operatorname{span}\{\mathbf{p}^n : n=1,2,\dots\}$.

Definition III. Let A be a normed algebra consisting of some complex-valued sequences and equipped with the coordinatewise operations. We say that A is a *sequence algebra* if it contains c_{00} as a dense subalgebra, and $\|\mathbf{p}^n\|=1$ for all n.

We see that the class of sequence algebras includes c_0, all l_p, $1\le p<\infty$, (but not l_∞), the Fourier algebras of discrete countable groups (after rearranging the relevant domains as sequences), and many other algebras.

The main result of the paper gives, within a certain reasonable class of normed modules over a sequence algebra, a full characterization of extremely flat modules with respect to that class. After some preliminary notes, we proceed to define this class.

Let A be a sequence algebra, and X a normed A-module. Often, when there is no danger of confusion, for $x\in X$ we shall write x_n instead of $\mathbf{p}^n\cdot x$ and call it the *nth coordinate* of x. Of course, $\mathbf{p}^n\cdot x_n=x_n$. Further, we set $X_n:=\{\mathbf{p}^n\cdot x : x\in X\}$ for every $n\in\mathbb{N}$. We see that X_n is a submodule of X. It will be called the *nth coordinate submodule* of X.

Definition IV. An A-module X is called *homogeneous* if, for every $x,y\in X$, the inequalities $\|x_n\|\le\|y_n\|$ for all $n\in\mathbb{N}$ imply that $\|x\|\le\|y\|$.

It immediately follows that for any elements x, y in a homogeneous module, the equalities $\|x_n\| = \|y_n\|$, $n \in \mathbb{N}$ imply that $\|x\| = \|y\|$. Thus in a homogeneous module the norm of an element is completely determined by the norms of its coordinates.

For many typical sequence algebras the class of homogeneous modules is fairly wide. In particular, it is easy to show that *all essential normed modules over* c_0*, consisting of complex-valued sequences, are homogeneous.* Moreover, l_p-sums, $1 \le p \le \infty$, of arbitrary families of normed spaces are obviously homogeneous l_q-modules for all $1 \le q < \infty$. The module l_2 over the Fourier algebra of a countable discrete Abelian group is also, of course, homogeneous. (In all examples we mean the coordinatewise outer multiplication.)

On the other hand, it is obvious that a homogeneous normed A-module X is necessarily faithful.

In this paper, we denote by $\mathcal{H}$ the class of all homogeneous normed A-modules, and by $\mathcal{H}_{\mathrm{es}}$ its subclass consisting of the essential modules.

Theorem I. *Let A be a sequence algebra, and Z be an essential* (*respectively, arbitrary*) *homogeneous normed A-module. Then Z is extremely flat relative to $\mathcal{H}$* (*respectively, $\mathcal{H}_{\mathrm{es}}$*) *if and only if, for every n, its nth coordinate submodule is isometrically isomorphic to a dense subspace of $L_1(\Omega_n, \mu_n)$ for some measure space (Ω_n, μ_n).*

Note that 'only if' part of this theorem is a rather easy corollary of the theorem of Grothendieck, cited above. Our proof of the 'if' part is more complicated, and it does not use the Grothendieck theorem. Indeed, the emphasis is shifted: the main thing now is to show that the answer depends not on the norm on the whole module but only on the norms of its coordinate subspaces.

In fact, we shall prove this theorem in a slightly stronger form; see Proposition 3.2 and Theorem 3.6 below.

Of course, if we wish to deal with Banach modules only, then in the criterion of extreme flatness which is an immediate corollary of Theorem I, we must replace the words 'dense subspace of $L_1(\Omega_n, \mu_n)$' by (just) '$L_1(\Omega_n, \mu_n)$'.

Remark. This theorem enables one to indicate various subclasses of $\mathcal{H}$ where the operation of projective tensor product has the so-called injective property: $\varphi \otimes_A \psi$ is isometric provided φ and ψ are. It follows from the factorization $\varphi \otimes_A \psi = (\varphi \otimes_A \mathbf{1})(\mathbf{1} \otimes_A \psi) = (\mathbf{1} \otimes_A \psi)(\varphi \otimes_A \mathbf{1})$. In particular, this is true for the subclass of $\mathcal{H}_{\mathrm{es}}$ consisting of the modules with all coordinate submodules isometrically isomorphic to dense subspaces of $L_1(\cdot)$ spaces.

As an easy corollary of Theorem I, we obtain

THEOREM II (see the end of Section 3). *Let A be a sequence algebra, and Z be an essential* (*respectively, arbitrary*) *homogeneous normed A-module. Then the dual module Z^* is extremely injective relative to $\mathcal{H}$* (*respectively, to $\mathcal{H}_{\text{es}}$*) *if and only if for every n the nth coordinate submodule Z_n is isometrically isomorphic to a dense subspace of $L_1(\Omega_n, \mu_n)$ for some measure space (Ω_n, μ_n).*

Thus, speaking informally, in both theorems the answer depends not on the norm on the whole module but only on the norms of its coordinate subspaces. In particular, all A-modules l_p, $1 \leq p < \infty$, (and all sequence modules over $A := c_0$) are extremely flat with respect to $\mathcal{H}$, whereas the same l_p and also l_∞ are extremely injective with respect to $\mathcal{H}$.

REMARK. Suppose that Z satisfies the conditions of the above theorem, and, for some n, the measure space (Ω_n, μ_n) is localizable in the sense of I. Segal [16, Def. 2.6]. Then, as shown in the cited paper (part of Theorem 5.1 there), the dual space $L_1(\Omega_n, \mu_n)^*$ is isometrically isomorphic to $L_\infty(\Omega_n, \mu_n)$. Therefore the same is true for $(Z_n)^*$ and hence, by Proposition 1.4, for the nth coordinate submodule of Z^*.

In both theorems we have assumed that some of the modules involved are essential. Such a condition cannot be omitted: a non-essential homogeneous normed module (being always extremely flat with respect to $\mathcal{H}_{\text{es}}$) need not be extremely flat with respect to $\mathcal{H}$. As a matter of fact, *the A-module l_∞* (apparently the first faithful non-essential module that comes to mind) *is not extremely flat with respect to the class of all homogeneous modules.* This is Theorem 4.3.

Let us make some comments on the proof of the main result. At the very beginning we observe that, under some conditions, tensor products of modules over a sequence algebra, and of their morphisms, can be described in a rather transparent and 'workable' form (Proposition 1.7). In particular, this is helpful in the principal preparatory step, Lemma 3.3, of somewhat technical character. At the end of our argument, we use the following fact: under a mild assumption on X or Z the (mere) injectivity of $\varphi : X \to Y$ implies the same property of $\varphi \otimes_A \mathbf{1}_Z$.

Thus we have come across another typical question of the theory of normed algebras, this time about the preservation, under tensor multiplication of modules, of the injectivity of given morphisms. Of course, it sounds similar to its well known pure algebraic prototype, which leads to the fundamental notion of (algebraic) flatness. But here we deal with bounded morphisms and a kind of functional-analytic tensor product. This profoundly affects the situation.

Even in the case when A is a 'very good' sequence algebra (c_0, say), and X, Y, Z are normed A-modules consisting of sequences, it can well be that a bounded morphism $\varphi : X \to Y$ is injective whereas $\varphi \otimes_A \mathbf{1} : X \otimes_A Z \to Y \otimes_A Z$ is not. However, if a given sequence algebra is indeed good and we are given arbitrary normed A-modules X, Y, Z and a *topologically injective* (in particular, isometric) morphism $\varphi : X \to Y$ then *the operator* $\varphi \otimes_A \mathbf{1} : X \otimes_A Z \to Y \otimes_A Z$ *is also injective.* (Note that at the same time it is not necessarily topologically injective.) This is Theorem 2.4.

Remark. We want to emphasize that we work in this paper, in a similar way to [10], [18], with the non-completed version of the projective tensor product. If we replace the latter by its completed version, Theorem 2.4 fails to be true. One can easily construct suitable counter-examples, taking some spaces without the approximation property.

1. Some preparations. We begin with a proposition of somewhat general character. In particular, it will enable us to derive Theorem II from Theorem I (cf. Introduction). This proposition actually appeared in [10, Prop. 9], but in a certain special case and in a slightly disguised form.

In what follows, A is a normed algebra, so far arbitrary, and $\mathbf{h}_A(\cdot,\cdot)$ denotes the space of all bounded morphisms between right normed modules. Such spaces are equipped with the operator norm.

Proposition 1.1. *Let X and Y be right normed A-modules, Z a left normed A-module, $\mathsf{i} : X \to Y$ an isometric morphism, and Z^* the right Banach A-module dual to Z. Then the following statements are equivalent:*

(i) *the operator $\mathsf{i} \otimes_A \mathbf{1}_Z : X \otimes_A Z \to Y \otimes_A Z$ is an isometry,*
(ii) *for every bounded morphism $\varphi : X \to Z^*$ of right A-modules, there exists a bounded morphism $\psi : Y \to Z^*$ of right modules such that the diagram*

$$\begin{tikzcd} X \arrow[r, "\mathsf{i}"] \arrow[d, "\varphi"] & Y \arrow[dl, "\psi"] \\ Z^* & \end{tikzcd}$$

is commutative and $\|\varphi\| = \|\psi\|$.

Proof. According to the functional-analytic version of the law of adjoint associativity (cf. Rieffel [13], or [9, Ch. 8.0]) the normed space $\mathbf{h}_A(X, Z^*)$ coincides with the space $(X \otimes_A Z)^*$ up to the isometric isomorphism taking a morphism $\varphi : X \to Z^*$ to the functional $f : X \otimes_{\mathcal{B}} Z \to \mathbb{C}$ well defined by $f(x \otimes_A z) = [\varphi(x)](z)$. Similarly, $\mathbf{h}_A(Y, Z^*)$ is identified with $(Y \otimes_A Z)^*$.

Moreover, one can easily check that we have a commutative diagram

$$\begin{tikzcd}
\mathbf{h}_A(Y, Z^*) \arrow[r, "\mathsf{i}_*"] \arrow[d] & \mathbf{h}_A(X, Z^*) \arrow[d] \\
(X \otimes_A Z)^* \arrow[r, "\mathsf{i}^\bullet"] & (Y \otimes_A Z)^*
\end{tikzcd}$$

Here the vertical arrows are isometric isomorphisms of normed spaces, acting as indicated above, i_* acts as $\beta \mapsto \beta\mathsf{i}$, and $\mathsf{i}^\bullet$ is the operator adjoint to $\mathsf{i} \otimes_A \mathbf{1}_Z : X \otimes_A Z \to Y \otimes_A Z$.

It is obvious that assertion (ii) is equivalent to the following statement: the operator i_* maps the closed unit ball in the domain space onto the closed unit ball in the range space. Because of the diagram above, this assertion is in turn equivalent to the statement that $\mathsf{i}^\bullet$ has the same property. But, as an obvious corollary (in fact, an equivalent formulation) of the Hahn–Banach theorem, an adjoint operator has the indicated property if and only if the original operator is isometric. The rest is clear. ∎

An immediate corollary is the Proposition that was formulated at the beginning of the Introduction.

As a byproduct, we have the following observation.

PROPOSITION 1.2. *Suppose that* X, Y, Z *and* i *are as before, and* Z_0 *is a dense submodule of* Z*. Then* $\mathsf{i} \otimes_A \mathbf{1}_Z$ *is an isometry if and only if the same is true of* $\mathsf{i} \otimes_A \mathbf{1}_{Z_0}$.

Proof. Indeed, the dual modules of Z and Z_0 coincide, and therefore (ii) above is valid if and only if it is valid after replacing Z by Z_0. ∎

Later we shall come across quite a few diagrams like the one above. To write them all down would take too much space. Therefore the following terminology is convenient. We shall say that the morphisms $\varphi : X_1 \to X_2$ and $\psi : Y_1 \to Y_2$, acting between normed A-modules, are *isometrically equivalent* if there exist isometric isomorphisms of A-modules, I and J, such that the diagram

$$\begin{tikzcd}
X_1 \arrow[r, "\varphi"] \arrow[d, "I"'] & X_2 \arrow[d, "J"] \\
Y_1 \arrow[r, "\psi"] & Y_2
\end{tikzcd} \tag{1.1}$$

is commutative (cf. [8]). In particular, we shall speak about the isometric equivalence of two operators ($\mathbb{C}$-modules). As to the isomorphisms I and J, we shall say that they *implement* the above equivalence.

From now on we concentrate on the case where A is a sequence algebra. We need some further notation and several elementary facts concerning A-modules and their tensor products.

Let X be a normed A-module, and X_n, $n = 1, 2, \ldots$, its coordinate submodules (see Introduction). We denote by $\alpha_n^X : X_n \to X$ the natural embeddings, and by $\beta_n^X : X \to X_n$ the projections $x \mapsto x_n$. Clearly, these are morphisms of A-modules that are isometries and, respectively, coisometries (= quotient maps).

Consider the pure algebraic A-module $\bigtimes_{n=1}^{\infty} X_n$, consisting of all sequences $(x_1, \ldots, x_n, \ldots)$ with $x_n \in X_n$ and endowed with the coordinatewise operations. Introduce the map

$$\sigma^X : X \to \bigtimes_{n=1}^{\infty} X_n : x \mapsto (x_1, \ldots, x_n, \ldots);$$

it is, of course, an A-module morphism. Since c_{00} is dense in A, σ^X is injective if and only if X is faithful. Moreover, $\mathrm{Ker}(\sigma^X)$ coincides with $\mathrm{Ann}(X)$, and hence it is closed. Therefore we can (and will) identify the submodule $\mathrm{Im}(\sigma^X)$ in $\bigtimes_{n=1}^{\infty} X_n$ with X^{red} (cf. Introduction) and endow it with the appropriate quotient norm.

If $x \in X_n$, then the sequence $\sigma^X(x) = (0, \ldots, 0, x, 0, \ldots)$ belongs to $(X^{\mathrm{red}})_n$. Taking into account that $\|y\| \geq \|x\|$ for all y with $\sigma^X(y) = \sigma^X(x)$, we immediately obtain

Proposition 1.3. *The birestriction $\sigma_n^X : X_n \to (X^{\mathrm{red}})_n$ of σ^X is an isometric isomorphism.*

Proof. Clear. ∎

Proposition 1.4. *For every n, the A-modules $(X^*)_n$ and $(X_n)^*$ are isometrically isomorphic.*

Proof. The morphisms $(\alpha_n^X)^* \alpha_n^{X^*} : (X^*)_n \to (X_n)^*$ and $\beta_n^{X^*} (\beta_n^X)^* : (X_n)^* \to (X^*)_n$ are contractive and inverse to each other. ∎

Now let Z be another A-module. Our object of interest is the A-module $X \otimes_A Z$.

Throughout the paper, $\otimes_{\mathrm{p}}$ stands for the non-completed projective tensor product of normed spaces (= $\mathbb{C}$-modules). The projective tensor norm will be denoted by $\| \cdot \|_{\mathrm{p}}$.

Proposition 1.5. *There exists an isometric isomorphism*

$$\rho_n^{X,Z} : X_n \otimes_{\mathrm{p}} Z_n \to (X \otimes_A Z)_n,$$

well defined by $x \otimes z \mapsto x \otimes_A z$.

Proof. Consider the contractive linear operators $\rho : X_n \otimes_{\mathrm{p}} Z_n \to X \otimes_A Z$ and $\pi : X \otimes_A Z \to X_n \otimes_{\mathrm{p}} Z_n$, associated with the contractive bilinear operator $X_n \times Z_n \to X \otimes_A Z : (x, z) \mapsto x \otimes_A z$ and the contractive balanced bilinear operator $X \times Z \to X_n \otimes_{\mathrm{p}} Z_n : (x, z) \mapsto \mathbf{p}^n \cdot x \otimes \mathbf{p}^n \cdot z$, respectively. Since $\pi\rho = \mathbf{1}$, it follows that ρ is an isometry (whereas π is a coisometry). Obviously, the image of ρ is exactly $(X \otimes_A Z)_n$. It remains to denote by $\rho_n^{X,Z}$ the relevant corestriction. ∎

Now we turn to the normed module $(X \otimes_A Z)^{\mathrm{red}}$ and to the coisometric morphism $\sigma^{X,Z} : X \otimes_A Z \to (X \otimes_A Z)^{\mathrm{red}}$, which is, by definition, the appropriate corestriction of $\sigma^{X \otimes_A Z}$ (cf. above). We want to describe them, up to isometric isomorphism and, respectively, isometric equivalence, in terms convenient for their study.

Consider the pure algebraic A-module $\mathsf{X}_{n=1}^{\infty}(X_n \otimes Z_n)$ with the coordinate-wise operations. For $x \in X$ and $z \in Z$ we shall denote by $x \odot z$ the sequence $(x_1 \otimes z_1, \dots, x_n \otimes z_n, \dots)$, belonging to this module. Denote by $X \odot Z$ the submodule of $\mathsf{X}_{n=1}^{\infty}(X_n \otimes Z_n)$ defined as the linear span of all such sequences.

The bilinear operator $X \times Z \to X \odot Z : (x, z) \mapsto x \odot z$ is clearly balanced. Therefore it gives rise to a linear operator and, obviously, a surjective A-module morphism $\odot_{X,Z} : X \otimes_A Z \to X \odot Z$, well defined by $x \otimes_A z \mapsto x \odot z$.

For $v \in X \odot Z$ we set

$$\|v\|_{\odot} := \inf \Big\{ \sum_{k=1}^{m} \|x^k\| \, \|z^k\| \Big\}, \tag{1.2}$$

where the infimum is taken over all representations of v in the form $\sum_{k=1}^{m} x^k \odot z^k$ with $x^k \in X$, $z^k \in Z$.

Proposition 1.6. *The function $v \mapsto \|v\|_{\odot}$ is a norm on $X \odot Z$. Moreover, with respect to this norm the module $X \odot Z$ is isometrically isomorphic to $(X \otimes_A Z)^{\mathrm{red}}$, and $\odot_{X,Z}$ is isometrically equivalent to $\sigma^{X,Z}$. In more detail there is a commutative diagram*

$$\begin{tikzcd} X \otimes_A Z \arrow[r, "\sigma^{X,Z}"] \arrow[d, "\mathbf{1}"'] & (X \otimes_A Z)^{\mathrm{red}} \arrow[d, "\iota^{X,Z}"] \\ X \otimes_A Z \arrow[r, "\odot_{X,Z}"] & X \odot Z \end{tikzcd} \tag{1.3}$$

where $\iota^{X,Z}$ is an isometric isomorphism of A-modules.

Proof. Since $\odot_{X,Z}$ is surjective, $X \odot Z$ is a seminormed module with respect to the seminorm $\|v\|' := \inf\{\|u\| : \odot_{X,Z}(u) = v\}$.

First, we show that $\|\cdot\|_{\odot} = \|\cdot\|'$. From (1.2) we easily see that $\|v\|' \le \|v\|_{\odot}$. On the other hand, for every $\varepsilon > 0$ we can take $u \in X \otimes_A Z$ with $\odot_{X,Z}(u) = v$ and $\|v\|' \ge \|u\| - \varepsilon$, and then represent u as $\sum_{k=1}^{m} x^k \otimes_A z^k$ so that $\|u\| > \sum_{k=1}^{m} \|x^k\| \, \|z^k\| - \varepsilon$. Then we evidently have $\|v\|' \ge \|v\|_{\odot} - 2\varepsilon$, and the reverse inequality follows.

Now take $u \in X \otimes_A Z$. Let $(\dots, u^n, \dots)$ be the sequence $\odot_{X,Z}(u)$. Obviously, $\rho_n^{X,Z}$ takes u^n to $u_n := \mathbf{p}^n \cdot u$. It easily follows that $\mathrm{Ker}(\sigma^{X,Z}) = \mathrm{Ker}(\odot_{X,Z})$. Since both $\sigma^{X,Z}$ and $\odot_{X,Z}$ are coisometries, there exists a unique isometric isomorphism $\iota_{X,Z}$ making the diagram (1.3) commutative. The rest is clear. ∎

Thus, by virtue of Propositions 1.3, 1.5 and 1.6, we have, for each n, a chain of isometric isomorphisms

$$X_n \otimes_{\mathrm{p}} Z_n \xrightarrow{\rho_n^{X,Z}} (X \otimes_A Z)_n \xrightarrow{\sigma_n^{X,Z}} (X \otimes_A Z)_n^{\mathrm{red}} \xrightarrow{\iota_n^{X,Z}} (X \odot Z)_n, \tag{1.4}$$

where the last map is the appropriate birestriction of $\iota^{X,Z}$. Denote by $\varkappa_n^{X,Z} : X_n \otimes_{\mathrm{p}} Z_n \to (X \odot Z)_n$ their composition. It is, of course, an isometric isomorphism of A-modules, well defined by taking $x \otimes z$ to $x \odot z$.

Proposition 1.7. *Suppose that at least one of the modules X and Z has the following property: there is a sequence $\mathbf{q}^m \in A$ consisting of finite sequences and such that for each element, say x, of our module we have $x = \lim_{m\to\infty} \mathbf{q}^m \cdot x$. Then $\odot_{X,Z}$ is an isometric isomorphism of A-modules.*

Before the proof, let us notice that for many sequence algebras every essential module over such an algebra has the indicated property. Of course, this is the case where $A = c_0$ or A is the Fourier algebra of a discrete countable Abelian group. However, more important for our aims is another sufficient condition, expressed in Proposition 3.1 below.

Proof. Since 'horizontal' morphisms in (1.3) are coisometric, it is sufficient to check that $\sigma^{X,Z}$ is injective. Suppose that for $u \in X \otimes_A Z$ we have $\odot_{X,Z}(u) = 0$ and hence $\sigma^{X,Z}(\mathbf{p}^n(u)) = 0$ for all n. Since $\mathbf{p}^n(u) \in (X \otimes_A Z)_n$, we see, by (1.4), that $\mathbf{p}^n(u) = 0$ for all n and hence $\mathbf{q}^m(u) = 0$ for all m. The rest is clear. ∎

The indicated assumption cannot be omitted, even when both modules are faithful:

Example 1.8. Consider $X := Z := l_\infty$ with the coordinatewise operations and uniform norm. Take the sequences $x := (1, 0, 1, 0, 1, 0, \dots) \in X$ and $z := (0, 1, 0, 1, 0, 1, \dots) \in Z$. Of course, $\odot_{X,Z}(x \otimes_A z) = 0$.

Now take two functionals $f, g : l_\infty \to \mathbb{C}$ of norm 1 such that $f(\xi) = g(\eta) = 0$ for $\xi, \eta \in c_0$ and $f(x) = g(z) = 1$; these are easily provided by the Hahn–Banach theorem. Then the bilinear functional $f \times g : X \times Z \to \mathbb{C} : (\xi, \eta) \mapsto f(\xi)g(\eta)$ is obviously balanced and contractive. Therefore it gives rise to the contractive functional $f \otimes_A g : X \otimes_A Z \to \mathbb{C}$, well defined by $\xi \otimes_A \eta \mapsto f(\xi)g(\eta)$. Since $(f \otimes_A g)(x \otimes_A z) = 1$, we have $x \otimes_A z \neq 0$. Thus $\odot_{X,Z}$ is not injective.

Now suppose that we have three A-modules X, Y and Z, so far arbitrary, and a bounded A-module morphism $\varphi : X \to Y$. The latter in an obvious way generates the sequence of its birestrictions $\varphi_n : X_n \to Y_n$.

Consider the bounded morphism $\varphi \otimes_A \mathbf{1} : X \otimes_A Z \to Y \otimes_A Z$; we recall that it is well defined by $x \otimes_A z \mapsto \varphi(x) \otimes_A z$. Clearly, $\varphi \otimes_A \mathbf{1}$ maps $\mathrm{Ann}(X \otimes_A Z)$ into $\mathrm{Ann}(Y \otimes_A Z)$. Therefore $\varphi \otimes_A \mathbf{1}$ gives rise to the bounded morphism $(\varphi \otimes_A \mathbf{1})^{\mathrm{red}} : (X \otimes_A Z)^{\mathrm{red}} \to (Y \otimes_A Z)^{\mathrm{red}}$, well defined by $(\varphi \otimes_A \mathbf{1})^{\mathrm{red}}(\sigma^{X,Z}(x \otimes_A z)) = \sigma^{Y,Z}(\varphi(x) \otimes_A z)$ for $x \in X, z \in Z$.

Combining this with Proposition 1.6, we obtain the commutative diagram

(1.5)
$$\begin{tikzcd} X \otimes_A Z \arrow[r, "\odot_{X,Z}"] \arrow[d, "\varphi\otimes_A\mathbf{1}"'] & X \odot Z \arrow[d, "\varphi\odot\mathbf{1}"] \\ Y \otimes_A Z \arrow[r, "\odot_{X,Z}"] & Y \odot Z \end{tikzcd}$$

where $\varphi \odot \mathbf{1}$ is well defined by $x \odot z \mapsto \varphi(x) \odot z$, that is, takes the sequence $(\dots, u_n, \dots)$ with $u_n \in X_n \otimes_{\mathrm{p}} Z_n$ to $(\dots, (\varphi_n \otimes \mathbf{1})u_n, \dots)$.

Note that we obviously have

(1.6)
$$\|\varphi \odot \mathbf{1}\| \le \|\varphi \otimes_A \mathbf{1}\| \le \|\varphi\|.$$

Being morphisms of A-modules, $\varphi \otimes_A \mathbf{1}$ and $\varphi \odot \mathbf{1}$ have well defined respective birestrictions $(\varphi \otimes_A \mathbf{1})_n$ and $(\varphi \odot \mathbf{1})_n$, for every n. Using the identifications in (1.4), for the pairs (X, Z) and (Y, Z), we easily obtain

PROPOSITION 1.9. *Both $(\varphi \otimes_A \mathbf{1})_n$ and $(\varphi \odot \mathbf{1})_n$ are isometrically equivalent to the operator $\varphi_n \otimes \mathbf{1} : X_n \otimes_{\mathrm{p}} Z_n \to Y_n \otimes_{\mathrm{p}} Z_n$.*

Proof. Clear. ∎

PROPOSITION 1.10. *Suppose that either both X and Y, or Z, satisfy the condition in Proposition 1.7. Then the morphisms $\varphi \otimes_A \mathbf{1}$ and $\varphi \odot \mathbf{1}$ are isometrically equivalent.* ∎

Proof. Clear. ∎

2. Tensoring injective morphisms. Let A be a sequence algebra, X, Y, Z normed A-modules, and $\varphi : X \to Y$ a bounded morphism. Suppose that φ is injective. When can we be sure that $\varphi \otimes_A \mathbf{1}$ is also injective?

If we ask the same about $\varphi \odot \mathbf{1}$, the situation is clear:

PROPOSITION 2.1. *Let φ be injective. Then so is $\varphi \odot \mathbf{1}$.*

Proof. Together with φ, its birestrictions φ_n are also injective. Therefore, for purely algebraic reasons, the same is true for the operators $\varphi_n \otimes \mathbf{1} : X_n \otimes_{\mathrm{p}} Z_n \to Y_n \otimes_{\mathrm{p}} Z_n$. It remains to recall the way $\varphi \odot \mathbf{1}$ acts. ∎

PROPOSITION 2.2. *Suppose that X or Z satisfies the condition of Proposition 1.7. Then, if φ is injective, so is $\varphi \otimes_A \mathbf{1}$.*

Proof. By Propositions 1.7 and 2.1, both $\odot_{X,Z}$ and $\varphi \odot \mathbf{1}$ in the commutative diagram (1.5) are injective. The rest is clear. ∎

There is another kind of condition, this time in terms of φ itself, that gives the same result. Suppose that φ is admissible, i.e. it has a bounded left inverse operator, and A is an amenable Banach algebra. In this case every A-module, in particular, our Z, is flat, that is, $\varphi \otimes_A \mathbf{1}$ is not only injective, but topologically injective; see, e.g., [6, Ch. VII]. (Actually, the cited book deals with the 'completed' theory, but it is easy to observe that the indicated property of $\varphi \otimes_A \mathbf{1}$ is valid in the 'non-completed' case as well.)

However, if we have just an injective morphism between two normed A-modules, even faithful, the situation is different:

EXAMPLE 2.3. Take $X := Z := l_\infty$ and set $Y := c_0$. Consider a sequence $(\zeta_1, \zeta_2, \ldots) \in c_0$ with non-zero terms and the map $\varphi : X \to Y : (\xi_1, \xi_2, \ldots.) \mapsto (\zeta_1\xi_1, \zeta_2\xi_2, \ldots.)$. Of course, φ is injective. Further, by Proposition 1.7 applied to Y, the lower horizontal arrow in (1.5) is an injective map. However, the upper arrow is not, as we know from Example 1.8. Therefore $\varphi \otimes_A \mathbf{1}$ cannot be injective.

Of course, such a φ is far from being admissible. But what can happen in the 'intermediate' case, when φ is not necessarily admissible, but at least topologically injective?

It is easy to show that $\varphi \otimes_A \mathbf{1}$ need not be *topologically* injective. Moreover, as a related phenomenon, in the 'completed' theory such a morphism need not even be injective. But the present paper deals with the 'non-completed' theory, and it turns out that for some class of sequence algebras we still have a positive result.

We recall the following elementary fact. If a normed algebra A has a bounded approximate identity e_ν, $\nu \in \Lambda$, and X is a normed, say left, A-module, then for every $x \in X_{\mathrm{es}}$ we have $x = \lim_\nu e_\nu \cdot x$.

THEOREM 2.4. *Suppose that a sequence algebra A has a bounded approximate identity e_ν, $\nu \in \Lambda$, X, Y, Z are normed A-modules, and $\varphi : X \to Y$ is a topologically injective morphism. Then $\varphi \otimes_A \mathbf{1}$ is also injective.*

Proof. Take $u \in X \otimes_A Z$, $u \neq 0$; we want to show that $(\varphi \otimes_A \mathbf{1})(u)$ is not 0. If $\odot_{X,Z}(u) \neq 0$, that is, $u \notin \mathrm{Ann}(X \otimes_A Z)$, then the desired fact follows from Proposition 2.1, combined with the commutative diagram (1.5). Thus we are allowed to assume that u lives in $\mathrm{Ann}(X \otimes_A Z)$.

Consider the quotient maps $\tau_X : X \to X_{\mathrm{an}}$ and $\tau_Z : Z \to Z_{\mathrm{an}}$ (cf. Introduction) and set, for brevity, $\tau := \tau_X \otimes \tau_Z : X \otimes Z \to X_{\mathrm{an}} \otimes Z_{\mathrm{an}}$. Recall that $X \otimes_A Z$, by its definition, is a quotient space of $X \otimes Z$ (actually, a quotient normed space of $X \otimes_{\mathrm{p}} Z$) and denote by γ the relevant quotient map. It is easy to see that $\mathrm{Ker}(\tau)$ is the algebraic sum of $X_{\mathrm{es}} \otimes Z$ and $X \otimes Z_{\mathrm{es}}$.

This obviously implies that

$$\gamma(\mathrm{Ker}(\tau)) \subseteq (X \otimes_A Z)_{\mathrm{es}}. \tag{2.1}$$

Fix an arbitrary $v \in X \otimes Z$ with $\gamma(v) = u$ and set $w := \tau(v)$. We claim that $w \neq 0$. Indeed, in the opposite case, by (2.1), we have $u \in (X \otimes_A Z)_{\mathrm{es}}$ and hence (see above), $u = \lim_\nu e_\nu \cdot u$. This, together with $u \in \mathrm{Ann}(X \otimes_A Z)$, gives $u = 0$, a contradiction.

Thus w, being a non-zero vector in $X_{\mathrm{an}} \otimes Z_{\mathrm{an}}$, can be represented as $w = \sum_{k=1}^n \widetilde{x}_k \otimes \widetilde{z}_k$ with $\widetilde{x}_k \in X_{\mathrm{an}}$ and $\widetilde{z}_k \in Z_{\mathrm{an}}$, where $\widetilde{x}_1 \neq 0$, and $\widetilde{z}_k$ are linearly independent.

Take an arbitrary $x_k \in X$ such that $\tau_X(x_k) = \widetilde{x}_k$ for $k = 1, \dots, n$. Our next claim is that $\varphi(x_1) \notin Y_{\mathrm{es}}$. Suppose the contrary. Then

$$\varphi(x_1) = \lim_\nu e_\nu \cdot \varphi(x_1) = \lim_\nu \varphi(e_\nu \cdot x_1).$$

But since φ is topologically injective, this implies that $x_1 = \lim_\nu e_\nu \cdot x_1$, and hence $x_1 \in X_{\mathrm{es}}$. Thus $\widetilde{x}_1 = 0$, a contradiction.

This claim implies, by a standard corollary of the Hahn–Banach theorem, that there exists a bounded functional $f : Y \to \mathbb{C}$ such that $f = 0$ on Y_{es}, and $f(\varphi(x_1)) = 1$. The same corollary provides a bounded functional $\widetilde{g} : Z_{\mathrm{an}} \to \mathbb{C}$ such that $\widetilde{g}(\widetilde{z}_1) = 1$ and $\widetilde{g}(\widetilde{z}_k) = 0$ for $k = 2, \dots, n$. Take an arbitrary $z_k \in Z$ with $\tau_Z(z_k) = \widetilde{z}_k$ for $k = 1, \dots, n$ and consider the bounded functional $g := \widetilde{g}\tau_Z : Z \to \mathbb{C}$. Then, of course, $g(z_1) = 1$ and $g(z_k) = 0$ for $k = 2, \dots, n$.

Now introduce the bounded bilinear functional $f \times g : Y \times Z \to \mathbb{C} : (y, z) \mapsto f(y)g(z)$. Since $f = 0$ on Y_{es} and $g = 0$ on Z_{es}, it is evidently balanced. Therefore it gives rise to a bounded linear functional, say $h : Y \otimes_A Z \to \mathbb{C}$, well defined by $h(y \otimes_A z) = f(y)g(z)$.

We easily see that $h = 0$ on $(Y \otimes_A Z)_{\mathrm{es}}$. At the same time the element $v - \sum_{k=1}^n x_k \otimes z_k$ belongs to $\mathrm{Ker}(\tau)$. Thus, by (2.1), we have $u - \sum_{k=1}^n x_k \otimes_A z_k \in (X \otimes_A Z)_{\mathrm{es}}$, and consequently $(\varphi \otimes_A \mathbf{1})(u) - \sum_{k=1}^n \varphi(x_k) \otimes_A z_k$ lies in $(Y \otimes_A Z)_{\mathrm{es}}$. Hence $h(\varphi \otimes_A \mathbf{1}(u)) = h(\sum_{k=1}^n \varphi(x_k) \otimes_A z_k)$, and the latter number is of course 1. It follows that $(\varphi \otimes_A \mathbf{1})(u) \neq 0$. ∎

3. Tensoring isometric morphisms. In this section we deal with homogeneous modules over sequence algebras, defined in the Introduction.

For every $N = 1, 2, \dots$ we set $\mathbf{P}^N := \sum_{n=1}^N \mathbf{p}^n \in A$. The following easy observation is done in [11], but, for the convenience of the reader, we repeat its short proof.

PROPOSITION 3.1. *If an A-module X is homogeneous, then for every $x \in X_{\mathrm{es}}$ we have*

$$x = \lim_{N \to \infty} \mathbf{P}^N \cdot x.$$

Proof. Fix x and $\varepsilon > 0$. It follows from Definition III that there is $y \in X$ of the form $\sum_{k=1}^{n} a^k \cdot z^k$ with $a^k \in c_{00}, z^k \in X$ such that $\|x - y\| < \varepsilon/2$. For all $n \in \mathbb{N}$ we have

$$\|x - \mathbf{P}^N \cdot x\| \leq \|x - y\| + \|y - \mathbf{P}^N \cdot y\| + \|\mathbf{P}^N \cdot y - \mathbf{P}^N \cdot x\|.$$

But, because of the choice of y, for some $M \in \mathbb{N}$ we have $y = \mathbf{P}^N \cdot y$ for all $N > M$. Moreover, the homogeneity of X implies that $\|\mathbf{P}^N \cdot (y - x)\| \leq \|y - x\|$. Therefore for all $N > M$ we have $\|x - \mathbf{P}^N \cdot x\| < \varepsilon$. ∎

Proposition 3.2. *Let Z be an A-module. Assume that, for any essential homogeneous A-modules X and Y and an isometric morphism $\mathsf{i} : X \to Y$ the morphism $\mathsf{i} \otimes_A \mathbf{1} : X \otimes_A Z \to Y \otimes_A Z$ is also isometric. Then, for every $n = 1, 2, \ldots,$ the coordinate submodule Z_n is, up to an isometric isomorphism of normed spaces, a dense subspace of $L_1(\Omega_n, \mu_n)$ for some measure space (Ω_n, μ_n).*

Proof. Suppose that, for a certain n, Z_n does not satisfy the stated condition. Then it easily follows from the criterion of Grothendieck [5, Thm. 1] that there are normed spaces X, Y and an isometric operator $\mathsf{i} : X \to Y$ such that the operator $\mathsf{i} \otimes_{\mathrm{p}} \mathbf{1} : X \otimes_{\mathrm{p}} Z_n \to Y \otimes_{\mathrm{p}} Z_n$ fails to be an isometry.

Set, for every $\xi = (\xi_1, \ldots, \xi_n, \ldots) \in A$, $x \in X$, $y \in Y$, $\xi \cdot x := \xi_n x$ and $\xi \cdot y := \xi_n y$. In this way we make X and Y A-modules that are essential and homogeneous. Moreover, i becomes an A-module morphism. It is sufficient, by virtue of Propositions 1.7 and 3.1, to show that the operator $\mathsf{i} \odot \mathbf{1} : X \odot Z \to Y \odot Z$ is not an isometry.

We see that $X_m = Y_m = 0$ for $m \neq n$. Hence $X \odot Z = (X \odot Z)_n$ and $Y \odot Z = (Y \odot Z)_n$. Therefore the isometric isomorphisms $\varkappa_n^{X,Z}$ and $\varkappa_n^{Y,Z}$ (see Section 1) act between $X_n \otimes_{\mathrm{p}} Z_n$ and $X \odot Z$, and, respectively, between $Y_n \otimes_{\mathrm{p}} Z_n$ and $Y \odot Z$. Moreover, these isometric isomorphisms obviously implement an isometric equivalence of the operators $\mathsf{i} \otimes_{\mathrm{p}} \mathbf{1}$ and $\mathsf{i} \odot \mathbf{1}$ (cf. (1.1)). Consequently, since the former is not an isometry, neither is the latter. ∎

Our principal aim is to show the converse.

The main step in our proof is the following technical lemma. In what follows, S is an arbitrary homogeneous normed A-module with the following properties:

(i) up to a linear isomorphism, we have $S = \bigoplus_{n=1}^{N} S_n$ for some $N \in \mathbb{N}$ (in other words, $\mathbf{P}^N \cdot x = x$ for every $x \in S$),

(ii) for every $n = 1, \ldots, N$, S_n is a normed subspace of $L_1(\Omega_n, \mu_n)$, for some measure space (Ω_n, μ_n), consisting of all step functions (= linear combinations of characteristic functions of μ_n-measurable subsets in Ω_n).

LEMMA 3.3. *Let X, Y be normed A-modules, Y be homogeneous and $\mathsf{i} : X \to Y$ a morphism. Let $u \in X \odot S$. Let $v := (\mathsf{i} \odot \mathbf{1}_S)(u) \in Y \odot S$ be represented as $v = \sum_{k=1}^{m} y^k \odot g^k$ with $y^k \in Y$, $g^k \in S$. Then for every $n = 1, \dots, N$ there exists a natural number M, $x^{kl} \in X_n$ and $g^{kl} \in S$, $k = 1, \dots, m$, $l = 1, \dots, M$ such that for*

$$y^{kl} := y_1^k + y_2^k + \cdots + y_{n-1}^k + \mathsf{i}_n(x_n^{kl}) + y_{n+1}^k + \cdots + y_N^k \tag{3.1}$$

we have

$$v = \sum_{k=1}^{m} \sum_{l=1}^{M} y^{kl} \odot g^{kl} \tag{3.2}$$

and

$$\sum_{k=1}^{m} \sum_{l=1}^{M} \|y^{kl}\| \, \|g^{kl}\| \leq \sum_{k=1}^{m} \|y^k\| \, \|g^k\|. \tag{3.3}$$

Proof. Let $\sum_{s=1}^{m'}{}' x^s \odot f^s$ be an arbitrary representation of u. Remembering what S_n is, we can find $M \in \mathbb{N}$ and a partition $\Omega_n = \bigsqcup_{l=1}^{M} \Delta_l$, where Δ_l, $l = 1, \dots, M$, are μ_n-measurable subsets of Ω_n such that all g_n^k, f_n^s are constant functions on each Δ_l. In particular, for every $k = 1, \dots, m$, g_n^k has the form $\sum_{l=1}^{M} \lambda^{kl} \chi_l$, where $\lambda^{kl} \in \mathbb{C}$ and χ_l is the characteristic function of Δ_l.

Now for every $k = 1, \dots, m$ and $l = 1, \dots, M$ we set

$$g^{kl} := \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|} g_1^k + \cdots + \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|} g_{n-1}^k + \lambda^{kl}\chi_l + \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|} g_{n+1}^k + \cdots + \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|} g_N^k, \tag{3.4}$$

provided $g_n^k \neq 0$, and $g^{kl} := 0$ otherwise. In the first case, since $\|\lambda^{kl}\chi_l\| = \left\| \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|} g_n^k \right\|$ and S is homogeneous, we see that

$$\|g^{kl}\| = \left\| \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|} g^k \right\|$$

for all k, l. But, living in $L_1(\cdot)$, we have $\sum_{l=1}^{M} \|\lambda^{kl}\chi_l\| = \|g_n^k\|$. Therefore for all k we have $\sum_{l=1}^{M} \|\lambda^{kl}\chi_l\| / \|g_n^k\| = 1$. Hence $g^k = \sum_{l=1}^{M} g^{kl}$ and

$$\|g^k\| = \sum_{l=1}^{M} \left\| \frac{\|\lambda^{kl}\chi_l\|}{\|g_n^k\|} g^k \right\| = \sum_{l=1}^{M} \|g^{kl}\|.$$

The same, of course, is true if $g_n^k = 0$.

From this we have

$$v = \sum_{k=1}^{m} \sum_{l=1}^{M} y^k \odot g^{kl} \quad \text{and} \quad \sum_{k=1}^{m} \sum_{l=1}^{M} \|y^k\| \, \|g^{kl}\| = \sum_{k=1}^{m} \|y^k\| \, \|g^k\|. \tag{3.5}$$

Let us concentrate on v_n. It follows from (3.5) and (3.4) that

$$v_n = \sum_{k=1}^{m} \sum_{l=1}^{M} y_n^k \otimes \lambda^{kl} \chi_l = \sum_{l=1}^{M} \Big(\sum_{k=1}^{m} \lambda^{kl} y_n^k \Big) \otimes \chi_l. \tag{3.6}$$

But, as we remember, $v = (\mathsf{i} \odot \mathbf{1}_S)(u)$, and u has the representation indicated above. Therefore $v = \sum_{s=1}^{m'} \mathsf{i}('x^s) \odot f^s$. Moreover, by the choice of Δ_l for all s we have $f_n^s = \sum_{l=1}^{M} \nu^{sl} \chi_l$ for some $\nu^{sl} \in \mathbb{C}$. Thus

$$v_n = \sum_{l=1}^{M} \Big(\sum_{s=1}^{m} \nu^{sl} \mathsf{i}_n('x^s) \Big) \otimes \chi_l = \sum_{l=1}^{M} \mathsf{i}_n(x^l) \otimes \chi_l, \tag{3.7}$$

where we have set $x^l := \sum_{s=1}^{m} \nu^{sl}('x^s)$.

But χ_l, $l = 1, \dots, M$, are linearly independent in S_n. Thus, comparing (3.7) and (3.6), we see that

$$\sum_{k=1}^{m} \lambda^{kl} y_n^k = \mathsf{i}_n(x^l) \quad \text{for all } l. \tag{3.8}$$

Now define

$$\alpha^{kl} := (\lambda^{kl})^{-1} \frac{\|\lambda^{kl} y_n^k\|}{\sum_{t=1}^{m} \|\lambda^{tl} y_n^t\|} \quad \text{provided} \quad \lambda^{kl} \sum_{t=1}^{m} \|\lambda^{tl} y_n^t\| \neq 0$$

and $\alpha^{kl} := 0$ otherwise. Finally, set $x_n^{kl} := \alpha^{kl} x^l$.

Take y^{kl} as in (3.1). Look at $v' := \sum_{k=1}^{m} \sum_{l=1}^{M} y^{kl} \odot g^{kl}$. By (3.1) and (3.5), $v'_{n'} = v_{n'}$ for all $n' \neq n$. As to v'_n, it is equal to

$$\sum_{k=1}^{m} \sum_{l=1}^{M} y_n^{kl} \otimes g_n^{kl} = \sum_{l=1}^{M} \sum_{k=1}^{m} \mathsf{i}_n(x_n^{kl}) \otimes \lambda^{kl} \chi_l = \sum_{l=1}^{M} \sum_{k=1}^{m} \mathsf{i}_n(\alpha^{kl} \lambda^{kl} x_n^l) \otimes \chi_l$$
$$= \sum_{l=1}^{M} \sum_{k=1}^{m} \mathsf{i}_n \Big(\frac{\|\lambda^{kl} y_n^k\|}{\sum_{t=1}^{m} \|\lambda^{tl} y_n^t\|} x_n^l \Big) \otimes \chi_l = \sum_{l=1}^{M} \mathsf{i}_n(x_n^l) \otimes \chi_l = \sum_{l=1}^{M} \sum_{k=1}^{m} \lambda^{kl} y_n^l \otimes \chi_l,$$

that is, by (3.8), to v_n. Thus v and v' have the same coordinates and hence, since $Y \odot S$ is essential, they coincide. The equality (3.2) follows.

It remains to obtain (3.3). For this, we want to show that for all l,

$$\|\mathsf{i}_n(x_n^{kl})\| \leq \|y_n^k\|. \tag{3.9}$$

If $\alpha^{kl} = 0$, this is immediate. Otherwise we have

$$\|\mathsf{i}_n(x_n^{kl})\| = \|\alpha^{kl} \mathsf{i}_n(x^{l_n})\| = \Big\| \frac{\|y_n^k\|}{\sum_{t=1}^{m} \|\lambda^{tl} y_n^t\|} \Big(\sum_{t=1}^{m} \lambda^{tl} y_n^t \Big) \Big\|,$$

and (3.9) follows from the triangle inequality for norms.

Now it is time to use that (not only S, but) Y is homogeneous. We have just shown that $\|y_n^{kl}\| \leq \|y_n^k\|$, and, of course, we have $\|y_{n'}^{kl}\| = \|y_{n'}^k\|$ for all

$n' \neq n$. Therefore

$$\|y^{kl}\| \leq \|y^k\|. \tag{3.10}$$

Consequently,

$$\sum_{k=1}^{m}\sum_{l=1}^{M} \|y^{kl}\|\,\|g^{kl}\| \leq \sum_{k=1}^{m}\sum_{l=1}^{M} \|y^k\|\,\|g^{kl}\|,$$

and, because of (3.5), we are done. ∎

Lemma 3.4. *Let X, Y, S be as in the previous lemma, and $\mathsf{i}: X \to Y$ an isometric morphism. Then the morphism $\mathsf{i} \otimes_A \mathbf{1}_S : X \otimes_A S \to Y \otimes_A S$ is also isometric.*

Proof. Of course, S is essential. Therefore, by Propositions 1.7 and 3.1, it is sufficient to show that $\mathsf{i} \odot \mathbf{1}_S : X \odot S \to Y \odot S$ is isometric.

Fix an arbitrary $u \in X \odot S$ and set $v := (\mathsf{i} \odot \mathbf{1})(u) \in Y \odot S$. Our task is to show that $\|u\| = \|v\|$.

Take an arbitrary representation $v = \sum_{k=1}^m y^k \odot g^k$ with $g^k \in S$. Set $n := 1$ in the previous lemma. Getting rid of double sums, we can say that this lemma gives us a representation

$$v = \sum_{k=1}^{m_1} y^{1k} \odot g^{1k},$$

where, for some $x_1^{1k} \in X_1$, $k = 1, \ldots, m_1$, and y_s^{1k}, $s = 2, \ldots, N$, we have

$$y^{1k} = \mathsf{i}_1(x_1^{1k}) + y_2^{1k} + y_3^{1k} + \cdots + y_N^{1k}$$

and

$$\sum_{k=1}^{m_1} \|y^{1k}\|\,\|g^{1k}\| \leq \sum_{k=1}^{m} \|y^k\|\,\|g^k\|.$$

Now apply Lemma 3.3 to the representation of v just obtained and $n := 2$. Looking at the form of the relevant y^{kl} in the situation when the role of y^k is played by y^{1k} and again getting rid of double sums, we obtain a representation

$$v = \sum_{k=1}^{m_2} y^{2k} \odot g^{2k},$$

where, for some $x_1^{2k} \in X_1$, $x_2^{2k} \in X_2$, $k = 1, \ldots, m_2$, and y_s^{1k}, $s = 3, \ldots, N$, we have

$$y^{2k} = \mathsf{i}_1(x_1^{2k}) + \mathsf{i}_2(x_2^{2k}) + y_3^{3k} + \cdots + y_N^{2k},$$

and

$$\sum_{k=1}^{m_1} \|y^{2k}\|\,\|g^{2k}\| \leq \sum_{k=1}^{m_1} \|y^{1k}\|\,\|g^{1k}\| \quad \text{(and hence} \quad \leq \sum_{k=1}^{m} \|y^k\|\,\|g^k\|).$$

Next we apply Lemma 3.3 to the last representation of v and $n := 3$, and so on. At the Nth step, again (the last time) getting rid of double sums, we come to a representation

$$v = \sum_{k=1}^{m_N} y^{Nk} \odot g^{Nk},$$

where, for some $x_1^{Nk} \in X_1, x_2^{Nk} \in X_2, \dots, x_N^{Nk} \in X_N$, $k = 1, \dots, m_N$, we have

$$y^{Nk} = \mathsf{i}_1(x_1^{Nk}) + \mathsf{i}_2(x_2^{Nk}) + \dots + \mathsf{i}_N(x_N^{Nk})$$

and

$$\sum_{k=1}^{m_N} \|y^{Nk}\| \, \|g^{2k}\| \le \sum_{k=1}^{m} \|y^k\| \, \|g^k\|.$$

Finally, define $x^k := x_1^{Nk} + \dots + x_N^{Nk} \in X$, $k = 1, \dots, m_N$. Obviously, $y^{Nk} = \mathsf{i}(x^k)$ and hence $(\mathsf{i} \odot \mathbf{1}_S)(\sum_{k=1}^{m_N} x^k \odot g^{Nk}) = v$. But $\mathsf{i} \odot \mathbf{1}_S$ is injective (see Proposition 2.1). Therefore $\sum_{k=1}^{m_N} x^k \odot g^{Nk}$ is exactly u. Recalling that i is isometric, we have

$$\|u\| \le \sum \|x^k\| \, \|g^{Nk}\| = \sum \|y^{Nk}\| \, \|g^{Nk}\|,$$

and hence

$$\|u\| \le \sum_{k=1}^{m} \|y^k\| \, \|g^k\|.$$

Taking the infimum as in (1.2), we have the estimate $\|u\| \le \|v\|$. Since, by (1.6), $\mathsf{i} \odot \mathbf{1}$ is contractive, the desired equality follows. ∎

LEMMA 3.5. *The assertion of the previous lemma remains true if we replace the module S by an arbitrary module Z such that*

(i) *there exists a natural N such that Z is linearly isomorphic to $\bigoplus_{n=1}^{N} Z_n$,*
(ii) *for every $n = 1, \dots, N$, Z_n is, up to an isometric isomorphism, a dense normed subspace of $L_1(\Omega_n, \mu_n)$ for some measure space (Ω_n, μ_n).*

Proof. Denote by $\bar{Z}$ and $\bar{Z}_n$, $n = 1, \dots, N$, the completions of the A-modules Z and Z_n, respectively. Take $z \in Z$. Obviously,

$$\max\{\|z_n\| : n = 1, \dots, N\} \le \|z\| \le \sum_{n=1}^{N} \|z_n\|.$$

Therefore a sequence z^m is a Cauchy sequence in Z if and only if for every $n = 1, \dots, N$ the sequence z_n^m is a Cauchy sequence in Z_n. It easily follows that $\bar{Z}$ is isometrically isomorphic to the algebraic direct sum $\bigoplus_{n=1}^{N} \bar{Z}_n$, endowed with the norm, well defined by $\|z\| = \lim_{m\to\infty} \|z^m\|$, where z^m is an arbitrary sequence in Z such that $\lim_{m\to\infty} z_n^m = z_n$ for every n. Obviously, $\bar{Z}_n$ is isometrically isomorphic to the space $L_1(\Omega_n, \mu_n)$, mentioned in the formulation. It easily follows that $\bar{Z}$ contains a dense submodule S satisfying the condition of Lemma 3.4. By virtue of that lemma, $\mathsf{i} \otimes_A \mathbf{1}_S$ is an isometry. Therefore, by Proposition 1.2, the same is true for $\mathsf{i} \otimes_A \mathbf{1}_{\bar{Z}}$, which in turn gives the desired property of $\mathsf{i} \otimes_A \mathbf{1}_Z$. ∎

Theorem 3.6. *Let Z be a homogeneous A-module such that, for every $n = 1, \ldots, N$, Z_n is, up to an isometric isomorphism, a dense normed subspace of $L_1(\Omega_n, \mu_n)$ for some measure space (Ω_n, μ_n). Further, let X and Y be two other homogeneous A-modules, and $\mathsf{i} : X \to Y$ an isometric morphism. Suppose that at least one of the modules X and Z is essential. Then the morphism $\mathsf{i} \otimes_A \mathbf{1} : X \otimes_A Z \to Y \otimes_A Z$ is also isometric.*

Proof. Fix $N \in \mathbb{N}$ for a time, and denote by Z^N the submodule $\{\mathbf{P}^N \cdot u : u \in Z\}$ of Z. Consider the diagram

$$\begin{array}{ccc} X \otimes_A Z^N & \xrightarrow{\mathbf{1}_X \otimes_A i} & X \otimes_A Z \\ {\scriptstyle \mathsf{i}'_N}\downarrow & & \downarrow{\scriptstyle \mathsf{i}'} \\ Y \otimes_A Z^N & \xrightarrow{\mathbf{1}_Y \otimes_A i} & Y \otimes_A Z \end{array}$$

where $\mathsf{i}'_N := \mathsf{i} \otimes_A \mathbf{1}_Z^N$, $\mathsf{i}' := \mathsf{i} \otimes_A \mathbf{1}_Z$, and $i : Z^N \to Z$ is the natural embedding. Since Z^N is, of course, essential, Lemma 3.5 together with Propositions 3.1, 1.7 and 1.10 implies that i'_N is an isometric morphism. Further, $\mathbf{1}_X \otimes_A i$ is contractive and has a contractive right inverse. The latter is $\mathbf{1}_X \otimes_A j$, where $j : Z \to Z^N$ acts as $z \mapsto \mathbf{P}^N \cdot z$. (It is contractive since Z is homogeneous.) Therefore $\mathbf{1}_X \otimes_A i$ is an isometry, and the same is true of $\mathbf{1}_Y \otimes_A i$.

Now take an arbitrary $u \in X \otimes_A Z$, represent it as a sum of elementary tensors and observe that for every $x \in X$ and $z \in Z$ we have

$$\mathbf{P}^N \cdot (x \otimes_A z) = \mathbf{P}^N \cdot x \otimes_A z = x \otimes_A \mathbf{P}^N \cdot z. \tag{3.11}$$

The second of these equalities implies that $\mathbf{P}^N \cdot u \in \mathrm{Im}(\mathbf{1}_X \otimes_A i)$. From this, since our diagram is obviously commutative and its three morphisms, mentioned above, are isometries, we have

$$\|\mathsf{i}'(\mathbf{P}^N \cdot u)\| = \|\mathbf{P}^N \cdot u\| \tag{3.12}$$

for all $N \in \mathbb{N}$. Further, both equalities in (3.11), combined with the condition on the pair (X, Z) and Proposition 3.1, imply that $u = \lim_{N\to\infty} \mathbf{P}^N \cdot u$. Hence, using (3.12), we see that

$$\|\mathsf{i}'(u)\| = \lim_{N\to\infty} \|\mathsf{i}'(\mathbf{P}^N \cdot u)\| = \lim_{N\to\infty} \|\mathbf{P}^N \cdot u\| = \|u\|. \ ∎$$

Combining this theorem with Proposition 3.2, we immediately obtain Theorem I of the Introduction, together with its corollaries for sequence modules and some other modules.

Finally, Theorem I easily yields the Hahn–Banach type theorem, formulated in the Introduction as Theorem II. Indeed, let A be a sequence algebra, and Z an essential (respectively, arbitrary) homogeneous normed A-module. In this case, according to Proposition 1.1 (more precisely, to its direct corollary, the Proposition, at the beginning of the Introduction), the dual module Z^* is extremely injective relative to $\mathcal{H}$ (respectively, $\mathcal{H}_{\mathrm{es}}$) if and only if Z is extremely flat relative to $\mathcal{H}$ (respectively, $\mathcal{H}_{\mathrm{es}}$). Then Theorem I works, and we are done.

4. A counter-example. Here we want to show that the condition in Theorem I, concerning the module being essential, cannot be omitted, even within the class of homogeneous modules. Namely, we shall show that the (non-essential) module l_∞ is not extremely flat with respect to the above mentioned class.

First let us make some observations of general character.

Let X be an A-module. A subset M of $\mathbb{N}$ is called the *support* of X whenever $X_n = 0$ if and only if $n \notin M$.

LEMMA 4.1. *Let X and Z be two modules that have non-intersecting supports. Then for every $x \in X$, $x' \in X_{\mathrm{es}}$, $z \in Z$, $z' \in Z_{\mathrm{es}}$ we have $x' \otimes_A z = x \otimes_A z' = 0$ in $X \otimes_A Z$.*

Proof. By Proposition 3.1, we have

$$x' \otimes_A z = \lim_{N\to\infty} \mathbf{P}^N \cdot x' \otimes_A z = \lim_{N\to\infty} \sum_{n=1}^{N} \mathbf{p}^n \cdot x' \otimes_A \mathbf{p}^n \cdot z.$$

But the condition on supports implies that, for every n, either $\mathbf{p}^n \cdot x'$ or $\mathbf{p}^n \cdot z$ is 0. ∎

For $x \in X$, we denote by $\widetilde{x}$ the coset $x + X_{\mathrm{es}} \in X_{\mathrm{an}}$.

PROPOSITION 4.2. *Let X and Z be as before. Then there exists an isometric isomorphism of normed spaces $I_{X,Z} : X \otimes_A Z \to X_{\mathrm{an}} \otimes_{\mathrm{p}} Z_{\mathrm{an}}$, well defined by $x \otimes_A z \mapsto \widetilde{x} \otimes \widetilde{z}$.*

Proof. Consider the bilinear operator $X \times Z \to X_{\mathrm{an}} \otimes_{\mathrm{p}} Z_{\mathrm{an}} : (x, z) \mapsto \widetilde{x} \otimes \widetilde{z}$; it is obviously contractive and balanced. Therefore it gives rise to a contractive operator $I_{X,Z}$, defined as stated.

Take $v \in X_{\mathrm{an}} \otimes_{\mathrm{p}} Z_{\mathrm{an}}$, represented, say, as $\sum_{k=1}^{n} \widetilde{x}_k \otimes \widetilde{z}_k$ with $x_k \in X$, $z_k \in Z$. Then we have $v = I_{X,Z}(u)$, where $u = \sum_{k=1}^{n} x_k \otimes_A z_k$ with arbitrary x_k, z_k, taken in the appropriate cosets. Obviously, we have to show that $\|u\| \le \|v\|$.

Take some $x'_k \in X_{\mathrm{es}}$, $z'_k \in Z_{\mathrm{es}}$. Lemma 4.1 implies that

$$u = \sum (x_k + x'_k) \otimes_A (z_k + z'_k).$$

Therefore

$$\|u\| \le \sum_{k=1}^{n} \|x_k + x'_k\| \, \|z_k + z'_k\|.$$

Since x'_k, z'_k can be chosen in an arbitrary way, we have

$$\|u\| \le \sum_{k=1}^{n} \|\widetilde{x}_k\| \, \|\widetilde{z}_k\|.$$

Finally, since the representation of v is also arbitrary, the very definition of the projective tensor norm gives the desired inequality. ∎

Now consider the normed quotient space ('ultraproduct') l_∞/c_0. Since it is not isometrically isomorphic to any space of the class $L_1(\Omega, \mu)$, the theorem of Grothendieck, cited in the Introduction, implies that there exist normed spaces E, F and an isometric operator $\widetilde{i} : E \to F$ such that the operator

$$\widetilde{i} \otimes_{\mathrm{p}} \mathbf{1} : E \otimes_{\mathrm{p}} (l_\infty/c_0) \to F \otimes_{\mathrm{p}} (l_\infty/c_0)$$

is not an isometry. Let us fix such E, F and $\widetilde{i}$.

In what follows, we shall need, apart from the already used tensor product $\otimes_{\mathrm{p}}$, the non-completed *injective* tensor product of normed spaces and bounded operators, denoted by $\otimes_{\mathrm{i}}$ (see, e.g., [3, Ch. I.4] or [15, Ch. 3]). The injective tensor norm will be denoted by $\|\cdot\|_{\mathrm{i}}$.

Consider the normed space $l_\infty \otimes_{\mathrm{i}} E$. Evidently, it is an A-module with the outer multiplication well defined by $\xi \cdot (\eta \otimes x) := \xi\eta \otimes x$ for $\xi \in A$, $\eta \in l_\infty$, $x \in E$.

This module is *contractive*: if $m_\xi : l_\infty \to l_\infty$ acts as $\eta \mapsto \xi\eta$, then, for every $u \in l_\infty \otimes_{\mathrm{i}} E$, we have $\xi \cdot u = (m_\xi \otimes_{\mathrm{i}} \mathbf{1}_E)(u)$, and hence

$$\|\xi \cdot u\|_{\mathrm{i}} \le \|m_\xi \otimes_{\mathrm{i}} \mathbf{1}_E\| \, \|u\| \le \|m_\xi\| \, \|\mathbf{1}_E\| \, \|u\| \le \|\xi\| \, \|u\|.$$

Moreover, the module $l_\infty \otimes_{\mathrm{i}} E$ is *homogeneous*. This fact can be deduced from the known properties of the operation $\otimes_{\mathrm{i}} C(\Omega)$ (see, e.g., ibid.) and the identification of l_∞ with $C(\beta\mathbb{N})$. But we prefer to give a simpler proof.

Obviously, it suffices to show that for $u \in l_\infty \otimes_{\mathrm{i}} E$ with $u = \sum_{k=1}^n \xi^k \otimes x^k$ we have

$$\|u\|_{\mathrm{i}} = \sup\{\|\mathbf{p}^n \cdot u\|_{\mathrm{i}} : n = 1, 2, \ldots\}.$$

Take $f \in (l_\infty)^*$ and $g \in E^*$ with $\|f\| = \|g\| = 1$. Then $(f \otimes g)(u) = f(\eta^g)$, where $\eta^g := \sum_{k=1}^n g(x^k)\xi^k$. Hence $|(f \otimes g)(u)| \le \|\eta^g\| = \sup\{|(\eta^g)_n| :$

$n = 1, 2, \ldots\}$. But for every n we have

$$|(\eta^g)_n| = \Big\| \sum_{k=1}^{n} \mathbf{p}^n \xi^k g(x^k) \Big\| = \Big\| (\mathbf{1} \otimes g)\Big(\sum_{k=1}^{n} \mathbf{p}^n \xi^k \otimes x^k \Big) \Big\| = \|(\mathbf{1} \otimes g)(\mathbf{p}^n \cdot u)\|$$
$$\leq \|\mathbf{p}^n \cdot u)\|.$$

Therefore the number $\|u\|_{\mathrm{i}}$, which is, by definition, $\sup\{|(f \otimes g)(u)| : f \in (l_\infty)^*, g \in E^*, \|f\| = \|g\| = 1\}$, does not exceed $\sup\{\|\mathbf{p}^n \cdot u\|_{\mathrm{i}} : n = 1, 2, \ldots\}$. Since the reverse inequality is obvious, we are done.

In the same way we define the contractive homogeneous A-module $l_\infty \otimes_{\mathrm{i}} F$. Finally, consider the operator $\mathbf{i} := \mathbf{1} \otimes_{\mathrm{i}} \widetilde{i} : l_\infty \otimes_{\mathrm{i}} E \to l_\infty \otimes_{\mathrm{i}} F$, which is evidently a morphism of A-modules. Because of the injectivity of $\otimes_{\mathrm{i}}$ (see, e.g., [3, Ch. I.4.3] or [15, p. 47]), $\mathbf{i}$ is an isometry.

From now on, it is convenient to use the notation X for $l_\infty \otimes_{\mathrm{i}} E$ and Y for $l_\infty \otimes_{\mathrm{i}} F$.

THEOREM 4.3. *The morphism $\mathbf{i} \otimes_A \mathbf{1} : X \otimes_A l_\infty \to Y \otimes_A l_\infty$ is not an isometry. As a corollary, the module l_∞ is not extremely flat with respect to the class of all homogeneous normed A-modules.*

Proof. We shall write Z instead of l_∞, and just $\mathbf{1}$ instead of $\mathbf{1}_Z$. Note that $Z_{\mathrm{an}} = l_\infty / c_0$.

Denote by Z^{od} and Z^{ev} the submodules of Z consisting of sequences with the zero even terms and, respectively, zero odd terms. Moreover, denote by $\mathbf{1}_{\mathrm{an}}$ and $\mathbf{1}_\bullet$ the identity operators on Z_{an} and $(Z^{\mathrm{ev}})_{\mathrm{an}}$, respectively. Our first claim is

1°. *The operator $\widetilde{i} \otimes_{\mathrm{p}} \mathbf{1}_\bullet : E \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}} \to F \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}}$ is not an isometry.*

Indeed, mapping the sequence $(0, \xi_2, 0, \xi_4, 0, \ldots)$ to $(\xi_2, \xi_4, \ldots)$, we obtain isometric isomorphisms of normed spaces (by no means of modules) $j : Z^{\mathrm{ev}} \to Z$, $j_{\mathrm{es}} : (Z^{\mathrm{ev}})_{\mathrm{es}} \to Z_{\mathrm{es}} = c_0$ and, passing to appropriate cosets, $j_{\mathrm{an}} : (Z^{\mathrm{ev}})_{\mathrm{an}} \to Z_{\mathrm{an}}$. Then we easily see that the operators $\widetilde{i} \otimes_{\mathrm{p}} \mathbf{1}_\bullet$ and $\widetilde{i} \otimes_{\mathrm{p}} \mathbf{1}_{\mathrm{an}}$ are isometrically equivalent. The rest is clear.

From now on we shall use the brief notation X^{od} for $Z^{\mathrm{od}} \otimes_{\mathrm{i}} E$, Y^{od} for $Z^{\mathrm{od}} \otimes_{\mathrm{i}} F$, $\mathbf{1}^{\mathrm{od}}$ for the identity operator on Z^{od}, and $\mathbf{i}^{\mathrm{od}}$ for $\mathbf{1}^{\mathrm{od}} \otimes_{\mathrm{i}} \widetilde{i} : X^{\mathrm{od}} \to Y^{\mathrm{od}}$. Similarly to what was said about X and Y, X^{od} and Y^{od} are contractive A-modules with respect to the same outer multiplication as for X and Y (cf. above), and $\mathbf{i}^{\mathrm{od}}$ is an isometric morphism of A-modules. Moreover, we introduce the operator $\mathbf{i}_{\mathrm{an}} : (X^{\mathrm{od}})_{\mathrm{an}} \to (Y^{\mathrm{od}})_{\mathrm{an}}$, which is well defined by mapping a coset $x + (X^{\mathrm{od}})_{\mathrm{es}}$ to $\mathbf{i}^{\mathrm{od}}(x) + (Y^{\mathrm{od}})_{\mathrm{es}}$.

Our next claim is

2°. *The operator $\mathbf{i}_{\mathrm{an}} \otimes_{\mathrm{p}} \mathbf{1}_\bullet : (X^{\mathrm{od}})_{\mathrm{an}} \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}} \to (Y^{\mathrm{od}})_{\mathrm{an}} \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}}$ is not an isometry.*

Denote the sequence $(1,0,1,0,1,\ldots) \in Z^{\mathrm{od}}$ by $\widetilde{1}^{\mathrm{od}}$. Consider the operator $\mathbf{s}_E : E \to (X^{\mathrm{od}})_{\mathrm{an}}$ taking a vector x to the coset $(\widetilde{1}^{\mathrm{od}} \otimes_{\mathrm{i}} x) + (X^{\mathrm{od}})_{\mathrm{es}})$, and then $\mathbf{s}_E \otimes_{\mathrm{p}} \mathbf{1}_\bullet : E \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}} \to (X^{\mathrm{od}})_{\mathrm{an}} \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}}$. First we shall show, as an intermediate step, that the latter operator is an isometry.

To this end, using the Hahn–Banach theorem, introduce the functional $h : Z^{\mathrm{od}} \to \mathbb{C}$ of norm 1 which maps the subspace $(Z^{\mathrm{od}})_{\mathrm{es}} = c_0 \cap Z^{\mathrm{od}}$ to 0 and $\widetilde{1}^{\mathrm{od}}$ to 1. It gives rise to the operator $\mathbf{t}_E^0 := h \otimes_{\mathrm{i}} \mathbf{1}_E : Z^{\mathrm{od}} \otimes_{\mathrm{i}} E \to \mathbb{C} \otimes_{\mathrm{i}} E$, that is, $\mathbf{t}_E^0 : X^{\mathrm{od}} \to E$. The latter evidently takes $(X^{\mathrm{od}})_{\mathrm{es}}$ to 0 and therefore generates an operator $\mathbf{t}_E := (X^{\mathrm{od}})_{\mathrm{an}} \to E$, well defined by mapping the coset $u + (X^{\mathrm{od}})_{\mathrm{es}}$, $u \in X^{\mathrm{od}}$, to $\mathbf{t}_E^0(u)$. Since $\mathbf{s}_E$ and $\mathbf{t}_E$ are, of course, contractive, so are $\mathbf{s}_E \otimes_{\mathrm{p}} \mathbf{1}_\bullet$ and $\mathbf{t}_E \otimes_{\mathrm{p}} \mathbf{1}_\bullet$. But the composition $(\mathbf{t}_E \otimes_{\mathrm{p}} \mathbf{1}_{\mathrm{an}})(\mathbf{s}_E \otimes_{\mathrm{p}} \mathbf{1}_{\mathrm{an}})$ is the identity operator on $E \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}}$. This implies that the former of the two factors is an isometry (and the latter is a coisometry).

In a similar way, we introduce the operator

$$\mathbf{s}_F \otimes_{\mathrm{p}} \mathbf{1}_\bullet : F \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}} \to (Y^{\mathrm{od}})_{\mathrm{an}} \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}}$$

and show that it is also an isometry. Consequently, in the diagram

$$\begin{tikzcd}
E \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}} \arrow[r, "\mathbf{s}_E \otimes_{\mathrm{p}} \mathbf{1}_\bullet"] \arrow[d, "\widetilde{i} \otimes_{\mathrm{p}} \mathbf{1}_\bullet"'] & (X^{\mathrm{od}})_{\mathrm{an}} \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}} \arrow[d, "\mathbf{i}_{\mathrm{an}} \otimes_{\mathrm{p}} \mathbf{1}_\bullet"] \\
F \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}} \arrow[r, "\mathbf{s}_F \otimes_{\mathrm{p}} \mathbf{1}_\bullet"] & (Y^{\mathrm{od}})_{\mathrm{an}} \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}}
\end{tikzcd}$$

the horizontal arrows are isometries. Further, our diagram is obviously commutative. It follows that the vertical arrows are simultaneously isometric or not. Therefore the present claim follows from the previous one.

We turn to the next claim.

3°. *The morphism* $\mathbf{i}^{\mathrm{od}} \otimes_A \mathbf{1}^{\mathrm{ev}} : X^{\mathrm{od}} \otimes_A Z^{\mathrm{ev}} \to Y^{\mathrm{od}} \otimes_A Z^{\mathrm{ev}}$ *is not isometric.*

The set of odd natural numbers is the support of both X^{od} and Y^{od} whereas the set of even natural numbers is the support of Z^{ev}. Therefore Proposition 4.2 provides isometric isomorphisms $I_{X^{\mathrm{od}},Z^{\mathrm{ev}}} : X^{\mathrm{od}} \otimes_A Z^{\mathrm{ev}} \to (X^{\mathrm{od}})_{\mathrm{an}} \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}}$ and $I_{Y^{\mathrm{od}},Z^{\mathrm{ev}}} : Y^{\mathrm{od}} \otimes_A Z^{\mathrm{ev}} \to (Y^{\mathrm{od}})_{\mathrm{an}} \otimes_{\mathrm{p}} (Z^{\mathrm{ev}})_{\mathrm{an}}$, well defined as stated. Looking at the appropriate commutative diagram, we see that these isomorphisms implement an isometric equivalence between the operators $\mathbf{i}^{\mathrm{od}} \otimes_A \mathbf{1}^{\mathrm{ev}}$ and $\mathbf{i}_{\mathrm{an}} \otimes_{\mathrm{p}} \mathbf{1}_\bullet$. The rest is clear.

4°. *The end of the proof.*

Let $\rho^{\mathrm{od}} : Z^{\mathrm{od}} \to Z$ and $\rho^{\mathrm{ev}} : Z^{\mathrm{ev}} \to Z$ be the natural embeddings. Set $\rho_X^{\mathrm{od}} := \rho^{\mathrm{od}} \otimes_{\mathrm{i}} \mathbf{1}_E$, $\rho_Y^{\mathrm{od}} := \rho^{\mathrm{od}} \otimes_{\mathrm{i}} \mathbf{1}_F$; these maps are obviously morphisms of

A-modules. Consider the diagram

$$\begin{tikzcd} X^{\mathrm{od}} \otimes_A Z^{\mathrm{ev}} \arrow[r, "\rho_X^{\mathrm{od}} \otimes_A \rho^{\mathrm{ev}}"] \arrow[d, "\mathbf{i}^{\mathrm{od}} \otimes_A \mathbf{1}^{\mathrm{ev}}"'] & X \otimes_A Z \arrow[d, "\mathbf{i} \otimes_A \mathbf{1}"] \\ Y^{\mathrm{od}} \otimes_A Z^{\mathrm{ev}} \arrow[r, "\rho_Y^{\mathrm{od}} \otimes_A \rho^{\mathrm{ev}}"] & Y \otimes_A Z \end{tikzcd}$$

Observe that its horizontal arrows are isometries. Indeed, introduce the morphisms $\sigma^{\mathrm{od}} : Z \to Z^{\mathrm{od}} : (\xi_1, \xi_2, \xi_3, \ldots) \mapsto (\xi_1, 0, \xi_3, 0, \xi_5, \ldots)$, $\sigma^{\mathrm{ev}} : Z \to Z^{\mathrm{ev}} : (\xi_1, \xi_2, \xi_3, \ldots) \mapsto (0, \xi_2, 0, \xi_4, 0, \xi_6, \ldots)$ and set $\sigma_X^{\mathrm{od}} := \sigma^{\mathrm{od}} \otimes_{\mathrm{i}} \mathbf{1}_E : Z \otimes_{\mathrm{i}} E \to Z^{\mathrm{od}} \otimes_{\mathrm{i}} E$. Obviously, the operator $\sigma_X^{\mathrm{od}} \otimes_A \sigma^{\mathrm{ev}}$ is contractive, and the same is true of $\rho_X^{\mathrm{od}} \otimes_A \rho^{\mathrm{ev}}$. But the composition

$$(\sigma_X^{\mathrm{od}} \otimes_A \sigma^{\mathrm{ev}})(\rho_X^{\mathrm{od}} \otimes_A \rho^{\mathrm{ev}}) = [(\sigma^{\mathrm{od}}\rho^{\mathrm{od}}) \otimes_{\mathrm{i}} \mathbf{1}_E] \otimes_A (\sigma^{\mathrm{ev}}\rho^{\mathrm{ev}})$$

is the identity operator on $X^{\mathrm{od}} \otimes_A Z^{\mathrm{ev}}$. This implies that the right factor, $\rho_X^{\mathrm{od}} \otimes_A \rho^{\mathrm{ev}}$, is an isometry (whereas the left factor is a coisometry). Similarly, $\rho_Y^{\mathrm{od}} \otimes_A \rho^{\mathrm{ev}}$ is an isometry as well.

Our diagram is clearly commutative, and, by the previous claim, its left vertical arrow is not an isometry. Hence the same is true of its right vertical arrow. The rest is clear. ∎

Remark. Extreme flatness is a recent stronger version of a much older notion of strict (or topological) flatness, mentioned in the Introduction. We recall that, to define a strictly flat module, one just has to replace the word 'isometric' by 'topologically injective' in Definition I (see, e.g., [7]).

If A is an amenable Banach algebra, then the A-module l_∞, as every normed module over such an algebra, is (just) flat in the standard sense of [6], [7], [14]. At the same time, by Theorem 4.3, it is not extremely flat. Here we want to note that one can show, using practically the same argument as in the proof of the last theorem, that it is not strictly flat either. The only difference is that at the very beginning one must use a somewhat stronger property of $Z := l_\infty/c_0$ than was employed before. Namely, there exist normed spaces E, F and a topologically injective operator $\mathbf{i} : E \to F$ such that $\mathbf{i} \otimes_{\mathrm{p}} \mathbf{1}_Z$ is not topologically injective. This is because l_∞/c_0, being, in the terminology of [3], an $\mathcal{L}_\infty^g$-space, cannot be an $\mathcal{L}_1^g$-space (ibid., Cor. 23.3(4)). This means, by Cor. 23.5(1) there, that the operation $\otimes_{\mathrm{p}} l_\infty/c_0$ 'does not respect subspaces isomorphically' or, in our terminology, l_∞/c_0 is not a strictly flat normed space ($\mathbb{C}$-module). The subsequent constructions and claims are, up to obvious modifications, the same.

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