STUDIA MATHEMATICA 206 (2) (2011)

Integrated version of the Post–Widder inversion formula for Laplace transforms

by

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Abstract. We establish an inversion formula of Post–Widder type for λ^{α}-multiplied vector-valued Laplace transforms ($\alpha > 0$). This result implies an inversion theorem for resolvents of generators of α-times integrated families (semigroups and cosine functions) which, in particular, provides a unified proof of previously known inversion formulae for α-times integrated semigroups.

1. Introduction. Let X be a Banach space and let $L^1_{\mathrm{loc}}([0,\infty);X)$ denote the vector space of functions $f : [0,\infty) \to X$ which are Bochner integrable on $[0,R]$ for all $R > 0$. For a function $f \in L^1_{\mathrm{loc}}([0,\infty);X)$, the Laplace transform $\hat{f}$ is given by

$$\hat{f}(\lambda) = \int_0^\infty f(t)e^{-\lambda t}\,dt$$

for those complex values λ for which the integral exists. It is a well known fact that any *Laplace transformable* function $f \in L^1_{\mathrm{loc}}([0,\infty);X)$ is determined by its Laplace transform, as the following theorem shows.

THEOREM 1.1 ([ABHN, Theorem 1.7.7]). *Let $f \in L^1_{\mathrm{loc}}([0,\infty);X)$ be such that $\hat{f}(\lambda)$ converges for some $\lambda \in \mathbb{C}$. Let $t > 0$ be a Lebesgue point of f. Then*

$$f(t) = \lim_{n\to\infty} (-1)^n \frac{1}{n!}\left(\frac{n}{t}\right)^{n+1} \hat{f}^{(n)}\left(\frac{n}{t}\right).$$

Recall that $t > 0$ is a *Lebesgue point* of a function $f \in L^1_{\mathrm{loc}}([0,\infty);X)$ if $\lim_{h\to 0} h^{-1}\int_t^{t+h} \|f(s)-f(t)\|\,ds = 0$. Every point of continuity is a Lebesgue point of f and almost all points are Lebesgue points of f (see [ABHN, p. 16]).

The above theorem provides us with a vector-valued version of the classical Post–Widder inversion formula for the Laplace transform; see [P, W].

2010 *Mathematics Subject Classification*: Primary 44A10; Secondary 47D62.
Key words and phrases: vector-valued Laplace transform, real inversion, Post–Widder formula, α-times integrated semigroup, α-times integrated cosine function.

DOI: 10.4064/sm206-2-4

Such a limit is known as a real inversion formula since only properties of $\hat{f}(\lambda)$ for large real λ are involved. In recent years, the Post–Widder formula has been fruitfully used in numerical applications (see for instance [MCPS, SB]).

The main result of this note, Theorem 2.1 below, is an integrated Post–Widder formula for λ^α-multiplied Laplace transforms (and Laplace–Stieltjes transforms) of vector-valued functions. This theorem allows us to obtain inversion formulae for resolvents of generators of (α-times) integrated semigroups and integrated cosine families of operators. Such formulae in particular recover and extend to α-times integrated semigroups other previously known results in the literature (see [C, VV]). The paper ends with a discussion of the canonical example of an integrated family, formed by the so-called Riesz kernels.

2. The main result. Let X be an arbitrary complex Banach space and let $f:(0,\infty)\to X$ be a measurable function such that

$$\sup_{t>0}\|t^{-\gamma}e^{-\omega t}f(t)\| =: M<\infty \tag{2.1}$$

for some $\gamma>-1$ and $\omega\ge 0$. Clearly, the Laplace transform $\hat{f}$ exists at least on the open right half-plane $\Re\lambda>\omega$.

The following is the main result of the paper.

THEOREM 2.1. *Let γ, ω and f be as above. Then, for every $\alpha\in(0,\gamma+1)$ and for any Lebesgue point $t>0$ of f,*

$$f(t)=\lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{(-1)^n}{n!}\left(\frac{n}{s}\right)^{n+1}\frac{d^n}{d\lambda^n}(\lambda^\alpha\hat{f}(\lambda))\Big|_{\lambda=n/s}\,ds.$$

This formula may be considered as an α-times integrated version of the Post–Widder formula. In the next lemma it is shown that the conditions on f and on α ensure that the Post–Widder approximant

$$L_{n,s}[\lambda^\alpha\hat{f}(\lambda)] := \frac{(-1)^n}{n!}\left(\frac{n}{s}\right)^{n+1}\frac{d^n}{d\lambda^n}(\lambda^\alpha\hat{f}(\lambda))\Big|_{\lambda=n/s}\qquad (s>0)$$

is Bochner integrable near the origin for n sufficiently large, so that the integral in Theorem 2.1 is actually convergent.

LEMMA 2.2. *Let $f:(0,\infty)\to X$, γ, ω and α be as in the assumptions of Theorem 2.1. Then the function $L_{n,(\cdot)}[\lambda^\alpha\hat{f}(\lambda)]$ is Bochner integrable in $(0,t)$ for every $t>0$ and every $n>\omega t$.*

Proof. First of all, notice that, due to the growth conditions on f, the integral $\int_0^\infty f(u)u^k e^{-\lambda u}\,du$ is Bochner convergent for every $\lambda>\omega$ and $k\ge 0$.

Now, take $t > 0$ and $n > \omega t$. Thus if $s \in (0,t)$ then $n > \omega s$, so we get

$$L_{n,s}[\lambda^\alpha \hat{f}(\lambda)] = \frac{(-1)^n}{n!} \sum_{k=0}^{n} C_{k,n}^\alpha \left(\frac{n}{s}\right)^{\alpha+1+k} \int_0^\infty f(u) u^k e^{-(n/s)u}\, du,$$

where $C_{k,n}^\alpha := (-1)^k \binom{n}{k} (n-k)! \binom{\alpha}{n-k}$ for $k = 0, \ldots, n$. Then note that, for the constant M appearing in (2.1),

$$\begin{aligned}\left(\frac{n}{s}\right)^{\alpha+1+k} & \int_0^\infty |f(u)| u^k e^{-(n/s)u}\, du \\ &\le M \left(\frac{n}{s}\right)^{\alpha+1+k} \int_0^\infty u^{\gamma+k} e^{-((n/s)-\omega)u}\, du \\ &= M \frac{(n/s)^{\alpha+1+k}}{((n/s)-\omega)^{\gamma+1+k}}\, \Gamma(\gamma+k+1) \\ &\le M\Gamma(\gamma+k+1) n^{\alpha+k+1} s^{\gamma-\alpha} \quad (k = 0, \ldots, n),\end{aligned}$$

provided that $\gamma > -1$. Therefore, the function $L_{n,s}[\lambda^\alpha \hat{f}(\lambda)]$ is integrable in $(0,t)$ whenever $\alpha \in (0, \gamma+1)$. ∎

Remark 2.3. In order to ensure the Bochner integrability of $L_{n,(\cdot)}[\lambda^\alpha \hat{f}(\lambda)]$ near the origin, it is enough to assume that the given function f is in $L^1_{\mathrm{loc}}([0,\infty); X)$, it is Laplace transformable, and its Laplace transform $\hat{f}$ satisfies

$$\int_R^\infty \lambda^{\alpha+k+1} \hat{f}^{(k)}(\lambda)\, d\lambda < \infty \qquad \text{for every } k \in \mathbb{N} \text{ and } R > 0.$$

Under these weaker assumptions, the inversion formula in Theorem 2.1 also holds.

Proof of Theorem 2.1. Let $t > 0$ be a Lebesgue point of f. Denote

$$\mathcal{I}_n(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} L_{n,s}[\lambda^\alpha \hat{f}(\lambda)]\, ds.$$

The convergence of this integral for $n > \omega t$ follows from Lemma 2.2. As before, write

$$L_{n,s}[\lambda^\alpha \hat{f}(\lambda)] = \frac{(-1)^n}{n!} \sum_{k=0}^{n} C_{k,n}^\alpha \left(\frac{n}{s}\right)^{\alpha+1+k} \int_0^\infty f(u) u^k e^{-(n/s)u}\, du$$

for $s \in (0,t)$ and $n > \omega t$. Using Fubini's Theorem we get

$$\mathcal{I}_n(t) = \frac{(-1)^n}{n!} \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n} C_{k,n}^\alpha \int_0^\infty u^k f(u) \mathcal{K}_n(u)\, du$$

where

$$\mathcal{K}_n(u) := \int_0^t (t-s)^{\alpha-1}\left(\frac{n}{s}\right)^{\alpha+k+1} e^{-(n/s)u}\,ds \quad (u>0).$$

Making the change of variable $z=(n/s)u-(n/t)u$, we obtain

$$\begin{aligned}\mathcal{K}_n(u) &= \frac{ne^{-(n/t)u}}{u^{\alpha+k}}t^{\alpha-k-1}\int_0^\infty z^{\alpha-1}(tz+nu)^k e^{-z}\,dz\\ &= \frac{ne^{-(n/t)u}}{u^{\alpha+k}}t^{\alpha-1}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{nu}{t}\right)^{k-i}\Gamma(\alpha+j).\end{aligned}$$

Then

$$\mathcal{I}_n(t) = \frac{(-1)^n}{n!}nt^{\alpha-1}\int_0^\infty u^{-\alpha}f(u)e^{-(n/t)u}\Phi_{n,t,\alpha}(u)\,du$$

where, for $u>0$,

$$\Phi_{n,t,\alpha}(u) := \sum_{k=0}^{n} C_{k,n}^{\alpha}\sum_{j=0}^{k}\binom{k}{j}\left(\frac{nu}{t}\right)^{k-j}\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)} = (-1)^n\left(\frac{nu}{t}\right)^n;$$

see [VV, Lemma 3.1] for the general formula. Hence we get

$$\mathcal{I}_n(t) = \frac{n^{n+1}}{n!}t^{\alpha-n-1}\int_0^\infty u^{n-\alpha}e^{-(n/t)u}f(u)\,du. \tag{2.2}$$

Notice that for every non-negative integer $n>\alpha+1$,

$$\frac{n^{n+1}}{n!}t^{\alpha-n-1}\int_0^\infty u^{n-\alpha}e^{-(n/t)u}\,du = \frac{n^\alpha}{n!}\Gamma(n+1-\alpha),$$

which tends to 1 ($n\to\infty$), since $\Gamma(u+1)\sim u^{u+1/2}e^{-u}\sqrt{2\pi}$ as $u\to\infty$ (see [T]). Thus, to obtain the assertion of the theorem, it is enough to check that

$$\mathcal{J}_n(t) := \mathcal{I}_n(t) - \frac{n^\alpha}{n!}\Gamma(n+1-\alpha)f(t) \to 0 \quad \text{as } n\to\infty.$$

To do so, set

$$G(s) := \int_t^s (f(u)-f(t))\,du = F(s)-F(t)-f(t)(s-t),$$

where $F(s) := \int_0^s f(u)\,du$ for $s\ge 0$. Then $\|F(s)\| \le \tilde{M}s^{\gamma+1}e^{\omega s}$ $(s\ge 0)$ for some $\tilde{M}>0$. This readily implies that the function G is exponentially bounded, that is, there exist some constants $\mu\ge 0$ and $C>0$ such that $\|G(s)\|\le Ce^{\mu s}$ for every $s\ge 0$. We may assume that $\mu\ge\omega$. On the other hand, the fact that t is a Lebesgue point of f implies that $\|G(s)\| = o(|s-t|)$ as $s\to t$.

By integration by parts, for $n > \max\{\mu t, \alpha\}$ we have

$$\begin{aligned}\mathcal{J}_n(t) &= \frac{n^{n+1}}{n!} t^{\alpha-n-1} \int_0^\infty u^{n-\alpha} e^{-(n/t)u} (f(u) - f(t))\, du \\ &= \frac{n^{n+1}}{n!} t^{\alpha-n-1} \int_0^\infty \left(\frac{nu^{n-\alpha}}{t} - (n-\alpha) u^{n-\alpha-1} \right) e^{-(n/t)u} G(u)\, du \\ &= \frac{n^{n+1}}{n!} t^{\alpha-n-1} \int_0^\infty \left(\frac{nu}{t} - (n-\alpha) \right) u^{n-\alpha-1} e^{-(n/t)u} G(u)\, du \\ &= \frac{n^{n+2}}{n!} \frac{1}{t} \int_0^\infty \left(y - \frac{n-\alpha}{n} \right) y^{n-\alpha-1} e^{-ny} G(ty)\, dy.\end{aligned}$$

Let now $\varepsilon > 0$ and choose $0 < \delta < 1$ such that if $|y-1| < \delta$ then

$$\frac{1}{t} \|G(ty)\| < \varepsilon |y-1|. \tag{2.3}$$

Divide $\mathcal{J}_n(t)$ into three integrals $\mathcal{J}_{1,n}(t)$, $\mathcal{J}_{2,n}(t)$ and $\mathcal{J}_{3,n}(t)$ whose intervals of integration are $(0, 1-\delta)$, $(1-\delta, 1+\delta)$ and $(1+\delta, \infty)$, respectively.

First, we are going to estimate $\mathcal{J}_{1,n}(t)$. Take $n > (\alpha+1)/\delta$. In this case, the function $y \mapsto y^{n-\alpha-1} e^{-ny}$ is increasing on $(0, 1-\delta)$, and therefore

$$\begin{aligned}\|\mathcal{J}_{1,n}(t)\| &\le \frac{n^{n+2}}{n!} \frac{1}{t} \int_0^{1-\delta} \left| y - \frac{n-\alpha}{n} \right| y^{n-\alpha-1} e^{-ny} \|G(ty)\|\, dy \\ &\le \frac{n^{n+2}}{n!} \frac{1}{t} (1-\delta)^{n-\alpha-1} e^{-n(1-\delta)} \int_0^{1-\delta} \|G(ty)\|\, dy =: a_n,\end{aligned}$$

where we have used that $\delta/(\alpha+1) \le (n-\alpha)/n - y < 1$ for all $y \in (0, 1-\delta)$. Then, by Stirling's formula,

$$a_n = O(n^{3/2}((1-\delta)e^\delta)^n) \quad \text{as } n \to \infty,$$

and therefore $a_n \to 0$ as $n \to \infty$, since $(1-\delta)e^\delta < 1$. Therefore, $\|\mathcal{J}_{1,n}(t)\| < \varepsilon$ for n large enough.

Now, applying to $\mathcal{J}_{2,n}(t)$ the estimate (2.3), we get

$$\begin{aligned}\|\mathcal{J}_{2,n}(t)\| &\le \varepsilon \frac{n^{n+2}}{n!} \int_{1-\delta}^{1+\delta} \left| y - \frac{n-\alpha}{n} \right| |y-1| y^{n-\alpha-1} e^{-ny}\, dy \\ &= \varepsilon \frac{n^{n+2}}{n!} \int_{1-\delta}^{1+\delta} \left| y - 1 + 1 - \frac{n-\alpha}{n} \right| |y-1| y^{n-\alpha-1} e^{-ny}\, dy \\ &\le \varepsilon \frac{n^{n+2}}{n!} \int_{1-\delta}^{1+\delta} \left((y-1)^2 + \left(1 - \frac{n-\alpha}{n} \right)(y+1) \right) y^{n-\alpha-1} e^{-ny}\, dy\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon \frac{n^{n+2}}{n!} \int_0^\infty \left(y^2 - \left(1 + \frac{n-\alpha}{n}\right) y + \left(2 - \frac{n-\alpha}{n}\right)\right) y^{n-\alpha-1} e^{-ny}\, dy \\
&= \varepsilon \frac{n^\alpha}{n!} \Bigg(\Gamma(n-\alpha+2) - \left(1 + \frac{n-\alpha}{n}\right) n\Gamma(n-\alpha+1) \\
&\qquad\qquad + \left(2 - \frac{n-\alpha}{n}\right) n^2 \Gamma(n-\alpha)\Bigg) \\
&= \varepsilon \frac{n^\alpha}{n!} (\Gamma(n-\alpha+1) + 2\alpha n\Gamma(n-\alpha)).
\end{aligned}
$$

Thus, the fact that $\lim_{n\to\infty} \frac{n^\beta}{n!}\Gamma(n-\beta+1) = 1$ for all $\beta \ge 0$ implies that

$$
\lim_{n\to\infty} \frac{n^\alpha}{n!}(\Gamma(n-\alpha+1) + 2\alpha n\Gamma(n-\alpha)) = 1 + 2\alpha.
$$

Hence, $\|\mathcal{J}_{2,n}(t)\| < 2(1+\alpha)\varepsilon$ for all sufficiently large n.

To estimate $\mathcal{J}_{3,n}(t)$, take $n_0 \in \mathbb{N}$ such that $n_0 > \mu t$ and let $n > n_0$. Thus, the function $y \mapsto y^{n-n_0-\alpha} e^{-(n-n_0)y}$ is decreasing on $(1+\delta, \infty)$. Then we have

$$
\begin{aligned}
\|\mathcal{J}_{3,n}(t)\| &\le \frac{n^{n+2}}{n!} \frac{1}{t} \int_{1+\delta}^\infty \left(y - \frac{n-\alpha}{n}\right) y^{n-\alpha-1} e^{-ny} \|G(ty)\|\, dy \\
&\le \frac{n^{n+2}}{n!} \frac{C}{t} \int_{1+\delta}^\infty y^{n-\alpha} e^{-ny} e^{\mu t y}\, dy \\
&= \frac{n^{n+2}}{n!} \frac{C}{t} \int_{1+\delta}^\infty y^{n-n_0-\alpha} e^{-(n-n_0)y} y^{n_0} e^{-(n_0-\mu t)y}\, dy \\
&\le \frac{n^{n+2}}{n!} \frac{C}{t} \frac{(1+\delta)^{n-n_0-\alpha}}{e^{(n-n_0)(1+\delta)}} \int_{1+\delta}^\infty y^{n_0} e^{-(n_0-\mu t)y}\, dy =: b_n.
\end{aligned}
$$

As before, Stirling's formula applies to show that $b_n \to 0$ as $n \to \infty$, and we find that $\|\mathcal{J}_{3,n}(t)\| < \varepsilon$ for large enough n. The proof is complete. ∎

Remark 2.4. There are some particular cases in which the inversion formula in Theorem 2.1 can be obtained as a consequence of Theorem 1.1—for example, when the function is the integral of order $\alpha > 0$ of a suitable function.

For $\alpha > 0$, set $j_\alpha(t) := t^{\alpha-1}\Gamma(\alpha)^{-1}$, $t > 0$. Let $g \in L^1_{\mathrm{loc}}([0,\infty); X)$ be an exponentially bounded function. Thus, $f := j_\alpha * g$ satisfies the assumptions of Theorem 2.1, where $*$ is the usual convolution on $\mathbb{R}^+$. Notice that $\lambda^\alpha \hat{f}(\lambda) = \hat{g}(\lambda)$ for appropriate complex values of λ. Therefore, by Theorem 1.1 and the dominated convergence theorem, for every $t > 0$,

$$f(t) = j_\alpha * \left(\lim_{n\to\infty} (-1)^n \frac{1}{n!}\left(\frac{n}{(\cdot)}\right)^{n+1} \hat{g}^{(n)}\left(\frac{n}{(\cdot)}\right)\right)(t)$$
$$= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{(-1)^n}{n!}\left(\frac{n}{s}\right)^{n+1} \hat{g}^{(n)}\left(\frac{n}{s}\right) ds$$
$$= \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{(-1)^n}{n!}\left(\frac{n}{s}\right)^{n+1} \frac{d^n}{d\lambda^n}(\lambda^\alpha \hat{f}(\lambda))\Big|_{\lambda=n/s} ds.$$

Thus the interest of Theorem 2.1 relies upon the fact that it provides an inversion formula for those functions $\varphi : (\omega,\infty) \to X$ which are not necessarily a Laplace transform, but such that $\lambda^{-\alpha}\varphi(\lambda)$ is a Laplace transform for some $\alpha > 0$; see [ABHN, Example 2.2.4]. Important classes of functions in this situation involve general α-times integrated semigroups or integrated cosine functions (see next section).

To end this section, we point out that there exists a well known version of Theorem 1.1 in which the Laplace–Stieltjes transform $\mathcal{L}_S$ of vector-valued Lipschitz continuous functions is considered. If $F : \mathbb{R}^+ \to X$ is a Lipschitz continuous function, that is,

$$\sup_{t,s\ge 0} \frac{\|F(t) - F(s)\|}{|t-s|} < \infty,$$

then the Laplace–Stieltjes transform of F is given by

$$\mathcal{L}_S(F)(\lambda) := -F(0) + \lambda \int_0^\infty e^{-\lambda t} F(t)\, dt$$

for those λ greater than the exponential growth bound of F. It follows from Theorem 1.1 that if $F : \mathbb{R}^+ \to X$ is a Lipschitz continuous function such that $F(0) = 0$ then

$$F(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} \frac{d^n}{d\lambda^n}\left(\frac{\mathcal{L}_S(F)(\lambda)}{\lambda}\right)\Big|_{\lambda=n/s}, \qquad t > 0.$$

See [ABHN, Theorem 2.3.1].

As a consequence of Theorem 2.1, we further obtain the following inversion formula for Laplace–Stieltjes transforms:

Corollary 2.5. *Let $F : \mathbb{R}^+ \to X$ be a Lipschitz continuous function such that $F(0) = 0$. Let $t > 0$. Then*

$$F(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \int_0^t \left(\frac{n}{s}\right)^{n+1} \frac{d^n}{d\lambda^n}(\mathcal{L}_S(F)(\lambda))\Big|_{\lambda=n/s} ds.$$

Proof. Under these assumptions, $\hat{F}(\lambda)=\lambda^{-1}\mathcal{L}_S(F)(\lambda)$ for λ large enough. Moreover, $\|F(t)\| \le Ct$ for every $t \ge 0$ and some $C > 0$. Then it suffices to apply Theorem 2.1 for $\alpha = 1$. ∎

3. Applications. We show here that Theorem 2.1 applies to α-times integrated semigroups and α-times integrated cosine families, obtaining in this way appropriate inversion formulae of Euler's type for these families.

3.1. Euler's exponential type formula for α-times integrated semigroups. Let X be a Banach space and let $\alpha > 0$. A strongly continuous family $(S_\alpha(t))_{t\ge0} \subseteq \mathcal{B}(X)$ of bounded operators on X is called an *α-times integrated semigroup* if $S_\alpha(0) = 0$ and

$$(3.1)\quad \Gamma(\alpha)S_\alpha(t)S_\alpha(s) = \int_t^{t+s} (t+s-r)^{\alpha-1}S_\alpha(r)\,dr - \int_0^{s}(t+s-r)^{\alpha-1}S_\alpha(r)\,dr$$

for every $s, t \ge 0$. Moreover, $(S_\alpha(t))_{t\ge0}$ is called *non-degenerate* if $S_\alpha(t)x = 0$ for all $t \ge 0$ implies $x = 0$. A 0-times integrated non-degenerate semigroup is a C_0-semigroup.

Assume that the function $S_\alpha : [0,\infty) \to \mathcal{B}(X)$ has a Laplace transform whenever $\lambda > \omega$ for some $\omega \in \mathbb{R}$. In this case, there exists a unique operator A on X satisfying $(\omega,\infty) \subseteq \rho(A)$ and such that

$$R(\lambda, A) := (\lambda - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t}S_\alpha(t)\,dt, \qquad \lambda > \omega.$$

Such an operator A is called the *generator* of $(S_\alpha(t))_{t\ge0}$. See [ABHN, H2] for the general theory of integrated semigroups.

COROLLARY 3.1. *Let $A : D(A) \subseteq X \to X$ be the generator of an α-times integrated semigroup $(S_\alpha(t))_{t\ge0}$ such that $\|S_\alpha(t)\| \le Ct^\gamma e^{\omega t}$, $t \ge 0$, for some $\gamma > \alpha - 1$ and $\omega \ge 0$. Then, for every $t > 0$ and every $x \in X$,*

$$S_\alpha(t)x = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left(\frac{n}{s}\right)^{n+1} R\left(\frac{n}{s}, A\right)^{n+1} x\,ds.$$

Proof. Let $x \in X$. Set $f(t) := S_\alpha(t)x$ for $t \ge 0$. Notice that f is continuous on $[0,\infty)$ since $(S_\alpha(t))_{t\ge0}$ is strongly continuous. By definition, $R(\lambda, A)x = \lambda^\alpha \hat{f}(\lambda)$ for λ large enough. Moreover, the resolvent equation gives us $((-1)^n/n!)(d^n/d\lambda^n)R(\lambda, A)x = R(\lambda, A)^{n+1}x$. Now, the claim follows directly from Theorem 2.1 since $\alpha \in (0, \gamma + 1)$. ∎

The above corollary extends previous results in this setting (see [C, Theorem 3.1] for n-times integrated semigroups, $n \in \mathbb{N}$, and [VV, Theorem 3.1] for exponentially bounded α-times integrated semigroups and $0 < \alpha < 1$), and provides a unified proof for them.

A large number of examples of α-times integrated semigroups satisfying the assumptions of Corollary 3.1 can be found in [H1].

3.2. α-times integrated cosine functions. Let X be a Banach space and let $\alpha > 0$. A strongly continuous family $(C_\alpha(t))_{t\geq 0} \subseteq \mathcal{B}(X)$ is an *α-times integrated cosine function* if $C_\alpha(0) = 0$ and

$$(3.2)\qquad 2\Gamma(\alpha)C_\alpha(t)C_\alpha(s) = \int_t^{t+s} (t+s-r)^{\alpha-1}C_\alpha(r)\,dr - \int_0^s (t+s-r)^{\alpha-1}C_\alpha(r)\,dr + \int_{t-s}^{t} (r-t+s)^{\alpha-1}C_\alpha(r)\,dr + \int_0^s (r+t-s)^{\alpha-1}C_\alpha(r)\,dr$$

for every $0 < s < t$. The family $(C_\alpha(t))_{t\geq 0}$ is called *non-degenerate* if $C_\alpha(t)x = 0$ for every $t \geq 0$ implies $x = 0$. If the Laplace transform of $C_\alpha(\cdot) : [0,\infty) \to \mathcal{B}(X)$ converges in (ω, ∞) for some $\omega \geq 0$, then there exists a unique operator A on X such that

$$\lambda R(\lambda^2, A) := \lambda(\lambda^2 - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t}C_\alpha(t)\,dt, \qquad \lambda > \omega.$$

See for example [EK, M2]. This operator A is called the *generator* of $(C_\alpha(t))_{t\geq 0}$. A 0-times integrated cosine function is the usual cosine function.

As a consequence of Theorem 2.1, one obtains the following result, which seems to be new.

Corollary 3.2. *Let $A : D(A) \subseteq X \to X$ be the generator of an α-times integrated cosine function $(C_\alpha(t))_{t\geq 0}$ for which there exist constants $\gamma > \alpha - 1$ and $\omega \geq 0$ satisfying $\|C_\alpha(t)\| \leq Ct^\gamma e^{\omega t}$ for $t \geq 0$. Then, for every $x \in X$ and $t > 0$,*

$$C_\alpha(t)x = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{(-1)^n}{n!} \left(\frac{n}{s}\right)^{n+1} \frac{d^n}{d\lambda^n}(\lambda R(\lambda^2, A))\bigg|_{\lambda=n/s} ds.$$

Proof. Similar to the proof of Corollary 3.1. ∎

Particular examples of generators of α-times integrated cosine functions are provided by those of α-times integrated semigroups. In fact, if an operator B on a Banach space is such that B and $-B$ are both generators of α-times integrated semigroups then $A = B^2$ is the generator of an α-times integrated cosine function; see [AK, EK]. In this case the explicit calculation of $(d^n/d\lambda^n)(\lambda R(\lambda^2, A))$ is simple:

$$\frac{d^n}{d\lambda^n}(\lambda R(\lambda^2, A)) = \frac{1}{2}[R(\lambda, -iB)^{n+1} + R(\lambda, iB)^{n+1}].$$

EXAMPLE 3.3. For $\alpha, t > 0$, the *Riesz kernel* is the function $R_t^{\alpha-1}$ defined by

$$R_t^{\alpha-1}(s) := \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\chi_{(0,t)}(s), \qquad s > 0.$$

These kernels play a central role in the study of Banach algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha e^{\omega t})$ of Sobolev type, which are in close relationship with α-times integrated semigroups and integrated cosine functions. Here, we are not concerned with these algebras, whose definition and first properties can be seen in [GM], for instance. Among these properties, we mention that the function $R_t^{\alpha-1}$ is a multiplier of the Banach algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha e^{\omega t})$ with respect to either the usual convolution product $*$ or the cosine convolution product $*_c$ on $\mathbb{R}^+$, which are given for $f, g \in \mathcal{T}_+^{(\alpha)}(t^\alpha e^{\omega t})$ by

$$f * g(t) := \int_0^t f(t-s)g(s)\,ds, \qquad t > 0,$$

and

$$f *_c g(t) := \frac{1}{2}\Big(f * g(t) + \int_t^\infty f(s-t)g(s)\,ds + \int_t^\infty g(s-t)f(s)\,ds\Big), \quad t > 0.$$

In both cases, as a multiplier, $\|R_t^{\alpha-1}\| \le Ct^\alpha e^{\omega t}$ $(t > 0)$.

In view of Theorem 2.1 we have the following.

COROLLARY 3.4. *Let $\alpha > 0$ and $\omega \ge 0$. Then for every $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha e^{\omega t})$ and $t > 0$ we have*

$$R_t^{\alpha-1} \bullet g = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left(\frac{n}{s}\right)^{n+1} e_{n/s}^{*(n+1)} \bullet g\,ds$$

in the norm of $\mathcal{T}_+^{(\alpha)}(t^\alpha e^{\omega t})$, where

$$e_\lambda^{*(n+1)}(r) = \frac{r^n}{n!}\,e_\lambda(r) \qquad (r \ge 0)$$

and $\bullet$ is either the usual convolution $*$ or the cosine convolution $*_c$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha e^{\omega t})$.

Proof. Note that for every $\lambda > \omega$ and $n \in \mathbb{N}$ one has

$$e_\lambda := e^{-\lambda(\cdot)} = \lambda^\alpha \int_0^\infty R_t^{\alpha-1} e^{-\lambda t}\,dt$$

and

$$\frac{(-1)^n}{n!}\,\frac{d^n}{d\lambda^n} e_\lambda = e_\lambda^{*(n+1)}.$$

Hence it is enough to take $f(t) = R_t^\alpha \bullet g$ in the formula of Theorem 2.1 to obtain the result. ∎

REMARK 3.5. The formula in the preceding corollary serves to illustrate Theorem 2.1 in a canonical situation, as regards α-times integrated semigroups. For simplicity, assume $\alpha > 1$. The equality

$$R_t^{\alpha-1}(r) = \lim_{n\to\infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(\frac{n}{s}\right)^{n+1} \frac{r^n}{n!}\, e^{-(n/s)r}\, ds, \qquad t > 0,$$

holds as a particular case of the fact that Theorem 1.1 remains true when one replaces functions like f with Dirac masses:

For $r > 0$,

$$\delta_r = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{\cdot}\right)^{n+1} \hat{\delta}_r^{(n)}\left(\frac{n}{\cdot}\right) = \lim_{n\to\infty} \left(\frac{n}{\cdot}\right)^{n+1} \frac{r^n}{n!} e^{-(n/\cdot)r}$$

in the sense of weak convergence of measures. In fact, for each continuous function F on $[0,\infty)$ with $\lim_{t\to\infty} F(t) = 0$ we have

$$F_n(r) := \frac{1}{n!} \int_0^\infty \left(\frac{n}{s}\right)^{n+1} r^n e^{-(n/s)r} F(s)\, ds = \frac{n^{n+1}}{n!} \int_0^\infty t^{n-1} e^{-nt} F\left(\frac{r}{t}\right) dt,$$

with

$$\frac{n^{n+1}}{n!} \int_0^\infty t^{n-1} e^{-nt}\, dt = 1.$$

Therefore

$$F_n(r) - F(r) = \frac{n^{n+1}}{n!} \int_0^\infty t^{n-1} e^{-nt} [F(r/t) - F(r)]\, dt,$$

and so by standard methods involving the partition of the integration domain $(0,\infty)$ into two parts $\{|t-1| \le \delta\}$ and $\{|t-1| > \delta\}$, for suitable small $\delta > 0$, one gets $\lim_{n\to\infty} F_n(r) = F(r)$. (In this connection, for the sake of completeness, let us point out that integration by parts gives us

$$\begin{aligned}\frac{n^{n+1}}{n!} \int_{1+\delta}^\infty t^{n-1} e^{-nt}\, dt &= \frac{1}{(n-1)!} \int_{n(1+\delta)}^\infty y^{n-1} e^{-y}\, dy \\ &= \sum_{k=1}^{n-1} \frac{(1+\delta)^{n-k} n^{n-k}}{(n-k)!} e^{-(1+\delta)n} + \int_{(1+\delta)n}^\infty e^{-y}\, dy,\end{aligned}$$

and this expression tends to 0 as $n \to \infty$ by the Stirling formula and the fact that $y \mapsto e^{-y}$ is integrable.)

Corollary 3.4 tells us that the above numerical limit holds indeed for convolution and in the norm of $\mathcal{T}_+^{(\alpha)}(t^\alpha e^{\omega t})$.

REMARK 3.6. If $S_\alpha(t)$ is an α-times integrated semigroup on a Banach space X satisfying

$$\|S_\alpha(t)\| \le Ct^\alpha \quad (t>0), \tag{3.3}$$

and

$$\lim_{t\to 0} \Gamma(\alpha+1)t^{-\alpha}S_\alpha(t)x = x \quad (x\in X), \tag{3.4}$$

then there exists a bounded Banach algebra homomorphism

$$\pi_\alpha\colon \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{B}(X)$$

with dense range, so that π_α extends to the multiplier algebra of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ with the usual convolution $*$, and we get $S_\alpha(t) = \pi_\alpha(R_t^{\alpha-1})$. Conversely, if we have a bounded Banach algebra homomorphism $\pi_\alpha\colon \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{B}(X)$ with dense range then $S_\alpha(t) := \pi_\alpha(R_t^{\alpha-1})$ is an α-times integrated semigroup satisfying (3.3) and (3.4).

For α-times integrated cosine functions there is a similar result, with the only difference that the homomorphism π_α has to be replaced with a homomorphism $\gamma_\alpha\colon \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{B}(X)$ with respect to the cosine convolution product $*_c$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

Actually, the family $(R_t^{\alpha-1})_{t\ge 0}$ is an α-times integrated semigroup and an α-times integrated cosine family since it satisfies the functional equations (3.1) and (3.2).

Notice that starting from Corollary 3.4, with a direct proof independent of Theorem 2.1, one can prove Corollary 3.1 and Corollary 3.2 (for $\gamma=\alpha$ and $\omega=0$) by just considering the image of $R_t^{\alpha-1}\bullet g$ and of its integral expression under the homomorphisms π_α and γ_α, respectively.

Thus it seems reasonable to consider the Riesz kernels as canonical integrated families, in the present setting. For all the above facts we refer the reader to [GM], [GMM] and [M2].

The authors would like to thank the referee for a careful reading of the text and valuable suggestions which have improved the paper.

This research has been partially supported by Projects MTM2010-16679 of the M.C.Y.T., Spain, and E-64 of the D.G. Aragón. The second author was also partially supported by a FPU Grant from the Ministry of Education of Spain.

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Received June 9, 2011
Revised version September 9, 2011 (7216)