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Ψ-pseudodifferential operators and estimates for maximal oscillatory integrals

by

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Abstract. We define a class of pseudodifferential operators with symbols $a(x,\xi)$ without any regularity assumptions in the x variable and explore their L^p boundedness properties. The results are applied to obtain estimates for certain maximal operators associated with oscillatory singular integrals.

1. Introduction. The theory of pseudodifferential operators, as developed by J. J. Kohn and L. Nirenberg [11] and L. Hörmander [8], has played a major role in the analysis of linear partial differential operators. Recall that a pseudodifferential operator is an operator given by

$$a(x,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}\,d\xi,$$

whose symbol $a(x,\xi)$ is assumed to be smooth in both spatial (x) and frequency (ξ) variables and satisfies certain growth conditions. An example is the $S^m_{\varrho,\delta}$ symbol class introduced in Hörmander [9], consisting of $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $m \in \mathbb{R}$, $0 \le \delta \le \varrho \le 1$.

An important problem in partial differential equations and harmonic analysis is the question of L^p boundedness of pseudodifferential operators, which has been extensively studied. Hörmander [9] showed that for $0 \le \delta < \varrho < 1$, operators with symbols in $S^{n(\varrho-1)/2}_{\varrho,\delta}$ are bounded on L^p for $1 < p < \infty$. A. Calderón and R. Vaillancourt [1] proved that if $a \in S^0_{\varrho,\varrho}$ for $0 \le \varrho < 1$, then the corresponding pseudodifferential operator is bounded on L^2. In a set of unpublished lecture notes, E. M. Stein showed that if $a(x,\xi) \in S^m_{\varrho,\delta}$

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and either $0 \le \delta < \varrho = 1$ or $0 < \delta = \varrho < 1$, then $a(x,D)$ is of weak type $(1,1)$ if $m = (\varrho - 1)n/2$. Furthermore, $a(x,D)$ is L^p bounded $(1 < p < \infty)$ if $(\varrho - 1)|1/2 - 1/p| \ge m/n$ (see also Stein [16]). In [5], C. Fefferman proved that if $a \in S^{n(\varrho-1)/2}_{\varrho,\delta}$ for $0 \le \delta < \varrho < 1$ then $a(x,D)$ is bounded from L^∞ to BMO.

There has also been work on operators with limited regularity and we wish to name only a couple of authors which have inspired us in our investigation. H. Kumano-go [12], Hörmander [10] and C. H. Ching [2] gave examples of bounded symbols which satisfy

$$|\partial_\xi^\alpha a(x,\xi)| \le C_\alpha \langle \xi \rangle^{-\varrho|\alpha|},$$

with $\varrho \le 1$, but the corresponding operator is not bounded on L^2. On the other hand, Nagase [14] was able to show that if $|\partial_\xi^\alpha a(x,\xi)| \le C_\alpha \langle \xi \rangle^{-|\alpha|}$, and $|\partial_\xi^\alpha a(x,\xi) - \partial_\xi^\alpha a(y,\xi)| \le C_\alpha |x-y|^\sigma \langle \xi \rangle^{-|\alpha|+\sigma\tau}$ for $|\alpha| \le n+1$, with $0 < \sigma \le 1$ and $0 \le \tau < 1$, then the associated pseudodifferential operator $a(x,D)$ is L^2 bounded. Later, R. Coifman and Y. Meyer [3] proved that if $a(x,\xi)$ satisfies the first condition of Nagase's theorem and

$$|\partial_\xi^\alpha a(x,\xi) - \partial_\xi^\alpha a(y,\xi)| \le C_\alpha \omega(|x-y|) \langle \xi \rangle^{-|\alpha|},$$

where ω satisfies the condition $\sum_{j=0}^\infty \omega(2^{-j})^2 < \infty$, then the L^p boundedness holds for $1 < p < \infty$.

In this paper we introduce the class of Ψ-pseudodifferential operators. The symbols that we are considering satisfy

$$|\partial_\xi^\alpha a(x,\xi)| \le C_\alpha \langle \xi \rangle^{m-\varrho|\alpha|}$$

for certain values of m and ϱ, but have no regularity assumption in the spatial variables x. Because of this lack of regularity, the corresponding operators do not have the required pseudo-local property. This means that these operators do not necessarily decrease the singular support when they act on distributions. Hence the name of Ψ-pseudodifferential operator is justified since these operators only look like pseudodifferential operators but they are "pseudo-pseudodifferential" operators. We establish the boundedness of these operators in certain L^p spaces and we show how they can be used as a tool in proving boundedness results for certain maximal operators in harmonic analysis.

In Section 2, we define $\Psi\Psi$DOs and discuss their boundedness. In Section 3, we give some applications to estimates for local maximal operators of oscillatory singular integrals of the type considered by S. Wainger [19], I. I. Hirschman [7], C. Fefferman [4], Fefferman and Stein [6] and A. Miyachi [13]. We also prove a theorem similar to P. Sjölin's [15] for oscillatory singular integrals but for the kernels considered in [4]. The proof of Sjölin's result is based on the multidimensional version of Carleson–Hunt's theorem, [15], but for the oscillatory singular kernels that we are considering, we can avoid the use of this difficult result by using $\Psi\Psi$DOs. To some extent, this can be viewed as a related investigation to that of Stein and Wainger [18]. The difference between our and their result is that they establish the L^2 boundedness of the global maximal operators associated to oscillatory singular integrals with polynomial phases where the degree of the polynomial is larger than or equal to 2 and the kernel is Calderón–Zygmund, while we prove the L^p boundedness ($2 \le p \le \infty$) of local maximal operators associated to oscillatory singular integrals with linear phases and Hirschman–Wainger kernels [7], [19].

The difference between local and global maximal operators is that in the local case the supremum is taken over a compact set but in the global case one takes the supremum over an unbounded open set.

2. Ψ-pseudodifferential operators and their L^p boundedness. In this section we define the Ψ-pseudodifferential operators which are crucial in all our further investigations.

DEFINITION 2.1. Let $a(x,\xi) \in C^\infty(\mathbb{R}^n_\xi)$ be a measurable function in x and let

$$\|\partial_\xi^\alpha a(x,\xi)\|_{L^\infty_x} \le C_\alpha \langle \xi \rangle^{m-\varrho|\alpha|}$$

for some $m \in \mathbb{R}$, $\varrho \le 1$ and some constant C_α. We denote by $L^\infty S^m_\varrho$ the class of symbols with this property.

Given this class of symbols we make the following definition.

DEFINITION 2.2. A *Ψ-pseudodifferential operator* ($\Psi\Psi$DO for short) is an operator $a(x,D)$ which is given by

$$a(x,D)f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} a(x,\xi)\widehat{f}(\xi)e^{i\langle x,\xi\rangle}\,d\xi, \qquad f \in \mathscr{S}.$$

Thus we do not assume any regularity in the spatial variable x and this will cause singularities for the Schwartz kernel $K(x,y)$ of the operator $a(x,D)$ which might go beyond the set $\{(x,x);\, x \in \mathbb{R}^n\}$. Our main concern, however, is the question of L^p boundedness of the $\Psi\Psi$DOs. To this end we start with the following:

PROPOSITION 2.3. *Let $a(x,\xi) \in L^\infty S^m_\varrho$, $0 \le \varrho \le 1$. Assume that $m < n(\varrho-1)/p$ and $1 \le p \le 2$. Then the operator $a(x,D)$ is bounded from L^p to L^p.*

Proof. We want to show that

$$\|a(x,D)u\|_{L^p} \lesssim \|u\|_{L^p}. \tag{2.1}$$

Here and below, $a \lesssim b$ means $a \leq Cb$ for some constant C. We will also denote all generic constants by C even though they might be different from line to line. Our strategy in proving the theorem is to use a Littlewood–Paley decomposition of the symbol $a(x,\xi)$. So let $\{\varphi_k\} \in C_0^\infty(\mathbb{R}^n)$ be a Littlewood–Paley partition of unity with $\operatorname{supp}\varphi_0 \subset \{\xi;\, |\xi| \leq 2\}$ and $\operatorname{supp}\varphi_k \subset \{\xi;\, |\xi| \sim 2^k\}$ for $k \geq 1$.

Furthermore, for all multi-indices α,

$$|\partial_\xi^\alpha \varphi_0(\xi)| \leq c_{\alpha,N}\langle\xi\rangle^{-N} \quad \text{for all } N, \tag{2.2}$$

$$|\partial_\xi^\alpha \varphi_k(\xi)| \leq c_\alpha 2^{-k|\alpha|} \quad \text{for some } c_\alpha > 0 \text{ and } k = 1, 2, \ldots, \tag{2.3}$$

and

$$\varphi_0(\xi) + \sum_{k=1}^{\infty} \varphi_k(\xi) = 1, \quad \forall\xi. \tag{2.4}$$

Using this, we decompose the symbol $a(x,\xi)$ as

$$a(x,\xi) = a_0(x,\xi) + \sum_{k=1}^{\infty} a_k(x,\xi) \tag{2.5}$$

with $a_k(x,\xi) = a(x,\xi)\varphi_k(\xi)$, $k = 0, 1, \ldots$.

We shall proceed by estimating the L^p norm of each term separately. So the first step is to establish the L^p boundedness of

$$a_0(x,D)u(x) = \frac{1}{(2\pi)^n}\int a_0(x,\xi)\widehat{u}(\xi)e^{i\langle x,\xi\rangle}\, d\xi. \tag{2.6}$$

By the definition of the symbol class $L^\infty S^m_\varrho$, inequality (2.2), and the Leibniz rule, we realize that

$$|\partial_\xi^\alpha a_0(x,\xi)| \leq c_{\alpha,N}\langle\xi\rangle^{-N} \quad \text{for all } N, \tag{2.7}$$

and all multi-indices α. Hence if we look at the Schwartz kernel of $a_0(x,D)$

$$K(x,y) = \frac{1}{(2\pi)^n}\int a_0(x,\xi)e^{i\langle x-y,\xi\rangle} d\xi, \tag{2.8}$$

and integrate by parts, we obtain, for all integers $M > 0$,

$$|K(x,y)| \lesssim \langle x-y\rangle^{-2M}\int |(1-\Delta_\xi)^M a_0(x,\xi)|\, d\xi. \tag{2.9}$$

Using (2.7) and choosing $M > n/2$, we see that both integrals $\int |K(x,y)|\, dx$ and $\int |K(x,y)|\, dy$ are convergent and Schur's lemma yields

$$\|a_0(x,D)u\|_{L^p} \lesssim \|u\|_{L^p}. \tag{2.10}$$

Now let us analyze $a_k(x,D)u(x) = (2\pi)^{-n}\int a_k(x,\xi)\widehat{u}(\xi)e^{i\langle x,\xi\rangle} d\xi$ for $k \geq 1$. Using the definition of $L^\infty S^m_\varrho$, inequality (2.3) and the Leibniz rule we realize that

$$|\partial_\xi^\alpha a_k(x,\xi)| \leq c_\alpha 2^{k(m-\varrho|\alpha|)} \quad \text{for some } c_\alpha > 0 \text{ and } k = 1, 2, \ldots \tag{2.11}$$

where we have also used the assumption $\varrho \le 1$. We note that $a_k(x,D)u(x)$ can be written as

$$a_k(x,D)u(x) = \int K_k(x,y)u(x-y)\,dy \tag{2.12}$$

with

$$K_k(x,y) = \frac{1}{(2\pi)^n}\int a_k(x,\xi)e^{i\langle y,\xi\rangle}\,d\xi = \check{a}_k(x,y), \tag{2.13}$$

where $\check{a}_k$ denotes the inverse Fourier transform of $a_k(x,\xi)$ with respect to ξ. One observes that

$$\begin{aligned} \|a_k(x,D)u\|_{L^p}^p &= \int\Big|\int K_k(x,y)u(x-y)\,dy\Big|^p\,dx \\ &= \int\Big|\int K_k(x,y)\sigma_k(y)\,\frac{1}{\sigma_k(y)}\,u(x-y)\,dy\Big|^p\,dx, \end{aligned} \tag{2.14}$$

with weight functions $\sigma_k(y)$ which will be chosen in a moment. Therefore, Hölder's inequality yields

$$\begin{aligned} &\|a_k(x,D)u\|_{L^p}^p \\ &\quad\le \int\Big\{\int|K_k(x,y)|^{p'}|\sigma_k(y)|^{p'}\,dy\Big\}^{p/p'}\Big\{\int\frac{|u(x-y)|^p}{|\sigma_k(y)|^p}\,dy\Big\}\,dx \end{aligned} \tag{2.15}$$

where $1/p + 1/p' = 1$. Now for an $l > n/p$, we define a class of weight functions by setting

$$\sigma_k(y) = \begin{cases} 2^{-k\varrho n/p}, & |y| \le 2^{-k\varrho}, \\ 2^{-k\varrho(n/p-l)}|y|^l, & |y| > 2^{-k\varrho}. \end{cases} \tag{2.16}$$

By Hausdorff–Young's theorem and the estimate (2.11), first for $\alpha = 0$ and then for $|\alpha| = l$, we have

$$\begin{aligned} \int 2^{-kp'\varrho n/p}|K_k(x,y)|^{p'}\,dy &\le 2^{-kp'\varrho n/p}\Big\{\int|a_k(x,\xi)|^p d\xi\Big\}^{p'/p} \\ &\lesssim 2^{-kp'\varrho n/p}\Big\{\int\limits_{|\xi|\sim 2^k} 2^{pmk}\,d\xi\Big\}^{p'/p} \\ &\lesssim 2^{kp'(m-n(\varrho-1)/p)}, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} &\int 2^{-k\varrho p'(n/p-l)}|K_k(x,y)|^{p'}|y|^{p'l}\,dy \\ &\quad\lesssim 2^{-k\varrho p'(n/p-l)}\Big\{\int|\nabla_\xi^l a_k(x,\xi)|^p\,d\xi\Big\}^{p'/p} \\ &\quad\lesssim 2^{-k\varrho p'(n/p-l)}\Big\{\int\limits_{|\xi|\sim 2^k} 2^{kp(m-\varrho l)}\,d\xi\Big\}^{p'/p} \lesssim 2^{kp'(m-n(\varrho-1)/p)}. \end{aligned} \tag{2.18}$$

Hence, splitting the integral into $|y| \leq 2^{-k\varrho}$ and $|y| > 2^{-k\varrho}$ yields

$$\Big\{\int |K_k(x,y)|^{p'} |\sigma_k(y)|^{p'}\, dy\Big\}^{p/p'} \lesssim \{2^{kp'(m-n(\varrho-1)/p)}\}^{p/p'} = 2^{kp(m-n(\varrho-1)/p)}. \tag{2.19}$$

Furthermore if we choose $l > n/p$ then

$$\int \frac{dy}{|\sigma_k(y)|^p} = 2^{k\varrho n} \int_{|y|\leq 2^{-k\varrho}} dy + 2^{k\varrho p(n/p-l)} \int_{|y|>2^{-k\varrho}} |y|^{-pl}\, dy = C(n), \tag{2.20}$$

a constant that only depends on the dimension n. Thus (2.14) yields

$$\|a_k(x,D)u\|_{L^p} \lesssim 2^{k(m-n(\varrho-1)/p)} \|u\|_{L^p}. \tag{2.21}$$

Summing up and using the bounds for $a_0(x,D)$ and $a_k(x,D)$, we obtain

$$\begin{aligned} \|a(x,D)u\|_{L^p} &\leq \|a_0(x,D)u\|_{L^p} + \sum_{k=1}^{\infty} \|a_k(x,D)u\|_{L^p} \\ &\lesssim \|u\|_{L^p} + \sum_{k=1}^{\infty} 2^{k(m-n(\varrho-1)/p)} \|u\|_{L^p} \end{aligned} \tag{2.22}$$

We observe that the series above converges if $m < n(\varrho-1)/p$. This ends the proof of the proposition.

Remark 2.4. If $0 \leq \varrho \leq 1$ then the L^2 boundedness result is sharp. In fact, following the example of Hörmander [10] one can see that if $m \geq n(\varrho-1)/2$ then there are symbols $a(x,\xi)$ in the Hörmander class $S^m_{\varrho,1}$ whose corresponding Kohn–Nirenberg quantization is not bounded on L^2. Since obviously $S^m_{\varrho,1} \subset L^\infty S^m_\varrho$, it follows at once that our L^2 result is sharp.

The condition $m < n(\varrho-1)/2$ also guarantees L^∞ boundedness. More precisely, we have

Proposition 2.5. *Let $a(x,\xi) \in L^\infty S^m_\varrho$. Assume that $m < n(\varrho-1)/2$ and $0 \leq \varrho \leq 1$. Then the operator $a(x,D)$ is bounded from L^∞ to L^∞.*

Proof. We will use the same technique as in the proof of Proposition 2.3. Observing that for $u \in L^\infty$,

$$\|a_k(x,D)u\|_{L^\infty} \leq \|u\|_{L^\infty} \int |K_k(x,y)|\, dy, \tag{2.23}$$

one only needs to estimate $\int |K_k(x,y)|\, dy$. We split the integral into

$$\begin{aligned} \int |K_k(x,y)|\, dy &= \int_{|y|\leq 2^{-k\varrho}} |K_k(x,y)|\, dy + \int_{|y|>2^{-k\varrho}} |K_k(x,y)|\, dy \\ &=: I_1 + I_2. \end{aligned} \tag{2.24}$$

To estimate I_1 we use the Cauchy–Schwarz inequality and (2.17) for the case $p = p' = 2$. Hence

$$I_1 \le \Big\{ \int_{|y|\le 2^{-k\varrho}} dy \Big\}^{1/2} \Big\{ \int |K_k(x,y)|^2\, dy \Big\}^{1/2} \lesssim 2^{k(m-n(\varrho-1)/2)}. \tag{2.25}$$

To estimate I_2 we use again the Cauchy–Schwarz inequality and (2.18) for the case $p = p' = 2$. This yields

$$\begin{aligned} I_2 &\le \Big\{ \int_{|y|>2^{-k\varrho}} |y|^{-2l}\, dy \Big\}^{1/2} \Big\{ \int |K_k(x,y)|^2 |y|^{2l}\, dy \Big\}^{1/2} \\ &\lesssim 2^{k(m-n(\varrho-1)/2)}. \end{aligned} \tag{2.26}$$

Thus

$$\|a_k(x,D)u\|_{L^\infty} \lesssim 2^{k(m-n(\varrho-1)/2)} \|u\|_{L^\infty} \tag{2.27}$$

for $k = 1, 2, \ldots$. Summing up and using the hypothesis on m and the bounds for $a_0(x,D)$ and $a_k(x,D)$, we obtain

$$\|a(x,D)u\|_{L^\infty} \lesssim \|u\|_{L^\infty} + \sum_{k=1}^{\infty} 2^{k(m-n(\varrho-1)/2)} \|u\|_{L^\infty} \lesssim \|u\|_{L^\infty}, \tag{2.28}$$

as desired.

Remark 2.6. For $0 \le \varrho \le 1$ the L^∞ result in Proposition 2.5 is sharp. In fact, if we consider the symbol $a(\xi) = \varphi(\xi)|\xi|^m e^{i|\xi|^{1-\varrho}} \in L^\infty S^m_\varrho$, where $\varphi = 0$ near zero and $\varphi = 1$ for large ξ, then it is known that the operator associated to this symbol does not map L^∞ to L^∞ if $m \ge n(\varrho-1)/2$ (see e.g. Miyachi [13]).

A consequence of the previous propositions is

Theorem 2.7. *Let $a(x,\xi) \in L^\infty S^m_\varrho$, $0 \le \varrho \le 1$. Then if $m < n(\varrho-1)/2$ then $a(x,D)$ is a bounded operator from L^p to L^p for all $p \in [2,\infty]$. For $\varrho = 1$ and $m < 0$ the range of p for which the operator is L^p bounded is $[1,\infty]$.*

Proof. This follows by interpolating the L^2 result of Proposition 2.3 with the L^∞ result of Proposition 2.5. The last claim follows from the interpolation between the L^1 result of Proposition 2.3 and L^∞ boundedness result of Proposition 2.5, which are valid for $m < 0$ and $\varrho = 1$.

We conclude this section by noting that the number of derivatives needed in the ξ variables need not be infinite. Indeed, by following our proofs one can reduce the number of derivatives to a finite number depending on the dimension n.

3. Applications to maximal operator estimates. In this section we consider certain maximal functions associated with strongly singular integral operators and oscillatory integrals. In each case we can establish boundedness in L^p for the local maximal operators.

Let us first consider singular integrals that are given by convolution with the distribution on $\mathbb{R}^n$ that away from the origin agrees with the function

$$K^{a,b}(x) = \frac{e^{i|x|^a}}{|x|^{n+b}}\,\varphi(x) \tag{3.1}$$

with $b \in \mathbb{R}$, $a \in (-\infty, 0] \cup (1, \infty)$ and φ a radial smooth compactly supported function, equal to 1 in the unit ball. Consider the strongly singular integral operator $Tf(x) = K^{a,b}(x) * f(x)$. This operator is bounded on L^p, provided that $a < 0$, $b < -an/2$, $1 < p < \infty$ and $|1/2 - 1/p| \le (an/2 + b)/an$ (see Hirschman [7], Wainger [19], Stein [17], C. Fefferman [4] and Miyachi [13]). Setting

$$\beta = \frac{an/2 + b}{a - 1} \quad \text{and} \quad K^{a,b}_t(x) = \frac{1}{t^{n-\beta}}\,K^{a,b}\left(\frac{x}{t}\right),$$

we are interested in the L^p estimates for the maximal operator

$$T_* f(x) := \sup_{0<t<1} |(K^{a,b}_t * f)(x)|. \tag{3.2}$$

Using the Fourier transform, one can write T_* as

$$T_* f(x) := \frac{1}{(2\pi)^n} \sup_{0<t<1} \left| \int (K^{a,b}_t)^\wedge(\xi) \widehat{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi \right|. \tag{3.3}$$

Since

$$(K^{a,b})^\wedge(\xi) = \frac{e^{i|\xi|^\alpha}}{|\xi|^\beta}\,\theta(\xi) \quad \text{with} \quad \alpha = \frac{a}{a-1},\ \beta = \frac{an/2 + b}{a - 1}$$

and $\theta(\xi)$ is a smooth function on $\mathbb{R}^n$, which vanishes near zero, and equals 1 outside a bounded set (this is essentially the case), the Fourier transform of $K^{a,b}_t$ is

$$(K^{a,b}_t)^\wedge(\xi) = \frac{e^{i|t\xi|^\alpha}}{|\xi|^\beta}\,\theta(t\xi). \tag{3.4}$$

Therefore to estimate T_* it is enough to estimate

$$\int \frac{e^{i|t(x)\xi|^\alpha}}{|\xi|^\beta} \theta(t(x)\xi) \widehat{f}(\xi) e^{i\langle x,\xi\rangle}\, d\xi$$

for an arbitrary measurable function $t(x) \in [0, 1]$. So we are dealing with estimates for a $\Psi\Psi DO$ with symbol

$$\frac{e^{i|t(x)\xi|^\alpha}}{|\xi|^\beta} \theta(t(x)\xi) \in L^\infty S^{-\beta}_{1-\alpha} = L^\infty S^{-(an/2+b)/(a-1)}_{1/(1-a)}.$$

Thus we have the following

THEOREM 3.1.

(i) *If* $a > 1$ *and* $b > a(n/p - n/2)$ *or* $a < 0$ *and* $b < a(n/p - n/2)$ *then* $T_* : L^p \to L^p$ *is bounded for* $1 \le p \le 2$.

(ii) *If* $a > 1$ *and* $b > 0$ *or* $a < 0$ *and* $b < 0$ *then* $T_* : L^p \to L^p$ *is bounded for* $2 \le p \le \infty$.

Proof. (i) follows from Proposition 2.3 and (ii) follows from Theorem 2.7.

Next let

$$T_\lambda^{a,b}(f)(x) = \int_{\mathbb{R}^n} e^{i\langle\lambda,y\rangle} K^{a,b}(y) f(x-y)\,dy, \tag{3.5}$$

with $\lambda \in \mathbb{R}^n$. It was shown by Sjölin [15] that for an appropriate Calderón–Zygmund kernel K, the operator

$$Cf(x) := \sup_{\lambda\in\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} e^{i\langle\lambda,y\rangle} K(y) f(x-y)\,dy \Big|$$

is bounded in $L^2(\mathbb{R}^n)$.

For $l < 1$ we define the maximal operator $T^* f(x) := \sup_{|\lambda|<l} |T_\lambda f(x)|$. Then we have

THEOREM 3.2. *T^* is bounded from $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $2 \le p \le \infty$, provided $a > 1$, $b > 0$.*

Proof. Once again, after linearization and using the Fourier transform, we are led to study a ΨΨDO

$$T(x,D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i|\xi-\lambda(x)|^\alpha}}{|\xi-\lambda(x)|^\beta}\, \theta(\xi-\lambda(x)) e^{i\langle x,\xi\rangle} \widehat{f}(\xi)\,d\xi \tag{3.6}$$

where $\lambda(x)$ is an arbitrary measurable function with $|\lambda(x)| < l < 1$, and α, β, θ as defined previously. A calculation shows that

$$\Big| \partial_\xi^\gamma \Big\{ \frac{e^{i|\xi-\lambda(x)|^\alpha}}{|\xi-\lambda(x)|^\beta} \theta(\xi-\lambda(x)) \Big\} \Big| \le C_{\gamma,l} \langle\xi\rangle^{\frac{an/2+b}{1-a} - \frac{|\gamma|}{1-a}} \quad \text{for } |\lambda| < l.$$

Now to prove the claim, apply Theorem 2.7.

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