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Unbounded $*$-representations of tensor product locally convex $*$-algebras induced by unbounded C^*-seminorms

by

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Abstract. The existence of unbounded $*$-representations of (locally convex) tensor product $*$-algebras is investigated, in terms of the existence of unbounded $*$-representations of the (locally convex) factors of the tensor product and vice versa.

1. Introduction. The study of (unbounded) $*$-representations is motivated by the Wightman formulation of quantum field theory and the representation theory of Lie algebras (see, for instance, [20]). In the Wightman formulation of quantum field theory, one assumes that the "smeared fields $\Phi(f)$" generate a $*$-algebra and that a field theory is a cyclic $*$-representation of this algebra satisfying some additional assumptions like Lorentz invariance and local commutativity (see [14, p. 88]). A question that researchers in the theory of $*$-representations often face is the following: under which conditions could one obtain the existence of well-behaved $*$-representations, in the sense that potential pathologies of the unbounded operators involved could be ruled out?

T. V. Powers introduced and studied in [14] and [15] a class of well-behaved self-adjoint $*$-representations of commutative $*$-algebras, called standard $*$-representations. In 2001 resp. 2002, S. J. Bhatt, A. Inoue and H. Ogi (see [7]) resp. K. Schmüdgen (see [19]) introduced independently a class of well-behaved $*$-representations in the sense mentioned above. S. J. Bhatt, A. Inoue and H. Ogi studied well-behaved $*$-representations by introducing the so-called "unbounded C^*-seminorms" (see Section 2). K. Schmüdgen studied well-behaved $*$-representations associated with a "compatible pair" $(\mathcal{A}, \mathcal{X})$ consisting of a $*$-algebra $\mathcal{A}$ with identity and a normed $*$-algebra $\mathcal{X}$ (not necessarily having an identity) which is an $\mathcal{A}$-module, so that a certain left action of $\mathcal{A}$ is defined on $\mathcal{X}$. A. Inoue re-

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lated in [11] the two concepts of well-behaved $*$-representations introduced in [7] and [19]. Moreover, in a series of joint papers (see, e.g., [3–6]) the well-behaved $*$-representations defined in [7] were studied in various ways.

The present paper aims to investigate the existence of unbounded $*$-representations of (topological) tensor product $*$-algebras in terms of unbounded C^*-seminorms and unbounded m^*-seminorms. In this aspect, the known theory of bounded $*$-representations of m^*-convex tensor product algebras (see, e.g., [9, Chapter VII]), the properties of the enveloping locally C^*-algebra (enveloping pro-C^*-algebra) of the latter (ibid.), as well as the methods developed in [7] and in [4, 5] play a significant rôle.

More precisely, Section 2 deals with the background material. In Section 3 the existence of unbounded $*$-representations of a tensor product $*$-algebra $\mathcal{A} \otimes \mathcal{B}$ is guaranteed by giving unbounded C^*-seminorms p, q on $\mathcal{A}$ and $\mathcal{B}$ resp. and vice versa (see Propositions 3.1, and 3.2). In Section 4 we construct well-behaved $*$-representations of $\mathcal{A} \otimes \mathcal{B}$ (as before) from given ones of $\mathcal{A}, \mathcal{B}$ and vice versa, using the so-called w-semifinite unbounded C^*-seminorms. Section 5 deals with the problem of Section 4 on Fréchet locally convex tensor product $*$-algebras using "naturally" defined unbounded m^*-seminorms on Fréchet locally convex $*$-algebras. We close this study with some applications and some comments concerning further investigation of this kind of problems.

2. Preliminaries. Throughout this paper we deal with complex associative algebras. All topological algebras we consider (save, of course, the seminormed ones) are supposed to be Hausdorff topological spaces.

An *unbounded* m^*-*(semi)norm* resp. C^*-*(semi)norm* on a $*$-algebra $\mathcal{A}$ is a submultiplicative $*$-(semi)norm resp. C^*-(semi)norm p defined on a $*$-subalgebra $\mathfrak{D}(p)$ of $\mathcal{A}$. Each unbounded C^*-(semi)norm is an unbounded m^*-(semi)norm (see [17]). Various examples can be found in [7, Section 7]. If p is an unbounded m^*-seminorm resp. unbounded C^*-seminorm on a $*$-algebra $\mathcal{A}$, the set

$$N_p \equiv \ker(p) = \{x \in \mathfrak{D}(p) : p(x) = 0\}$$

is a $*$-ideal in $\mathfrak{D}(p)$, while the set

$$\mathcal{I}_p = \{x \in \mathfrak{D}(p) : ax \in \mathfrak{D}(p)\ \forall a \in \mathcal{A}\} \tag{2.1}$$

is the largest left ideal of $\mathcal{A}$ contained in $\mathfrak{D}(p)$. A key tool for the construction of an unbounded $*$-representation of $\mathcal{A}$ in terms of an unbounded C^*-seminorm p on $\mathcal{A}$ is the condition $\mathcal{I}_p \not\subset N_p$ (cf. [7, Remark 2.3]). If $\mathcal{C}(\mathbb{R})$ is the $*$-algebra of all continuous functions on $\mathbb{R}$, and $\mathcal{C}_{\mathrm{b}}(\mathbb{R})$ its $*$-subalgebra consisting of all bounded continuous functions on $\mathbb{R}$, then the supremum norm $\|\cdot\|_\infty$ on $\mathcal{C}_{\mathrm{b}}(\mathbb{R})$ is an unbounded C^*-(semi)norm on $\mathcal{C}(\mathbb{R})$, and $\mathcal{I}_{\|\cdot\|_\infty} = \mathcal{C}_{\mathrm{c}}(\mathbb{R})$, the algebra of all continuous functions on $\mathbb{R}$ with compact support.

For $\mathcal{A}$ and p as above, denote by $\mathcal{A}_p$ the Banach $*$-algebra resp. C^*-algebra which is the completion of the normed $*$-algebra $(\mathfrak{D}(p)/N_p)[\|\cdot\|_p]$ under the m^*-norm resp. C^*-norm $\|x+\ker p\|_p := p(x)$, $x \in \mathfrak{D}(p)$, induced by p. We shall use the notation $x_p = x + \ker p$, $x \in \mathfrak{D}(p)$. If p is an unbounded C^*-seminorm p on $\mathcal{A}$, then $\mathcal{A}_p$ as a C^*-algebra (the *enveloping C^*-algebra* of the C^*-seminormed algebra $\mathfrak{D}(p)[p]$) has a (bounded) faithful $*$-representation Π_p on a Hilbert space $\mathcal{H}_{\Pi_p}$. It is shown in [7, Proposition 2.2] that Π_p gives rise to an unbounded $*$-representation π_p of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\pi_p}$ such that $\|\overline{\pi_p(x)}\| \le p(x)$ for all $x \in \mathfrak{D}(p)$ and $\|\overline{\pi_p(x)}\| = p(x)$ for all $x \in \mathcal{I}_p$. From the definition of π_p (ibid.), it follows that π_p is nontrivial, that is, $\mathcal{H}_{\pi_p} \neq \{0\}$, if and only if $\mathcal{I}_p \not\subset N_p$. Examples of unbounded C^*-seminorms that satisfy this condition are given in [6, Section 6]. Based on the above, we fix the following notation:

$$\mathrm{Rep}(\mathcal{A}_p) = \{\text{all faithful nondegenerate } *\text{-representations } \Pi_p \text{ of } \mathcal{A}_p\},$$
$$\mathrm{Rep}(\mathcal{A}, p) = \{\pi_p : \Pi_p \in \mathrm{Rep}(\mathcal{A}_p)\},$$

i.e., $\mathrm{Rep}(\mathcal{A}, p)$ denotes all nontrivial $*$-representations π_p of $\mathcal{A}$, deriving from the elements Π_p of $\mathrm{Rep}(\mathcal{A}_p)$, and

$$\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p) = \{\pi_p \in \mathrm{Rep}(\mathcal{A}, p) : \mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}\}.$$

An unbounded C^*-seminorm p on $\mathcal{A}$ is called *weakly semifinite* (briefly, *w-semifinite*) if $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p) \neq \emptyset$; an element π_p in $\mathrm{Rep}^{\mathrm{WB}}(\mathcal{A}, p)$ is called a *well-behaved $*$-representation* of $\mathcal{A}$ defined by p (see [6, p. 4] and [7, p. 54]).

For topological tensor product ($*$-)algebras we refer the reader to [9, 13, 21]. For convenience we fix some notation. If $\mathcal{A}[\tau_{\mathcal{A}}]$, $\mathcal{B}[\tau_{\mathcal{B}}]$ are locally convex $*$-algebras with continuous multiplication, denote by $\mathcal{A} \underset{\pi}{\otimes} \mathcal{B}$ their *projective tensor product* and by $\mathcal{A} \widehat{\otimes} \mathcal{B}$ their *completed projective tensor product* (see [9, 13]). Suppose that $\tau_{\mathcal{A}}$, $\tau_{\mathcal{B}}$ are respectively defined by the families $\{p\}$, $\{q\}$ of $*$-seminorms. Then, the topology π on $\mathcal{A} \otimes \mathcal{B}$ is determined by the $*$-seminorms $\{r\}$ such that

$$r(z) = \inf\Big\{\sum_i p(x_i)q(y_i) : z = \sum_i x_i \otimes y_i\Big\}, \quad z \in \mathcal{A} \otimes \mathcal{B}, \tag{2.2}$$

where the infimum is taken over all representations $\sum_i x_i \otimes y_i$ of $z \in \mathcal{A} \otimes \mathcal{B}$. If $\mathcal{A}[p]$, $\mathcal{B}[q]$ are (semi)normed algebras we shall denote the r as in (2.2) by $\|\cdot\|_\gamma$, and $\mathcal{A} \underset{\pi}{\otimes} \mathcal{B}$ by $\mathcal{A} \underset{\gamma}{\otimes} \mathcal{B}$ [9, 21].

Furthermore, if $\mathcal{A}[p]$, $\mathcal{B}[q]$ are C^*-seminormed algebras and $R(\mathcal{A}[p])$ resp. $R(\mathcal{B}[q])$ denote the sets of all p-continuous bounded $*$-representations of $\mathcal{A}[p]$ resp. q-continuous bounded $*$-representations of $\mathcal{B}[q]$, then the corresponding *minimal* and *maximal* C^*-seminorms on $\mathcal{A} \otimes \mathcal{B}$ denoted by $r_{\min}$ resp. $r_{\max}$

are defined as follows:

$$\begin{aligned} r_{\min}(z) &:= \sup\{\|(\pi_1 \otimes \pi_2)(z)\| : (\pi_1, \pi_2) \in R(\mathcal{A}[p]) \times R(\mathcal{B}[q])\}, \\ r_{\max}(z) &:= \sup\{\|\pi(z)\| : \pi \in R(\mathcal{A}[p] \mathbin{\widehat{\otimes}} \mathcal{B}[q])\}, \end{aligned} \tag{2.3}$$

for each $z \in \mathcal{A} \otimes \mathcal{B}$ (see [21, pp. 206, 207] and [9, Section 31]). Note that for any π_1, π_2, π as before, one has

$$\|(\pi_1 \otimes \pi_2)(z)\| \leq \|z\|_\gamma, \quad \|\pi(z)\| \leq \|z\|_\gamma, \quad \forall z \in \mathcal{A} \otimes \mathcal{B}.$$

The corresponding C^*-seminormed tensor product algebras under $r_{\min}$ and $r_{\max}$ will be denoted by $\mathcal{A} \underset{\min}{\otimes} \mathcal{B}$ resp. $\mathcal{A} \underset{\max}{\otimes} \mathcal{B}$. In the case when $\mathcal{A}, \mathcal{B}$ are C^*-algebras, the C^*-algebra tensor product under the minimal and maximal C^*-crossnorms $\|\cdot\|_{\min}$ resp. $\|\cdot\|_{\max}$ will be denoted by $\mathcal{A} \underset{\min}{\widetilde{\otimes}} \mathcal{B}$ resp. $\mathcal{A} \underset{\max}{\widetilde{\otimes}} \mathcal{B}$.

Finally, if $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces, their Hilbert space tensor product will be denoted by $\mathcal{H}_1 \mathbin{\widetilde{\otimes}} \mathcal{H}_2$.

3. Unbounded $*$-representations of tensor product $*$-algebras. In this section, we construct unbounded C^*-seminorms on a tensor product $*$-algebra, in terms of given unbounded C^*-seminorms on the factors, and vice versa.

Let $\mathcal{A}$, $\mathcal{B}$ be $*$-algebras and p, q unbounded C^*-seminorms on $\mathcal{A}$, $\mathcal{B}$ resp. with domains $\mathfrak{D}(p)$, $\mathfrak{D}(q)$. Consider the corresponding C^*-seminormed algebras $\mathfrak{D}(p)[p]$, $\mathfrak{D}(q)[q]$ and the $*$-subalgebra $\mathfrak{D}(r) := \mathfrak{D}(p)[p] \otimes \mathfrak{D}(q)[q]$ of $\mathcal{A} \otimes \mathcal{B}$. Then the minimal and maximal C^*-seminorms $r_{\min}$, $r_{\max}$ on $\mathfrak{D}(r)$ (see (2.3)) are unbounded C^*-seminorms on $\mathcal{A} \otimes \mathcal{B}$ defined by p, q.

Using the very definitions, one proves easily that (see (2.1))

$$\mathcal{I}_{r_{\min}} = \mathcal{I}_p \otimes \mathcal{I}_q = \mathcal{I}_{r_{\max}}.$$

In particular,

$$\mathcal{I}_{r_{\min}} \not\subset N_{r_{\min}} \;\Leftrightarrow\; \mathcal{I}_p \not\subset N_p \text{ and } \mathcal{I}_q \not\subset N_q.$$

Indeed, suppose that $\mathcal{I}_{r_{\min}} \not\subset N_{r_{\min}}$, but $\mathcal{I}_p \subset N_p$. Then $p(x) = 0$ for all $x \in \mathcal{I}_p$, while for each $z \in \mathfrak{D}(r)$ and $(\pi_1, \pi_2) \in R(\mathfrak{D}(p)[p]) \times R(\mathfrak{D}(q)[q])$,

$$\|(\pi_1 \otimes \pi_2)(z)\| \leq \|z\|_\gamma = \inf\Big\{\sum_i p(x_i)q(y_i) : z = \sum_i x_i \otimes y_i\Big\}.$$

Therefore $r_{\min}(z) = 0$ for all $z \in \mathcal{I}_{r_{\min}}$, which contradicts our hypothesis. Hence, $\mathcal{I}_p \not\subset N_p$ and similarly $\mathcal{I}_q \not\subset N_q$.

Conversely, suppose that $\mathcal{I}_p \not\subset N_p$, $\mathcal{I}_q \not\subset N_q$, but $\mathcal{I}_{r_{\min}} \subset N_{r_{\min}}$. It follows that $r_{\min}(x \otimes y) = 0$ for all $(x, y) \in \mathcal{I}_p \times \mathcal{I}_q$, that is, $p(x)q(y) = 0$ for all $(x, y) \in \mathcal{I}_p \times \mathcal{I}_q$, so that $p(x) = 0$ for all $x \in \mathcal{I}_p$ or $q(y) = 0$ for all $y \in \mathcal{I}_q$, which is a contradiction. Hence, $\mathcal{I}_{r_{\min}} \not\subset N_{r_{\min}}$.

Note that $\|\cdot\|_\lambda \preceq r_{\min} \preceq r_{\max} \preceq \|\cdot\|_\gamma$, where $\|\cdot\|_\lambda$ is the injective $*$-seminorm on $\mathfrak{D}(p)[p] \otimes \mathfrak{D}(q)[q]$ (see [21, pp. 206–208] and [9, Proposition 31.3]), so that

$$r_{\min}(x \otimes y) = p(x)q(y) = r_{\max}(x \otimes y), \quad \forall x \in \mathfrak{D}(p) \text{ and } y \in \mathfrak{D}(q).$$

Summing up the above and taking into account the discussion at the beginning of Section 2, we are led to the following

PROPOSITION 3.1. *If $\mathcal{A}$, $\mathcal{B}$ are $*$-algebras and p, q unbounded C^*-seminorms on $\mathcal{A}$, $\mathcal{B}$ resp. such that $\mathcal{I}_p \not\subset N_p$ and $\mathcal{I}_q \not\subset N_q$, then the tensor product $*$-algebra $\mathcal{A}\otimes\mathcal{B}$ admits an unbounded $*$-representation deriving from an unbounded C^*-seminorm r on $\mathcal{A}\otimes\mathcal{B}$ induced by p, q with $\mathcal{I}_r \not\subset N_r$.*

Suppose now that $\mathcal{A}$, $\mathcal{B}$ are $*$-algebras with identities $e_{\mathcal{A}}$, $e_{\mathcal{B}}$ resp., and $\mathcal{A}_0$, $\mathcal{B}_0$ $*$-subalgebras of $\mathcal{A}$, $\mathcal{B}$ resp. with $e_{\mathcal{A}} \in \mathcal{A}_0$ and $e_{\mathcal{B}} \in \mathcal{B}_0$. Let r be an unbounded C^*-seminorm on the $*$-algebra $\mathcal{A}\otimes\mathcal{B}$ such that $\mathfrak{D}(r) := \mathcal{A}_0\otimes\mathcal{B}_0$. Then the relations

$$p(x) := r(x \otimes e_{\mathcal{B}}), \quad \forall x \in \mathcal{A}_0 \quad \text{and} \quad q(y) := r(e_{\mathcal{A}} \otimes y), \quad \forall y \in \mathcal{B}_0$$

define unbounded C^*-seminorms on $\mathcal{A}$, $\mathcal{B}$ resp. with $\mathfrak{D}(p) := \mathcal{A}_0$ and $\mathfrak{D}(q) := \mathcal{B}_0$. Moreover,

$$r(x \otimes y) = r((x \otimes e_{\mathcal{B}})(e_{\mathcal{A}} \otimes y)) \le p(x)q(y), \quad \forall x \otimes y \in \mathfrak{D}(r),$$
$$r(z) \le \|z\|_\gamma, \quad \forall z \in \mathfrak{D}(r),$$
$$\mathcal{I}_r = \mathcal{I}_p \otimes \mathcal{I}_q \text{ and } \mathcal{I}_r \not\subset N_r \;\Rightarrow\; \mathcal{I}_p \not\subset N_p \text{ and } \mathcal{I}_q \not\subset N_q.$$

So, we can state

PROPOSITION 3.2. *Let $\mathcal{A}$, $\mathcal{B}$ be $*$-algebras with identities $e_{\mathcal{A}}$, $e_{\mathcal{B}}$ resp. Let $\mathcal{A}_0$, $\mathcal{B}_0$ be $*$-subalgebras of $\mathcal{A}$, $\mathcal{B}$ resp. such that $e_{\mathcal{A}} \in \mathcal{A}_0$ and $e_{\mathcal{B}} \in \mathcal{B}_0$. If r is an unbounded C^*-seminorm on $\mathcal{A}\otimes\mathcal{B}$ with $\mathfrak{D}(r) := \mathcal{A}_0\otimes\mathcal{B}_0$ and $\mathcal{I}_r \not\subset N_r$, then $\mathcal{A}$, $\mathcal{B}$ admit unbounded $*$-representations induced by unbounded C^*-seminorms p, q on $\mathcal{A}$, $\mathcal{B}$ deriving from r such that $\mathfrak{D}(p) = \mathcal{A}_0$, $\mathfrak{D}(q) = \mathcal{B}_0$, $\mathcal{I}_p \not\subset N_p$, $\mathcal{I}_q \not\subset N_q$ and $r(x \otimes y) \le p(x)q(y)$ for all $(x,y) \in \mathcal{A}_0 \times \mathcal{B}_0$.*

4. Well-behaved $*$-representations of tensor product $*$-algebras. As mentioned in Section 1, unbounded $*$-representations may show pathologies. So, naturally one wishes to have conditions under which unbounded $*$-representations exhibit good behaviour. In [4, 5] various conditions have been elaborated that yield the existence of so-called well-behaved $*$-representations (see Section 2). In the present section, using some of these results we investigate the construction of well-behaved $*$-representations of tensor product $*$-algebras defined by unbounded C^*-seminorms or unbounded m^*-seminorms.

THEOREM 4.1. *Let $\mathcal{A}$, $\mathcal{B}$ be $*$-algebras and π_p, π_q be well-behaved $*$-representations of $\mathcal{A}$, $\mathcal{B}$ induced by w-semifinite unbounded C^*-seminorms p, q on $\mathcal{A}$, $\mathcal{B}$ resp. such that $\mathcal{I}_p \not\subset N_p$ and $\mathcal{I}_q \not\subset N_q$. Then $\mathcal{A}\otimes\mathcal{B}$ admits a* *well-behaved $*$-representation π_r induced by a w-semifinite unbounded C^*-seminorm r on $\mathcal{A} \otimes \mathcal{B}$ with $\mathcal{I}_r \not\subset N_r$, and $\pi_r = \pi_p \otimes \pi_q$ on $\mathcal{H}_p \otimes \mathcal{H}_q$, where $\mathcal{H}_p$, $\mathcal{H}_q$ are the Hilbert spaces associated with π_p, π_q resp.*

Proof. For convenience we give the construction of π_p (see [7, p. 57]) in the following diagram (for the notation, see Section 2):

$$\begin{array}{ccc} \mathcal{A} \hookleftarrow \mathfrak{D}(p)[p] \longrightarrow & (\mathfrak{D}(p)/N_p)[\|\cdot\|_p] & \hookrightarrow \mathcal{A}_p \\ \pi_p \downarrow & & \downarrow \Pi_p \\ \mathcal{L}^\dagger(\mathfrak{D}(\pi_p)) & & \mathcal{B}(\mathcal{H}_p) \end{array}$$

where Π_p is a nondegenerate faithful $*$-representation of $\mathcal{A}_p$ on a Hilbert space $\mathcal{H}_p$. Set $\mathcal{H}_{\pi_p} := \overline{\mathfrak{D}(\pi_p)}^{\|\cdot\|}$ with $\|\cdot\|$ the Hilbert space norm on $\mathcal{H}_p$ and $\mathfrak{D}(\pi_p) := \langle \pi_p(x + N_p)\xi : x \in \mathcal{I}_p, \xi \in \mathcal{H}_p\rangle$, where $\langle \cdots \rangle$ means linear span. $\mathcal{L}^\dagger(\mathfrak{D}(\pi_p))$ is the $*$-algebra in which π_p takes values (see [10, p. 8]), and

$$\pi_p(a)(\xi) = \pi_p(a)\Big(\sum_k \Pi_p(x_k + N_p)\xi_k\Big) := \sum_k \Pi_p(ax_k + N_p)\xi_k, \qquad \forall a \in A.$$

Since π_p is well-behaved we have $\mathcal{H}_p = \mathcal{H}_{\pi_p}$. If π_q is the corresponding well-behaved $*$-representation of $\mathcal{B}$ we similarly have $\mathcal{H}_q = \mathcal{H}_{\pi_q}$. Consider now the unbounded C^*-seminorm $r = r_{\max}$ on $\mathfrak{D}(r) := \mathfrak{D}(p) \otimes \mathfrak{D}(q)$ (see (2.3)). Then, by [9, p. 392, Theorem 31.7], we get the isomorphism

$$\mathcal{A}_p \underset{\max}{\widetilde{\otimes}} \mathcal{B}_q = (\mathcal{A} \otimes \mathcal{B})_r$$

of C^*-algebras, with $(\mathcal{A}\otimes\mathcal{B})_r$ the enveloping C^*-algebra of the C^*-seminormed algebra $\mathfrak{D}(r)[r]$, constructed as $\mathcal{A}_p$ in Section 2. Take now a faithful nondegenerate bounded $*$-representation Π_q of the C^*-algebra $\mathcal{B}_q$. The bounded $*$-representation $\Pi_p \otimes \Pi_q$ of $\mathcal{A}_p \underset{\max}{\otimes} \mathcal{B}_q$ in $\mathcal{H}_p \widetilde{\otimes} \mathcal{H}_q$ is faithful nondegenerate and $\|\cdot\|_{\max}$-continuous, so that (see also [9, p. 375, Proposition 30.2]) it extends (uniquely up to equivalence) to a (continuous) faithful nondegenerate $*$-representation of $\mathcal{A}_p \underset{\max}{\widetilde{\otimes}} \mathcal{B}_q = (\mathcal{A}\otimes\mathcal{B})_r$ in $\mathcal{H}_p \widetilde{\otimes} \mathcal{H}_q$, also denoted by $\Pi_p \otimes \Pi_q$. Let now π_r be the (unbounded) $*$-representation of $\mathcal{A} \otimes \mathcal{B}$ deriving from the unbounded C^*-seminorm $r = r_{\max}$ through the faithful nondegenerate $*$-representation $\Pi_p \otimes \Pi_q$ of $(\mathcal{A} \otimes \mathcal{B})_r$. We shall show that π_r is well-behaved, which equivalently means that $\overline{\mathfrak{D}(\pi_r)}^{\|\cdot\|} = \mathcal{H}_p \widetilde{\otimes} \mathcal{H}_q$. Take the well-behaved $*$-representation of $\mathcal{B}$ deriving from the w-semifinite unbounded C^*-seminorm q. Consider $\pi_p \otimes \pi_q$ on $\mathcal{A} \otimes \mathcal{B}$ with $\mathfrak{D}(\pi_p \otimes \pi_q) := \mathfrak{D}(\pi_p) \otimes \mathfrak{D}(\pi_q)$. It is easily seen that

$$\mathfrak{D}(\pi_p) \otimes \mathfrak{D}(\pi_q) \subset \mathfrak{D}(\pi_r) \subset \mathcal{H}_p \widetilde{\otimes} \mathcal{H}_q.$$

Taking now the $\|\cdot\|$-closures in $\mathcal{H}_p \widetilde{\otimes} \mathcal{H}_q$ and using the continuity of the tensor map $\otimes$ we obtain

$$\overline{\mathfrak{D}(\pi_p)}^{\|\cdot\|} \otimes \overline{\mathfrak{D}(\pi_q)}^{\|\cdot\|} \subset \overline{\mathfrak{D}(\pi_r)}^{\|\cdot\|} \subset \mathcal{H}_p \widetilde{\otimes} \mathcal{H}_q,$$

where $\overline{\mathfrak{D}(\pi_p)}^{\|\cdot\|} = \mathcal{H}_p$ and $\overline{\mathfrak{D}(\pi_q)}^{\|\cdot\|} = \mathcal{H}_q$, so that closing again we finally get $\overline{\mathfrak{D}(\pi_r)}^{\|\cdot\|} = \mathcal{H}_p \widetilde{\otimes} \mathcal{H}_q$, which equivalently means that r is w-semifinite and π_r well-behaved. We note further that

$$\pi_r = \pi_p \otimes \pi_q \quad \text{on } \mathcal{H}_p \otimes \mathcal{H}_q$$

since

$$(\mathfrak{D}(p) \underset{\max}{\otimes} \mathfrak{D}(q))/N_r = \mathfrak{D}(r)/N_r = \mathfrak{D}(p)/N_p \underset{\max}{\otimes} \mathfrak{D}(q)/N_q,$$

where the second equality is an isometric isomorphism. ∎

Given two $*$-algebras $\mathcal{A}$, $\mathcal{B}$ endowed with unbounded m^*-seminorms p, q resp., recall that $\|\cdot\|_\gamma$ denotes the projective unbounded tensor m^*-seminorm on $\mathcal{A} \otimes \mathcal{B}$ with $\mathfrak{D}(\|\cdot\|_\gamma) := \mathfrak{D}(p) \otimes \mathfrak{D}(q)$.

THEOREM 4.2. *Let $\mathcal{A}$, $\mathcal{B}$ be $*$-algebras with identities $e_\mathcal{A}$ resp. $e_\mathcal{B}$ and p, q unbounded m^*-seminorms on $\mathcal{A}$, $\mathcal{B}$ resp. with $e_\mathcal{A} \in \mathfrak{D}(p)$, $e_\mathcal{B} \in \mathfrak{D}(q)$ and $\mathcal{I}_{\|\cdot\|_\gamma} \neq \{0\}$. Suppose that $\mathcal{A} \otimes \mathcal{B}$ admits a well-behaved $*$-representation π_r deriving from a w-semifinite unbounded C^*-seminorm r on $\mathcal{A} \otimes \mathcal{B}$ with $\mathfrak{D}(r) = \mathfrak{D}(\|\cdot\|_\gamma)$, $r \leq \|\cdot\|_\gamma$ and $\mathcal{I}_r \not\subset N_r$. Then $\mathcal{A}$, $\mathcal{B}$ admit well-behaved $*$-representations π_{r_i}, $i = 1, 2$, deriving from w-semifinite unbounded C^*-seminorms r_1, r_2 resp. with $\mathfrak{D}(r_1) = \mathfrak{D}(p)$, $r_1 \leq p$ and $\mathcal{I}_{r_1} \not\subset N_{r_1}$, resp. $\mathfrak{D}(r_2) = \mathfrak{D}(q)$, $r_2 \leq q$ and $\mathcal{I}_{r_2} \not\subset N_{r_2}$.*

Proof. By our assumption on $\mathcal{A} \otimes \mathcal{B}$, [4, Corollary 3.6] implies that the enveloping C^*-algebra $\mathcal{E}(\mathfrak{D}(\|\cdot\|_\gamma))$ of $\mathfrak{D}(\|\cdot\|_\gamma)$ is nontrivial with $\mathcal{I}_{p\otimes q} \not\subset \ker\mu$ for some $\mu \in R(\mathfrak{D}(\|\cdot\|_\gamma))$. But by [9, p. 413, Theorem 32.4],

$$\mathcal{E}(\mathfrak{D}(\|\cdot\|_\gamma)) = \mathcal{E}(\mathfrak{D}(p) \widehat{\otimes} \mathfrak{D}(q)) = \mathcal{E}(\mathfrak{D}(p)) \underset{\max}{\widetilde{\otimes}} \mathcal{E}(\mathfrak{D}(q)),$$

and since $\mathcal{E}(\mathfrak{D}(\|\cdot\|_\gamma))$ is nontrivial the same is true for $\mathcal{E}(\mathfrak{D}(p))$ and $\mathcal{E}(\mathfrak{D}(q))$. Define now $\mu_p(x) := \mu(x \otimes e_\mathcal{B})$ for all $x \in \mathfrak{D}(p)$. Then

$$\|\mu_p(x)\| = \|\mu(x \otimes e_\mathcal{B})\| \leq \|x \otimes e_\mathcal{B}\|_\gamma = p(x)q(e_\mathcal{B}), \quad \forall x \in \mathfrak{D}(p).$$

Therefore, $\mu_p \in R(\mathfrak{D}(p))$. In the same way, one defines $\mu_q \in R(\mathfrak{D}(q))$. Clearly,

$$\mu(x \otimes y) = \mu_p(x)\mu_q(y), \quad \forall (x, y) \in \mathfrak{D}(p) \times \mathfrak{D}(q).$$

So, if $\mathcal{I}_p \subset \ker\mu_p$, then $\mu_p(x) = 0$ for all $x \in \mathcal{I}_p$, so that $\mu(x \otimes y) = 0$ for all $x \otimes y \in \mathcal{I}_p \otimes \mathcal{I}_q = \mathcal{I}_r$, which implies $\mathcal{I}_{p\otimes q} \subset \ker\mu$, a contradiction. Hence, $\mathcal{I}_p \not\subset \ker\mu_p$ and in the same way $\mathcal{I}_q \not\subset \ker\mu_q$. Now, by [4, Corollary 3.6] there are well-behaved $*$-representations of $\mathcal{A}$, $\mathcal{B}$, say π_{r_1}, π_{r_2}, induced by

w-semifinite unbounded C^*-seminorms r_1 and r_2 on $\mathcal{A}$, $\mathcal{B}$ resp. such that $\mathfrak{D}(r_1) = \mathfrak{D}(p)$, $r_1 \leq p$, $\mathcal{I}_{r_1} \not\subset N_{r_1}$ and $\mathfrak{D}(r_2) = \mathfrak{D}(q)$, $r_2 \leq q$, $\mathcal{I}_{r_2} \not\subset N_{r_2}$. ■

REMARK. Let $\bar{r} = r_{\max}$ be induced by r_1, r_2 on $\mathfrak{D}(\bar{r}) = \mathfrak{D}(p) \otimes \mathfrak{D}(q)$. Theorem 4.1 implies that $\bar{r}$ is w-semifinite with $\mathcal{I}_{\bar{r}} \not\subset N_{\bar{r}}$; the corresponding well-behaved $*$-representation $\pi_{\bar{r}}$ coincides with $\pi_{r_1} \otimes \pi_{r_2}$ on $\mathcal{H}_{r_1} \otimes \mathcal{H}_{r_2}$.

5. Well-behaved $*$-representations of locally convex tensor product $*$-algebras. In this section, we investigate the existence of well-behaved $*$-representations of tensor product locally convex $*$-algebras defined by unbounded C^*-seminorms.

A locally convex algebra with a continuous involution is called a *locally convex $*$-algebra*. A *Fréchet $*$-algebra* is a metrizable complete locally convex $*$-algebra. Let $\mathcal{A}[\tau]$ be a metrizable locally convex $*$-algebra with identity $e_{\mathcal{A}}$. We may always suppose that τ is defined by a sequence $(p_n)_{n\in\mathbb{N}}$ of seminorms with the properties:

(i) $p_1 \leq p_2 \leq \cdots$;
(ii) $p_n(xy) \leq p_{n+1}(x)p_{n+1}(y)$, $\forall x, y \in A$, $\forall n \in \mathbb{N}$;
(iii) $p_n(x^*) = p_n(x)$, $\forall x \in A$, $\forall n \in \mathbb{N}$;
(iv) $p_n(e_{\mathcal{A}}) = 1$, $\forall n \in \mathbb{N}$.

Such a family $(p_n)_{n\in\mathbb{N}}$ will be called a *defining sequence of seminorms* for $\mathcal{A}[\tau]$.

Concerning properties (iii) and (iv) see resp. [9, p. 32, Theorem 3.7] and [8, p. 241, Corollary and Theorem 3].

Suppose now that $\mathcal{A}[\tau_{\mathcal{A}}]$ is a Fréchet $*$-algebra with $(p_n)_{n\in\mathbb{N}}$ a defining sequence of seminorms. Set

$$\mathfrak{D}(p_\infty^{\mathcal{A}}) = \{x \in \mathcal{A} : \sup_n p_n(x) < \infty\} \quad \text{with} \quad p_\infty^{\mathcal{A}}(x) := \sup_n p_n(x),\ x \in \mathfrak{D}(p_\infty^{\mathcal{A}}).$$

Then $\mathfrak{D}(p_\infty^{\mathcal{A}})[p_\infty^{\mathcal{A}}]$ is a Banach $*$-subalgebra of $\mathcal{A}[\tau_{\mathcal{A}}]$. If $\mathcal{B}[\tau_{\mathcal{B}}]$ is a second Fréchet $*$-algebra with $(q_n)_{n\in\mathbb{N}}$ a defining family of seminorms and $p_\infty^{\mathcal{B}}$ the corresponding m^*-norm on $\mathfrak{D}(p_\infty^{\mathcal{B}})$, recall that (see Section 2) $\|\cdot\|_\gamma$ denotes the projective tensor m^*-norm on $\mathfrak{D}(p_\infty^{\mathcal{A}}) \otimes \mathfrak{D}(p_\infty^{\mathcal{B}})$. In this connection, we have

PROPOSITION 5.1. *Let $\mathcal{A}[\tau_{\mathcal{A}}]$, $\mathcal{B}[\tau_{\mathcal{B}}]$ be unital Fréchet $*$-algebras with defining sequences of seminorms $(p_n)_{n\in\mathbb{N}}$ resp. $(q_n)_{n\in\mathbb{N}}$ and identities $e_{\mathcal{A}}$ resp. $e_{\mathcal{B}}$. Then the following are equivalent*:

(1) *The metrizable locally convex tensor product $*$-algebra $\mathcal{A} \underset{\pi}{\otimes} \mathcal{B}$ admits a well-behaved $*$-representation deriving from a w-semifinite unbounded C^*-seminorm r on $\mathcal{A} \underset{\pi}{\otimes} \mathcal{B}$ with $r \leq \|\cdot\|_\gamma$ on $\mathfrak{D}(r) := \mathfrak{D}(p_\infty^{\mathcal{A}}) \otimes \mathfrak{D}(p_\infty^{\mathcal{B}})$ and $\mathcal{I}_r \not\subset N_r$.*

(2) $\mathcal{A}[\tau_{\mathcal{A}}]$, $\mathcal{B}[\tau_{\mathcal{B}}]$ *have well-behaved* $*$*-representations deriving from w-semifinite unbounded* C^**-seminorms* $r_{\mathcal{A}}$, $r_{\mathcal{B}}$ *on* $\mathcal{A}[\tau_{\mathcal{A}}]$, $\mathcal{B}[\tau_{\mathcal{B}}]$ *resp., with* $r_i \le p^i_\infty$ *on* $\mathfrak{D}(p^i_\infty)$ *and* $\mathcal{I}_{r_i} \not\subset N_{r_i}$, $i = \mathcal{A},\ \mathcal{B}$.

Proof. (1)⇒(2). Denote by $(r_n)_{n\in\mathbb{N}}$ the defining sequence of seminorms for the projective tensorial topology π on $\mathcal{A}\otimes\mathcal{B}$ (see (2.2)). Consider now

$$\mathfrak{D}(r_\infty) = \{z\in\mathcal{A}\otimes\mathcal{B} : \sup_n r_n(z) < \infty\} \quad \text{with} \quad r_\infty(z) := \sup_n r_n(z),\ z\in\mathfrak{D}(r_\infty).$$

Then $\mathfrak{D}(r_\infty)$ is a normed $*$-algebra. Property (iv) of $p_n,\ q_n,\ n\in\mathbb{N}$, implies that

$$e_{\mathcal{A}} \in \mathfrak{D}(p^{\mathcal{A}}_\infty), \quad e_{\mathcal{B}} \in \mathfrak{D}(p^{\mathcal{B}}_\infty) \quad \text{and} \quad e_{\mathcal{A}}\otimes e_{\mathcal{B}} \in \mathfrak{D}(r_\infty).$$

The elementary tensor $e_{\mathcal{A}}\otimes e_{\mathcal{B}}$ is also an identity of the Fréchet $*$-algebra $\mathcal{A}\widehat{\otimes}\mathcal{B}$. Thus, the assertion follows by applying Theorem 4.2 with $p^{\mathcal{A}}_\infty$ in place of p and $p^{\mathcal{B}}_\infty$ in place of q.

(2)⇒(1). By [4, Theorem 3.5] there exist p^i_∞-continuous representable positive linear forms f_i on $\mathfrak{D}(p^i_\infty)$ with $\mathcal{I}_{p^i_\infty} \not\subset \ker f_i$, $i = \mathcal{A},\mathcal{B}$. Let $f = f_{\mathcal{A}}\otimes f_{\mathcal{B}}$. Then f is a $\|\cdot\|_\gamma$-continuous positive linear form on $\mathfrak{D}(p^{\mathcal{A}}_\infty) \underset{\gamma}{\otimes} \mathfrak{D}(p^{\mathcal{B}}_\infty)$, which uniquely extends to a (continuous) positive linear form on $\mathfrak{D}(p^{\mathcal{A}}_\infty) \mathbin{\widehat{\otimes}} \mathfrak{D}(p^{\mathcal{B}}_\infty)$, also denoted by f [12]. Moreover, $\mathcal{I}_{p^i_\infty} \ne \{0\}$, $i = \mathcal{A},\mathcal{B}$, so $\mathcal{I}_{p^{\mathcal{A}}_\infty\otimes p^{\mathcal{B}}_\infty} \ne \{0\}$, since $\mathcal{I}_{p^{\mathcal{A}}_\infty\otimes p^{\mathcal{B}}_\infty} = \mathcal{I}_{p^{\mathcal{A}}_\infty}\otimes\mathcal{I}_{p^{\mathcal{B}}_\infty}$. Now, by the GNS-construction f is representable on $\mathfrak{D}(p^{\mathcal{A}}_\infty) \mathbin{\widehat{\otimes}} \mathfrak{D}(p^{\mathcal{B}}_\infty)$, hence also on $\mathfrak{D}(p^{\mathcal{A}}_\infty) \underset{\gamma}{\otimes} \mathfrak{D}(p^{\mathcal{B}}_\infty)$, and clearly f is $\|\cdot\|_\gamma$-continuous with $\mathcal{I}_{p^{\mathcal{A}}_\infty\otimes p^{\mathcal{B}}_\infty} \not\subset \ker f$. If $\mathcal{I}_{p^{\mathcal{A}}_\infty\otimes p^{\mathcal{B}}_\infty} \subset \ker f$, then either $\mathcal{I}_{p^{\mathcal{A}}_\infty} \subset \ker f_{\mathcal{A}}$ or $\mathcal{I}_{p^{\mathcal{B}}_\infty} \subset \ker f_{\mathcal{B}}$, which is a contradiction. So, from [4, Theorem 3.5] we have (1). ∎

For the next result we need some extra concepts. Let $\mathcal{A}$ be a $*$-algebra with $\varrho_{\mathcal{A}}(a^*a) < \infty$ for all $a\in\mathcal{A}$, where $\varrho_{\mathcal{A}}$ denotes the spectral radius. Then the quantity

$$p_{\mathcal{A}}(a) := \varrho_{\mathcal{A}}(a^*a)^{1/2}, \qquad a\in\mathcal{A},$$

is called the *Pták function*. V. Pták [16] gave a number of important characterizations of hermiticity in Banach $*$-algebras through various properties of this function. A $*$-algebra $\mathcal{A}$ is called *hermitian* if $\mathrm{sp}_{\mathcal{A}}(a) \subset \mathbb{R}$ for all $a^* = a$ in $\mathcal{A}$, where $\mathrm{sp}_{\mathcal{A}}(a)$ is the spectrum of $a\in\mathcal{A}$.

A C^*-seminorm p on a $*$-algebra $\mathcal{A}$ is called *spectral* if $\varrho_{\mathcal{A}}(a) \le p(a)$ for all $a\in\mathcal{A}$. In every hermitian Banach $*$-algebra, the Pták function is a spectral C^*-seminorm [16]. Let now $p^{\mathcal{A}}_\infty$, $p^{\mathcal{B}}_\infty$ and r_∞ be as before, but with r_∞ defined through the defining family of seminorms for the Fréchet $*$-algebra $\mathcal{A}\widehat{\otimes}\mathcal{B}$, so that $\mathfrak{D}(r_\infty)$ will now be a Banach $*$-subalgebra of $\mathcal{A}\widehat{\otimes}\mathcal{B}$. Then we have the following

THEOREM 5.2. *Let $\mathcal{A}[\tau_{\mathcal{A}}]$ and $\mathcal{B}[\tau_{\mathcal{B}}]$ be Fréchet $*$-algebras with identities $e_{\mathcal{A}}$ resp. $e_{\mathcal{B}}$ and $\mathcal{A}\,\widehat{\otimes}\,\mathcal{B}$ the Fréchet $*$-algebra* (with identity), *corresponding to the completed projective tensor product of $\mathcal{A}[\tau_{\mathcal{A}}]$ and $\mathcal{B}[\tau_{\mathcal{B}}]$. Then*:

(1) *If $\mathfrak{D}(r_\infty)$ is hermitian and $\mathcal{I}_{p_\infty^{\mathcal{A}}} \otimes \mathcal{I}_{p_\infty^{\mathcal{B}}} \not\subset N_{p_{\mathfrak{D}(r_\infty)}}$, then $\mathcal{A}[\tau_{\mathcal{A}}]$, $\mathcal{B}[\tau_{\mathcal{B}}]$ have well-behaved $*$-representations $\pi_{r_{\mathcal{A}}}$, $\pi_{r_{\mathcal{B}}}$ deriving from w-semi-finite unbounded C^*-seminorms $r_{\mathcal{A}}$ resp. $r_{\mathcal{B}}$ with $\mathfrak{D}(r_i) = \mathfrak{D}(p_\infty^i)$ and spectral unbounded C^*-seminorms $r'_{\mathcal{A}}$, $r'_{\mathcal{B}}$ with $\mathfrak{D}(r'_i) = \mathfrak{D}(p_\infty^i)$ such that $\|\overline{\pi_{r_i}(x)}\| \leq r'_i(x) \leq p_\infty^i(x)$ for all $x \in \mathfrak{D}(p_\infty^i)$, $i = \mathcal{A}, \mathcal{B}$.*

(2) *If either of $\mathcal{A}[\tau_{\mathcal{A}}]$, $\mathcal{B}[\tau_{\mathcal{B}}]$ is commutative and both $\mathfrak{D}(p_\infty^i)$, $i = \mathcal{A}, \mathcal{B}$, are hermitian with $\mathcal{I}_{p_\infty^i} \not\subset N_{p_{\mathfrak{D}(p_\infty^i)}}$, $i = \mathcal{A}, \mathcal{B}$, then $\mathcal{A}\,\widehat{\otimes}\,\mathcal{B}$ has a well behaved $*$-representation π_r of the kind of π_{r_i}, $i = \mathcal{A}, \mathcal{B}$, as in* (1).

Proof. (1) By the very definitions (see beginning of this section) we have $e_{\mathcal{A}} \in \mathfrak{D}(p_\infty^{\mathcal{A}})$ and $e_{\mathcal{B}} \in \mathfrak{D}(p_\infty^{\mathcal{B}})$. Moreover,

$$r_\infty(e_{\mathcal{A}} \otimes y) = \sup_n q_n(y), \quad \forall y \in \mathfrak{D}(p_\infty^{\mathcal{B}}),$$
$$r_\infty(x \otimes e_{\mathcal{B}}) = \sup_n p_n(x), \quad \forall x \in \mathfrak{D}(p_\infty^{\mathcal{A}}).$$

It follows that

$$r_\infty(x \otimes y) = r_\infty((x \otimes e_{\mathcal{B}})(e_{\mathcal{A}} \otimes y)) \leq p_\infty^{\mathcal{A}}(x) p_\infty^{\mathcal{B}}(y), \quad \forall (x, y) \in \mathfrak{D}(p_\infty^{\mathcal{A}}) \times \mathfrak{D}(p_\infty^{\mathcal{B}}).$$

Therefore, $\mathfrak{D}(p_\infty^{\mathcal{A}}) \otimes \mathfrak{D}(p_\infty^{\mathcal{B}}) \subset \mathfrak{D}(r_\infty)$. In particular, $\mathfrak{D}(p_\infty^i)$, $i = \mathcal{A}, \mathcal{B}$, is isometrically imbedded into $\mathfrak{D}(r_\infty)$, so it is hermitian as a closed $*$-subalgebra of a hermitian Banach $*$-algebra. Thus, the Pták function $p_{\mathfrak{D}(p_\infty^i)}$ is a C^*-seminorm on $\mathfrak{D}(p_\infty^i)$ with $p_{\mathfrak{D}(p_\infty^i)} \leq p_\infty^i$, $i = \mathcal{A}, \mathcal{B}$ [16]. Moreover,

$$\mathcal{I}_{p_\infty^i} \not\subset N_{p_{\mathfrak{D}(p_\infty^i)}}, \quad i = \mathcal{A}, \mathcal{B}.$$

Indeed, suppose $\mathcal{I}_{p_\infty^{\mathcal{A}}} \subset N_{p_{\mathfrak{D}(p_\infty^{\mathcal{A}})}}$, i.e. $p_{\mathfrak{D}(p_\infty^{\mathcal{A}})}(x) = 0$ for all $x \in \mathcal{I}_{p_\infty^{\mathcal{A}}}$. But the topology $\|\cdot\|_\gamma$ is finer than the $r_\infty(\cdot)$-topology on $\mathfrak{D}(p_\infty^{\mathcal{A}}) \otimes \mathfrak{D}(p_\infty^{\mathcal{B}})$, therefore

$$\mathfrak{D}(p_\infty^{\mathcal{A}})\,\widehat{\otimes}\,\mathfrak{D}(p_\infty^{\mathcal{B}}) \subset \overline{\mathfrak{D}(p_\infty^{\mathcal{A}}) \otimes \mathfrak{D}(p_\infty^{\mathcal{B}})}^{r_\infty} \subset \mathfrak{D}(r_\infty).$$

Hence, $\varrho_{\mathfrak{D}(r_\infty)}(z) \leq \varrho_{\mathfrak{D}(p_\infty^{\mathcal{A}})\widehat{\otimes}\mathfrak{D}(p_\infty^{\mathcal{B}})}(z)$ for all $z \in \mathfrak{D}(p_\infty^{\mathcal{A}})\,\widehat{\otimes}\,\mathfrak{D}(p_\infty^{\mathcal{B}})$. So, for any $(x, y) \in \mathfrak{D}(p_\infty^{\mathcal{A}}) \times \mathfrak{D}(p_\infty^{\mathcal{B}})$ we obtain (see also [9, p. 407, Corollary 31.21])

$$\begin{aligned} p_{\mathfrak{D}(r_\infty)}(x \otimes y)^2 &= \varrho_{\mathfrak{D}(r_\infty)}(x^*x \otimes y^*y) \leq \varrho_{\mathfrak{D}(p_\infty^{\mathcal{A}})\widehat{\otimes}\mathfrak{D}(p_\infty^{\mathcal{B}})}(x^*x \otimes y^*y) \\ &= \varrho_{\mathfrak{D}(p_\infty^{\mathcal{A}})}(x^*x)\varrho_{\mathfrak{D}(p_\infty^{\mathcal{B}})}(y^*y) = p_{\mathfrak{D}(p_\infty^{\mathcal{A}})}(x)^2 p_{\mathfrak{D}(p_\infty^{\mathcal{B}})}(y)^2, \end{aligned}$$

which implies $p_{\mathfrak{D}(r_\infty)}(x \otimes y) = 0$ whenever $x \in \mathcal{I}_{p_\infty}^{\mathcal{A}}$; thus $\mathcal{I}_{p_\infty^{\mathcal{A}}} \otimes \mathcal{I}_{p_\infty^{\mathcal{B}}} \subset N_{p_{\mathfrak{D}(r_\infty)}}$, a contradiction. The same is true if $\mathcal{I}_{p_\infty^{\mathcal{B}}} \subset N_{p_{\mathfrak{D}(p_\infty^{\mathcal{B}})}}$. The assertion now follows from [5, Theorem 3.5].

(2) Since both $\mathfrak{D}(p_\infty^{\mathcal{A}})$, $\mathfrak{D}(p_\infty^{\mathcal{B}})$ are commutative and hermitian Banach $*$-algebras, it follows from [12, p. 65, Theorem III.3] (and/or [9, p. 447, Theorem 34.15]) that $\mathfrak{D}(p_\infty^{\mathcal{A}}) \widehat{\otimes} \mathfrak{D}(p_\infty^{\mathcal{B}})$ is also hermitian. Let $p = \|\cdot\|_\gamma$ with $\mathfrak{D}(p) = \mathfrak{D}(p_\infty^{\mathcal{A}}) \widehat{\otimes} \mathfrak{D}(p_\infty^{\mathcal{B}})$. Then p is an unbounded m^*-(semi)norm of $\mathcal{A} \widehat{\otimes} \mathcal{B}$ with

$$p_{\mathfrak{D}(p)} \le p \quad \text{on } \mathfrak{D}(p) \quad \text{and} \quad \mathcal{I}_p \not\subset N_{p_{\mathfrak{D}(p)}}. \tag{5.1}$$

Indeed, from the assumptions in (2) there are $x \in \mathcal{I}_{p_\infty^{\mathcal{A}}}$ and $y \in \mathcal{I}_{p_\infty^{\mathcal{B}}}$ such that $p_{\mathfrak{D}(p_\infty^{\mathcal{A}})}(x) \neq 0$ and $p_{\mathfrak{D}(p_\infty^{\mathcal{B}})}(y) \neq 0$. Let $z = x \otimes y$. Then $x \otimes y \in \mathcal{I}_{p_\infty^{\mathcal{A}}} \otimes \mathcal{I}_{p_\infty^{\mathcal{B}}} \subset \mathcal{I}_p$ and (see [9, p. 407, Corollary 31.21])

$$\begin{aligned} p_{\mathfrak{D}(p)}(x \otimes y)^2 &= \varrho_{\mathfrak{D}(p)}(x^*x \otimes y^*y) = \varrho_{\mathfrak{D}(p_\infty^{\mathcal{A}})}(x^*x)\varrho_{\mathfrak{D}(p_\infty^{\mathcal{B}})}(y^*y) \\ &= p_{\mathfrak{D}(p_\infty^{\mathcal{A}})}(x)^2 p_{\mathfrak{D}(p_\infty^{\mathcal{B}})}(y)^2; \end{aligned}$$

therefore $p_{\mathfrak{D}(p)}(x \otimes y) \neq 0$, which means that the second statement in (5.1) is true. The assertion now follows from [5, Theorem 3.5]. ∎

Applications. (1) Let $\mathcal{A}$ be an arbitrary hermitian Banach $*$-algebra with identity and $\mathcal{C}(\mathbb{R}, \mathcal{A})$ the Fréchet $*$-algebra of all continuous functions from $\mathbb{R}$ to $\mathcal{A}$ with the topology of compact convergence. Then Theorem 5.2(2) applies with $\mathcal{C}(\mathbb{R})$ in place of $\mathcal{A}$, and $\mathcal{A}$ in place of $\mathcal{B}$.

One gets a special case by replacing the real numbers $\mathbb{R}$ with the natural numbers $\mathbb{N}$.

Note that $\mathcal{C}(\mathbb{R}, \mathcal{A}) = \mathcal{C}(\mathbb{R}) \widehat{\widehat{\otimes}} \mathcal{A}$, where the latter is the completion with respect to the *injective tensorial topology* and the equality holds up to a $*$-isomorphism of Fréchet $*$-algebras (see [13, p. 391, Theorem 1.1]). That the tensorial topology is not the one considered in Theorem 5.2 does not affect the proof of statement (2) of this theorem (cf. e.g., [9, Corollary 34.16]). Moreover, in the present case, $\mathfrak{D}(p_\infty^{\mathcal{C}(\mathbb{R})}) = \mathcal{C}_{\mathrm{b}}(\mathbb{R})$, the C^*-algebra of all bounded continuous functions on $\mathbb{R}$, while $\mathfrak{D}(p_\infty^{\mathcal{A}}) = \mathcal{A}$ with $p = \|\cdot\|_\lambda$, the injective tensor norm [21], and $\mathfrak{D}(p) = \mathcal{C}_{\mathrm{b}}(\mathbb{R}) \widehat{\widehat{\otimes}} \mathcal{A}$.

(2) Theorem 5.2(2) also applies for $\mathcal{C}(\mathbb{R} \times \mathbb{R}) = \mathcal{C}(\mathbb{R}) \widehat{\widehat{\otimes}} \mathcal{C}(\mathbb{R})$ and/or $\mathcal{C}(\mathbb{R} \times \mathbb{N})$ (for the preceding identification, see [13, p. 392, Corollary 1.1]).

(3) Theorem 4.1 applies to any pair $\mathcal{A}$, $\mathcal{B}$ of algebras described in [7, p. 75, Example 7.1(1), (a) and (b)] and [4, Examples 3.9(1)–(5)].

One does not expect that algebras of complex-valued or vector-valued smooth or analytic functions have unbounded $*$-representations. Take, for instance, the Fréchet $*$-algebra $\mathcal{O}(\mathbb{C})$ of all entire functions on the complex plane $\mathbb{C}$. Then $\mathfrak{D}(p_\infty^{\mathcal{O}(C)}) = \mathbb{C}$ and $\mathcal{I}_{p_\infty^{\mathcal{O}(C)}} = \{0\}$. Therefore, neither $\mathcal{O}(\mathbb{C})$ nor $\mathcal{O}(\mathbb{C}, \mathcal{A})$, $\mathcal{A}$ a Fréchet $*$-algebra with identity, may have unbounded $*$-representations.

Remark. Investigating the existence of unbounded $*$-representations of (topological) tensor product $*$-algebras induced by unbounded C^*-seminorms can go further by using positive linear forms and tensor products of so-called GB^*-algebras (cf. [1]). To the best of our knowledge nothing is known on tensor products of GB^*-algebras. Because of the importance of GB^*-algebras in the theory of unbounded $*$-representations, the study of these topics seems to be interesting.

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