Haar measure and continuous representations of locally compact abelian groups

by

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Abstract. Let $L(X)$ be the algebra of all bounded operators on a Banach space $X$, and let $\theta : G \to L(X)$ be a strongly continuous representation of a locally compact and second countable abelian group $G$ on $X$. Set $\sigma^1(\theta(g)) := \{\lambda/|\lambda| \mid \lambda \in \sigma(\theta(g))\}$, where $\sigma(\theta(g))$ is the spectrum of $\theta(g)$, and let $\Sigma_\theta$ be the set of all $g \in G$ such that $\sigma^1(\theta(g))$ does not contain any regular polygon of $T$ (by a regular polygon we mean the image under a rotation of a closed subgroup of the unit circle $T$ different from $\{1\}$). We prove that $\theta$ is uniformly continuous if and only if $\Sigma_\theta$ is a non-null set for the Haar measure on $G$.

1. Introduction. A characterization of uniform continuity for strongly continuous groups was given in [7]. Indeed the authors proved that a strongly continuous one-parameter group $(T(t))_{t \in \mathbb{R}}$ on a Banach space $X$ is uniformly continuous if and only if $\{t \in \mathbb{R} \mid \sigma^1(T(t)) \neq \mathbb{T}\}$ is non-meager, where $\mathbb{T}$ denotes the unit circle of $\mathbb{C}$ and $\sigma^1(T(t)) := \{\lambda/|\lambda| \mid \lambda \in \sigma(T(t))\}$, well defined since $T(t)$ is invertible. The following generalization of this result was obtained in [1]: if $G$ is a second countable and locally compact abelian group then either $\theta$ is uniformly continuous or $\Sigma_\theta := \{g \in G \mid \text{there is no } P \in \mathcal{P} \text{ with } P \subseteq \sigma^1(\theta(g))\}$ is meager, where $\mathcal{P}$ is the set of regular polygons of $\mathbb{T}$. So when the representation is not uniformly continuous, the angular distribution of the spectrum of $\theta(g)$ is rather dispersed, except for $g$ in a meager set in $G$.

In the present work, we are interested in another condition, obtained by replacing meager set by null set.

Example 1.1. Let $(T(t))_{t \in \mathbb{R}}$ be the translation group on $L^2(\mathbb{R})$ defined by $(T(t)f)(x) = f(x + t)$. This one-parameter group is strongly continuous, not uniformly continuous and for all $t \neq 0$, $\sigma(T(t)) = \mathbb{T}$, thus $\Sigma_\theta = \{0\}$ is indeed a null set.

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Following J. Esterle (see [2], [3]) we define a representation of a topological
group $G$ on a Banach algebra $A$ to be a map $\theta: G \to A$ such that $\theta(1) = I$,
where 1 and $I$ respectively denote the unit elements of $G$ and $A$, and $\theta(uv) =
\theta(u)\theta(v)$.

In [2] the author established a zero-$\sqrt{3}$ law for representations of locally
compact abelian groups: if $\theta: G \to A$ is a locally bounded representation of
such a group on a Banach algebra then either $\theta$ is uniformly continuous or
$\limsup_{g \to 1} \rho(\theta(g) - I) \geq \sqrt{3}$, where $\rho$ denotes the spectral radius.

As a consequence of our results we find, but only in the case of strongly
continuous representations of locally compact and second countable abelian
groups, that either $\theta$ is uniformly continuous or $\liminf_{g \to 1, g \in G \setminus M} \rho(\theta(g) - I)
\geq \sqrt{2}$ where $M$ is a null set in $G$.

2. Characterization of uniform continuity. For a locally bounded
representation of a locally compact abelian group $G$, there are some argu-
ments, based on Gelfand–Hille’s theorem, Shilov’s idempotent theorem and
the standard structure theorem for locally compact abelian groups (see [2])
that allow us to go from spectral continuity (that is, $\lim_{g \to 1} \rho(\theta(g) - I) = 0$)
to uniform continuity.

Furthermore R. Phillips (see [5]) proved that the continuity for one-
parameter groups can be read through the characters, in the sense that if $T: \mathbb{R} \to A$ is a locally bounded representation of $\mathbb{R}$ on a commutative Ba-
nach algebra $A$ then its uniform continuity is equivalent to the continuity of
t $\in \mathbb{R} \mapsto \chi(T(t))$ for all $\chi \in \hat{A}$, where $\hat{A}$ denotes the character space of $A$.

However, going from the continuity through each character (that is, $\chi \circ T$
continuous for all $\chi \in \hat{A}$) to the uniform condition on $\hat{A}$: $\lim_{t \to 0} \rho(\theta(t) - 1) = 0$ required, in the case of $\mathbb{R}$, an analytical argument difficult to adapt to
a general group.

Therefore in order to generalize this result from $\mathbb{R}$ to any locally compact
abelian group, we had to use in [11] the Phillips theorem and the standard
structure theorem for locally compact abelian groups, and to deal separately
with compact groups and euclidean groups $\mathbb{R}^n$.

Here, we present a direct proof of this generalization and also a simplified
proof of the Phillips result.

In what follows we denote by $\mathcal{V}(1)$ the family of all neighborhoods of the
unit element of $G$.

**Lemma 2.1.** Let $\theta$ be a locally bounded representation of a topological
abelian group $G$ on a Banach algebra $A$. Then for all $\epsilon > 0$ there exists $V_\epsilon \in \mathcal{V}(1)$ such that for all $g \in V_\epsilon$,
\[ \sigma(\theta(g)) \subseteq \{ z \in \mathbb{C} \mid 1 - \epsilon \leq |z| \leq 1 + \epsilon \} . \]
Proof. Since $\theta$ is locally bounded, there exist $M > 1$ and $V \in \mathcal{V}(1)$ such that for all $g \in V$, $\|\theta(g)\| \leq M$. By the continuity of the product, for all $n \geq 1$ there exists $V_n \in \mathcal{V}(1)$ such that for all $g \in V_n$, $\|\theta(g^n)\| \leq M$ and $\|\theta(g^{-n})\| \leq M$. Since $\sigma(\theta(g^n)) = \{1/\lambda \mid \lambda \in \sigma(\theta(g^n))\}$ we obtain
$$\sigma(\theta(g^n)) \subseteq \{z \in \mathbb{C} \mid 1/M \leq |z| \leq M\},$$
and since $\sigma(\theta(g^n)) = (\sigma(\theta(g)))^n$, we have
$$(1/M)^{1/n} \leq |z| \leq M^{1/n}$$
for all $g \in V_n$ and $z \in \sigma(\theta(g))$. This yields the desired conclusion. ■

Proposition 2.2. Let $\theta$ be a locally bounded representation of a locally compact abelian group $G$ on a commutative Banach algebra $A$. The following assertions are equivalent:

(i) $\theta$ is uniformly continuous.
(ii) $\theta$ is spectrally continuous.
(iii) For all $\chi \in \hat{A}$, $\chi \circ \theta$ is continuous.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): Clear.

(iii) $\Rightarrow$ (ii): Let $V \in \mathcal{V}(1)$ be compact and symmetric. Then $H = \bigcup_{n \in \mathbb{N}} V^n$ is a locally compact and $\sigma$-compact subgroup of $G$. We have $H \in \mathcal{V}(1)$, hence it suffices to show that $\theta_H := \theta|_H$ is spectrally continuous.

We first show that $\{\chi \circ \theta_H/|\chi \circ \theta_H| \mid \chi \in \hat{A}\}$ is compact in $C(H, \mathbb{T})$ equipped with the topology of compact convergence. Since $H$ is $\sigma$-compact, this topology is metrizable, thus it suffices to check that $\{\chi \circ \theta_H/|\chi \circ \theta_H| \mid \chi \in \hat{A}\}$ is sequentially compact. So let $(\chi_n \circ \theta_H/|\chi_n \circ \theta_H|)_{n \in \mathbb{N}}$ be a sequence in $\{\chi \circ \theta_H/|\chi \circ \theta_H| \mid \chi \in \hat{A}\}$. The Gelfand space $\hat{A}$ is compact, and thus $\{\chi \circ \theta_H/|\chi \circ \theta_H| \mid \chi \in \hat{A}\} \subseteq C(H, \mathbb{T})$ is compact for the product topology and so is the set of restrictions to $V$ that we denote $\{\chi \circ \theta_V/|\chi \circ \theta_V|\}$.

By hypothesis we have $\{\chi \circ \theta_V/|\chi \circ \theta_V| \mid \chi \in \hat{A}\} \subseteq C(V, \mathbb{T})$, and thus we can apply Eberlein–Šmulian’s theorem (see [9, p. 296]): $\{\chi \circ \theta_V/|\chi \circ \theta_V| \mid \chi \in \hat{A}\}$ is sequentially compact in $\mathbb{T}^V$, therefore we can extract a subsequence $(\chi_{n_k} \circ \theta_V/|\chi_{n_k} \circ \theta_V|)_{k \in \mathbb{N}}$ that converges to an element $\chi \circ \theta_V/|\chi \circ \theta_V|$, that is, for all $g \in V$, $\chi_{n_k}(\theta_V(g))/|\chi_{n_k}(\theta_V(g))| \to \chi(\theta_V(g))/|\chi(\theta_V(g))|$.

But since it concerns restrictions of morphisms, the convergence extends from $V$ to $H = \bigcup_{n \in \mathbb{N}} V^n$, and using the dominated convergence theorem, we find that for all $f \in L^1(H)$, $\hat{f}(\chi_{n_k} \circ \theta_H/|\chi_{n_k} \circ \theta_H|) \to \hat{f}(\chi \circ \theta_H/|\chi \circ \theta_H|)$ (where $L^1(H)$ denotes the $L^1$-space of $H$ with respect to a Haar measure $m$ and $\hat{f}$ denotes the Fourier transform of $f$).

Since in the dual group $\hat{H}$ the topology of compact convergence on $H$ coincides with the weak* topology that $\hat{H}$ inherits as a subset of $L^\infty(H)$, we conclude that $\chi_{n_k} \circ \theta_H/|\chi_{n_k} \circ \theta_H| \to \chi \circ \theta_H/|\chi \circ \theta_H|$, which proves the compactness.
Then, by Ascoli’s theorem, \( \{\chi \circ \theta_H / |\chi \circ \theta_H| \mid \chi \in \hat{A}\} \) is equicontinuous; so for all \( \epsilon > 0 \) there exists \( W_\epsilon \in \mathcal{V}(1) \) in \( H \) such that for all \( h \in W_\epsilon \),
\[
\sup_{\chi \in \hat{A}} \left| \chi \circ \theta_H(h) / |\chi \circ \theta_H(h)| - 1 \right| < \epsilon.
\]

Lemma 2.1 yields \( V_\epsilon \in \mathcal{V}(1) \) such that for all \( h \in V_\epsilon \) and all \( \chi \in \hat{A} \),
\[
\chi \circ \theta_H(h) \in \{ z \in \mathbb{C} \mid 1 - \epsilon \leq |z| \leq 1 + \epsilon \},
\]
and thus for all \( h \in W_\epsilon \cap V_\epsilon \) and all \( \chi \in \hat{A} \),
\[
|\chi \circ \theta_H(h) - 1| \leq |\chi \circ \theta_H(h) - \chi \circ \theta_H(h) / |\chi \circ \theta_H(h)|| + |\chi \circ \theta_H(h) / |\chi \circ \theta_H(h)| - 1|
\]
\[
\leq 2\epsilon,
\]
that is, \( \rho(\theta_H(h) - I) \leq 2\epsilon \), and \( \theta_H \) is spectrally continuous.

(ii)\( \Rightarrow \)(i): See Theorem 3.3 in [2]. ■

3. Preliminary results. Let \( G \) be a topological group and \( \varphi : G \to \mathbb{T} \)
a morphism. Define
\[
\Gamma_\varphi := \{ \lambda \in \mathbb{T} \mid \text{there is a net} (g_i) \text{ in} G \text{ converging to} 1 \text{ such that} \varphi(g_i) \to \lambda \}
\]
\[
= \bigcap_{\mathcal{V}(1)} \varphi(W)
\]
(see [2]). Then:

- \( \Gamma_\varphi \) is a closed subgroup of \( \mathbb{T} \) (thus \( \Gamma_\varphi = \Gamma_k \) the group of \( k \)th roots of unity for some \( k \geq 1 \), or \( \Gamma_\varphi = \mathbb{T} \)).
- \( \varphi \) is continuous if and only if \( \Gamma_\varphi = \{1\} \).
- If the group locally admits division by every \( n \geq 1 \) (in the sense that for every \( n \in \mathbb{N} \) there exist \( V \in \mathcal{V}(1) \), a compact subset \( W \) of \( G \) containing 1 and a map \( \psi : V \to W \) such that \( \psi(1) = 1 \) and \( \psi^n(u) = u \) for every \( u \in V \), then one can easily check that \( \Gamma_\varphi \) is divisible, thus either \( \Gamma_\varphi = \mathbb{T} \) or \( \Gamma_\varphi = \{1\} \).

**Lemma 3.1.** Let \( \Gamma \) be a subset of \( \mathbb{T} \), and \( V \) an open subset of \( \mathbb{T} \) such that \( \lambda V \cap \Gamma \neq \emptyset \) for all \( \lambda \in \mathbb{T} \). Then there exists a compact set \( K \subseteq V \) such that \( \lambda K \cap \Gamma \neq \emptyset \) for all \( \lambda \in \mathbb{T} \).

**Proof.** Since \( V \) is open in \( \mathbb{T} \), there exists a sequence \( (O_n)_{n \in \mathbb{N}} \) of relatively compact open sets in \( V \) with \( \overline{O_n} \subseteq O_{n+1} \) for all \( n \in \mathbb{N} \) and \( V = \bigcup_{n \in \mathbb{N}} O_n \). It suffices to show that there exists an element of the sequence \( (\overline{O_n})_{n \in \mathbb{N}} \) intersected by every \( \lambda \Gamma \).

If it is not true then for all \( n \in \mathbb{N} \) there exists \( \lambda_n \in \mathbb{T} \) such that \( \lambda_n \Gamma \cap \overline{O_n} = \emptyset \), thus \( \lambda_n \Gamma \subseteq \mathbb{T} \setminus \overline{O_n} \subset \mathbb{T} \setminus O_n =: F_n \). As the sequence \( (O_n)_{n \in \mathbb{N}} \) is
increasing, \((F_n)_{n \in \mathbb{N}}\) is a decreasing sequence of closed sets such that
\[
\bigcap_{n \in \mathbb{N}} F_n = \bigcap_{n \in \mathbb{N}} T \setminus O_n = T \setminus \bigcup_{n \in \mathbb{N}} O_n = T \setminus V.
\]
Moreover, since \(T\) is compact, we can suppose that \((\lambda_n)_{n \in \mathbb{N}}\) is convergent. Denote by \(\lambda\) its limit, let \(\mu \in \Gamma\) and \(N \in \mathbb{N}\). For all \(k \geq N\), we have \(\lambda_k \mu \in \lambda_k \Gamma \subseteq F_k \subseteq F_N\), and so \(\lambda \mu \in F_N = F_N\). Finally \(\lambda \Gamma \subseteq \bigcap_{n \in \mathbb{N}} F_n = T \setminus V\), which is a contradiction. 

**Lemma 3.2.** Let \(G\) be a locally compact and second countable abelian group and \(m\) a Haar measure on \(G\). If \(A \subseteq G\) is measurable with \(m(A) > 0\), then for every \(N \geq 1\) there exists \(U_1 \in \mathcal{V}(1)\) such that for all \((g_1, \ldots, g_N) \in U_1^N\),
\[
m \left( A \cap \bigcap_{i=1}^N g_i A \right) > 0.
\]

**Proof.** As \(G\) is \(\sigma\)-finite, we can assume that \(m(A) < \infty\). Let \(\beta \in ]0, 1[\) and \(\alpha = \beta/(N + 1)\). We know, by regularity of \(m\), that there exist \(K \subseteq A \subseteq U\) with \(K\) and \(U\) respectively compact and open satisfying \(m(K) \geq (1 - \alpha)m(U)\); since \(K\) is compact, there exist \(U_1 \in \mathcal{V}(1)\) such that \(U_1 K \subseteq U\). Let us check by finite induction on \(k\) that for all \(k \in \{1, \ldots, N\}\) and for all \((g_1, \ldots, g_k) \in U_1^k\),
\[
m \left( K \cap \bigcap_{i=1}^k g_i K \right) \geq (1 - (k + 1)\alpha)m(U).
\]
We have \(m(K \cap g_1 K) \geq m(K) + m(g_1 K) - m(U)\) for \(K \cup g_1 K \subseteq U\), and since \(m\) is translation-invariant, we obtain \(m(K \cap g_1 K) \geq (1 - 2\alpha)m(U)\). Suppose that \(m(K \cap \bigcap_{i=1}^k g_i K) \geq (1 - (k + 1)\alpha)m(U)\) for some \(1 \leq k < N\); then
\[
m \left( K \cap \bigcap_{i=1}^{k+1} g_i K \right) = m \left( g_1 K \cap \left( K \cap \bigcap_{i=2}^{k+1} g_i K \right) \right)
\[
\geq m(g_1 K) + m \left( K \cap \bigcap_{i=2}^{k+1} g_i K \right) - m(U).
\]
Thus, by induction hypothesis and invariance of \(m\),
\[
m(g_1 K) + m \left( K \cap \bigcap_{i=2}^{k+1} g_i K \right) - m(U)
\[
\geq m(K) + (1 - (k + 1)\alpha)m(U) - m(U)
\[
\geq (1 - (k + 2)\alpha)m(U),
\]
which is the expected result.
In particular, for all \((g_1, \ldots, g_N) \in U_1^N\),
\[
m(A \cap \bigcap_{i=1}^N g_i A) \geq (1 - (N + 1)\alpha)m(U) = (1 - \beta)m(U) \geq (1 - \beta)m(A) > 0.\]

**Lemma 3.3.** Let \(\varphi\) be a morphism from a locally compact abelian group \(G\) into \(\mathbb{T}\), and \(m\) a Haar measure on \(G\). If \(V\) is an open subset of \(\mathbb{T}\) such that \(\lambda V \cap \Gamma_\varphi \neq \emptyset\) for all \(\lambda \in \mathbb{T}\), and if \(A \subseteq G\) is measurable with \(m(A) > 0\), then \(\varphi(A) \cap V \neq \emptyset\).

**Proof.** Suppose that \(\varphi(A) \cap V = \emptyset\). Let us prove that there exist a symmetric and open \(V_0 \in \mathcal{V}(1)\) and an open set \(V_1\) in \(\mathbb{T}\) such that for all \(\lambda \in \mathbb{T}\),
\[
\lambda V_1 \cap \Gamma_\varphi \neq \emptyset \quad \text{and} \quad V_0 V_1 \subseteq V.
\]
By Lemma 3.1 there exists a compact set \(K \subseteq V\) such that \(\lambda K \cap \Gamma_\varphi \neq \emptyset\) for all \(\lambda \in \mathbb{T}\); thus if \(\pi : \mathbb{T} \times \mathbb{T} \to \mathbb{T}\) denotes the product on \(\mathbb{T} \times \mathbb{T}\), the compact set \(\{1\} \times K\) is a subset of the open set \(\pi^{-1}(V)\) and so there exist a symmetric open unit-neighborhood \(V_0\) and an open set \(V_1\) containing \(K\) such that
\[
V_0 \times V_1 \subseteq \pi^{-1}(V) \quad \text{and} \quad V_0 V_1 \subseteq V.
\]
Since \(\lambda V_1 \cap \Gamma_\varphi \neq \emptyset\) for all \(\lambda \in \mathbb{T}\), we can easily deduce that \(\mathbb{T} = \bigcup_{\lambda \in \Gamma_\varphi} \lambda V_1\), and then by compactness there exists \(N \geq 1\) such that \(\mathbb{T} = \bigcup_{i=1}^N \lambda_i V_1\) with \(\lambda_i \in \Gamma_\varphi\).

By Lemma 3.2 there exists \(U_1 \in \mathcal{V}(1)\) in \(G\) such that for all \((g_1, \ldots, g_N) \in U_1^N\), \(m(A \cap \bigcap_{i=1}^N g_i A) > 0\) and thus \(A \cap \bigcap_{i=1}^N g_i A \neq \emptyset\).

Let \(i \in \{1, \ldots, N\}\) and \(\lambda_i \in \Gamma_\varphi\). Since \(\Gamma_\varphi \subseteq \varphi(U_1)\), there exists \(g_i \in U_1\) such that \(\lambda_i \varphi(g_i)^{-1} \in V_0\), that is, \(\lambda_i \in \varphi(g_i)V_0\), and thus
\[
\mathbb{T} = \bigcup_{i=1}^N \lambda_i V_1 \subseteq \bigcup_{i=1}^N \varphi(g_i) V_0 V_1 \subseteq \bigcup_{i=1}^N \varphi(g_i) V,
\]
so \(\mathbb{T} = \bigcup_{i=1}^N \varphi(g_i) V\) and \(G = \varphi^{-1}(\mathbb{T}) = \bigcup_{i=1}^N g_i \varphi^{-1}(V)\).

Let \(g \in A \cap \bigcap_{i=1}^N g_i A \subseteq G\). There exists \(i_0 \in \{1, \ldots, N\}\) such that \(g \in g_{i_0} \varphi^{-1}(V)\), thus \(g_{i_0}^{-1} g \in \varphi^{-1}(V)\). Since \(g \in g_{i_0} A\), we obtain \(g_{i_0}^{-1} g \in A \cap \varphi^{-1}(V)\), which is a contradiction. ■

**Proposition 3.4.** Let \(G\) be a locally compact abelian group, \(m\) a Haar measure on \(G\), \(\mathcal{K}(\mathbb{T})\) the space of all compact subsets of \(\mathbb{T}\) equipped with the Hausdorff metric and \(\omega : G \to \mathcal{K}(\mathbb{T})\) a Borel map. Let \((\varphi_i)_{i \in I}\) be a family of morphisms from \(G\) into \(\mathbb{T}\) such that \(\varphi_i(g) \in \omega(g)\) for all \(i \in I\) and \(g \in G\). For \(i \in I\) set
\[
\Omega_{\varphi_i} := \{g \in G \mid \forall \lambda \in \mathbb{T}, \lambda \Gamma_\varphi \subseteq \omega(g)\}.
\]
Then the set \(\bigcup_{i \in I} \Omega_{\varphi_i}\) has measure zero.
\textbf{Proof.} The proof is in two steps.

**STEP 1.** Suppose that $\bigcup_{i \in I} \Gamma_{\varphi_i}$ is infinite. Then the family of groups $\Gamma_{\varphi_i}$ contains elements of arbitrarily large order, and so for each nonempty open set $U$ of $\mathbb{T}$ there exists $i \in I$ such that $\lambda \Gamma_{\varphi_i} \cap U \neq \emptyset$ for every $\lambda \in \mathbb{T}$.

Let $g \in G$ be such that $\omega(g) \neq \mathbb{T}$. Then the open set $\mathbb{T} \setminus \omega(g)$ is non-empty and thus there exists $i \in I$ such that $\lambda \Gamma_{\varphi_i} \not\subseteq \omega(g)$ for all $\lambda \in \mathbb{T}$, so that $g \in \Omega_{\varphi_i}$. Since the other inclusion is obvious, we obtain $\bigcup_{i \in I} \Omega_{\varphi_i} = \{g \in G \mid \omega(g) \neq \mathbb{T}\}$, which is measurable as the inverse image of an open set in $\mathcal{K}(\mathbb{T})$.

Let $\mathcal{V} = \{V_n \mid n \in \mathbb{N}\}$ be a basis of open subsets on $\mathbb{T}$. Set $A_n = \{g \in G \mid \omega(g) \cap V_n = \emptyset\}$. Then the set $\{g \in G \mid \omega(g) \neq \mathbb{T}\} = \bigcup_{n \in \mathbb{N}} A_n$ is a Borel subset of $\mathbb{T}$.

If $m(\{g \in G \mid \omega(g) \neq \mathbb{T}\}) > 0$ then there exists $n_0 \in \mathbb{N}$ such that $m(A_{n_0}) > 0$; since $\bigcup_{i \in I} \Gamma_{\varphi_i}$ is infinite, there exists $i_0 \in I$ such that $\lambda V_{n_0} \cap \Gamma_{\varphi_{i_0}} \neq \emptyset$ for all $\lambda \in \mathbb{T}$, but $\varphi_{i_0}(g) \in \omega(g)$ for all $g \in G$, hence $\varphi_{i_0}^{-1}(V_{n_0}) \cap A_{n_0} = \emptyset$, which contradicts Lemma 3.3; so $\bigcup_{i \in I} \Omega_{\varphi_i}$ has measure zero.

**STEP 2.** Suppose that $\bigcup_{i \in I} \Gamma_{\varphi_i}$ is finite, thus $\{\Gamma_{\varphi_i} \mid i \in I\} = \{\Gamma_{p_j}\}_{j=1}^m$. For all $j \in \{1, \ldots, m\}$, define $\Omega_j = \{g \in G \mid \forall \lambda \in \mathbb{T}, \lambda \Gamma_{p_j} \not\subseteq \omega(g)\}$ and let us check that $\Omega_j$ has measure zero.

Let $\mathcal{W} = \{W_n \mid n \in \mathbb{N}\}$ be the (countable) set of finite unions $W_n$ of elements of $\mathcal{V}$ such that $\lambda \Gamma_{p_j} \cap W_n \neq \emptyset$ for all $\lambda \in \mathbb{T}$.

Let $g \in \Omega_j$. For all $\lambda \in \mathbb{T}$, we have $\lambda \Gamma_{p_j} \not\subseteq \omega(g)$, thus $\lambda \Gamma_{p_j} \cap \mathbb{T} \setminus \omega(g) \neq \emptyset$, and by Lemma 3.1, there exists a compact subset $K \subseteq \mathbb{T} \setminus \omega(g)$ such that $\lambda K \cap \Gamma_{p_j} \neq \emptyset$ for all $\lambda \in \mathbb{T}$.

Since $K$ is compact and $\mathbb{T} \setminus \omega(g)$ is open, there exists $W_n \in \mathcal{W}$ such that $K \subseteq W_n \subseteq \mathbb{T} \setminus \omega(g)$, so $g \in B_n := \{g \in G \mid W_n \cap \omega(g) = \emptyset\}$. Therefore $\Omega_j = \bigcup_{n \in \mathbb{N}} B_n$ (the other inclusion is obvious). But for all $n \in \mathbb{N}$, $B_n$ is measurable since $\omega$ is Borel, and since $\{C \in \mathcal{K}(\mathbb{T}) \mid C \cap W_n = \emptyset\}$ is a Borel subset of $\mathcal{K}(\mathbb{T})$, $\Omega_j$ is measurable too.

If $m(\Omega_j) > 0$ then there exists $n_0 \in \mathbb{N}$ such that $m(B_{n_0}) > 0$; but there exists a morphism $\varphi$ in the family $(\varphi_i)_{i \in I}$ such that $\varphi_i = \Gamma_{p_j}$ and since for all $g \in G$, $\varphi(g) \in \omega(g)$, we find that $\varphi^{-1}(W_{n_0}) \cap B_{n_0} = \emptyset$, whereas $\lambda \Gamma_{\varphi} \cap W_{n_0} \neq \emptyset$ for all $\lambda \in \mathbb{T}$, which contradicts Lemma 3.3.

Accordingly $\bigcup_{i \in I} \Omega_{\varphi_i} = \bigcup_{j=1}^m \Omega_j$ has measure zero. \hfill \blacksquare

**4. The main result and consequences.** Let $G$ be a locally compact abelian group, $X$ a Banach space and $\theta : G \to \mathcal{L}(X)$ a strongly continuous representation of $G$ on $X$. We are interested in the distribution of the arguments of the elements of the spectrum $\sigma(\theta(g))$ when $\theta$ is not uniformly continuous. We write:
• $A_\theta$ for the closed subalgebra of $\mathcal{L}(X)$ generated by $\theta(G)$ (so $A_\theta$ is commutative),
• $\sigma_{A_\theta}(\theta(g))$ for the spectrum of $\theta(g)$ in $A_\theta$,
• $\hat{A}_\theta$ for the character space of $A_\theta$,
• $K^1 = \{\lambda/|\lambda| \mid \lambda \in K\}$ for $K \subseteq \mathbb{C}^*$.

We have the following results:

**Lemma 4.1.**

(i) For all $g \in G$, $\sigma^1(\theta(g)) = \sigma_{A_\theta}^1(\theta(g))$.

(ii) For all $\chi \in \hat{A}_\theta$, the map $g \mapsto |(\chi \circ \theta)(g)|$ is a continuous morphism from $G$ into $(\mathbb{R}^+\times, \times)$.

**Proof.** (i) We have $\sigma(\theta(g)) \subseteq \sigma_{A_\theta}(\theta(g))$, thus $\sigma^1(\theta(g)) \subseteq \sigma_{A_\theta}^1(\theta(g))$. Moreover we know that $\partial \sigma_{A_\theta}(\theta(g)) \subseteq \sigma(\theta(g))$ and since $0 \notin \sigma_{A_\theta}(\theta(g))$ it is clear that $(\partial \sigma_{A_\theta})^1(\theta(g)) = \sigma_{A_\theta}^1(\theta(g))$ (every half-line from the origin that intersects $\sigma_{A_\theta}(\theta(g))$ intersects also its boundary by connectedness), hence $\sigma_{A_\theta}^1(\theta(g)) \subseteq \sigma^1(\theta(g))$.

(ii) $\theta$ is locally bounded so there exist $M > 1$ and an open $V \in \mathcal{V}(1)$ such that $||\theta(g)|| \leq M$ for all $g \in V$. Then $|\chi \circ \theta(g^{-1})| \leq M$ for all $g \in V$, thus $1/M \leq |\chi \circ \theta(g)| \leq M$ for all $g \in V$. Therefore $\Gamma_{|\chi \circ \theta|}$ is a bounded multiplicative subgroup of $(\mathbb{R}^+\times, \times)$, that is, $\Gamma_{|\chi \circ \theta|} = \{1\}$, which shows that $|\chi \circ \theta|$ is continuous. ■

Recall two useful results:

**Lemma 4.2** (see [7]). Let $X$ be a Banach space, $T \in \mathcal{L}(X)$, and $Y$ a $T$-invariant closed subspace of $X$. Then $\rho^\infty(T) \subseteq \rho^\infty(T|_Y)$ where $\rho^\infty$ denotes the unbounded connected component of the resolvent set $\rho$. If $0 \in \rho^\infty(T)$ then $\sigma^1(T|_Y) \subseteq \sigma^1(T)$.

**Proposition 4.3** (see [8] or [10]). If $X$ is a separable Banach space, then the map $T \mapsto \sigma(T)$ (respectively $T \mapsto \sigma^1(T)$) from $\mathcal{L}(X)$ into $\mathcal{K}(\mathbb{C})$ (respectively $\mathcal{K}(\mathbb{T})$) is Borel (where $\mathcal{K}(\mathbb{C})$ and $\mathcal{K}(\mathbb{T})$ are equipped with the Hausdorff topology and $\mathcal{L}(X)$ with the strong operator topology).

For $\chi \in \hat{A}_\theta$, we denote by $\chi_1$ the morphism from $G$ into $\mathbb{T}$ defined by $\chi_1(g) := (\chi \circ \theta)(g)/|(\chi \circ \theta)(g)|$ and we set:

$$\Omega_\chi := \{g \in G \mid \forall \lambda \in \mathbb{T}, \lambda \Gamma_{\chi_1} \not\subseteq \sigma^1(\theta(g))\},$$

$$\Omega := \bigcup_{\chi \in \hat{A}_\theta} \Omega_\chi,$$

$$\Sigma_\theta := \{g \in G \mid \text{there is no } P \in \mathcal{P} \text{ with } P \subseteq \sigma^1(\theta(g))\},$$

where $\mathcal{P}$ is the set of regular polygons of $\mathbb{T}$. 
THEOREM 4.4. Let $G$ be a locally compact and second countable abelian group, $m$ a Haar measure on $G$, $X$ a Banach space, and $\theta : G \to \mathcal{L}(X)$ a strongly continuous representation of $G$ on $X$. Then $\Omega$ is a null set for $m$.

Proof. Note that $\chi_1(g) \in \sigma^1(\theta(g))$ for all $\chi \in \hat{A}_\theta$.

**Step 1.** Suppose that $X$ is separable. By Proposition 4.3 and the strong continuity of $\theta$, the map $g \mapsto \sigma^1(\theta(g))$ from the locally compact abelian group $G$ into $\mathcal{K}(\mathbb{T})$ is Borel, so the result is a consequence of Proposition 3.4.

**Step 2.** Suppose that $X$ is not separable. If $\theta$ is uniformly continuous then for all $\chi \in \hat{A}$, $\chi \circ \theta$ and $\chi_1$ are continuous by Proposition 2.2 and Lemma 4.1, hence $\Gamma_{\chi_1} = \{1\}$ and $\Omega_\chi = \emptyset$.

If $\theta$ is not uniformly continuous, there exist $\delta > 0$ and a sequence $(g_n)_{n \in \mathbb{N}}$ in $G$ such that $\lim_n g_n = 1$ and $\|\theta(g_n) - I\| > \delta$. So there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of unit vectors in $X$ such that $\|\theta(g_n)x_n - x_n\| > \delta$ for all $n \in \mathbb{N}$. Now set $Y := \text{span}(\bigcup_{n \in \mathbb{N}}\{\theta(g)x_n \mid g \in G\})$; since $G$ is separable and $\theta$ strongly continuous, $\{\theta(g)x_n \mid g \in G\}$ is separable, thus $Y$ is separable, and clearly $Y$ is $(\theta(g))_{g \in G}$-invariant. Using the first step we conclude that $\bigcup_{\chi \in \hat{A}_\theta} \Omega_{\chi,Y}$ has measure zero, where

$$\Omega_{\chi,Y} := \{g \in G \mid \forall \lambda \in \mathbb{T}, \lambda \Gamma_{\chi_1} \not\subseteq \sigma^1(\theta(g)|Y)\}.$$  

If $g \in \Omega_\chi$, then $\sigma^1(\theta(g)) \neq \mathbb{T}$, thus $0 \in \rho_\infty(\theta(g))$, and by Lemma 4.2, $\sigma^1(\theta(g)|Y) \subseteq \sigma^1(\theta(g))$, further $g \in \Omega_{\chi,Y}$, that is, $\Omega_\chi \subseteq \Omega_{\chi,Y}$.

Accordingly $\bigcup_{\chi \in \hat{A}_\theta} \Omega_\chi$ is a null set for $m$. ■

**Remark.** If $\theta$ is uniformly continuous, the theorem is uninteresting because $\Omega_\chi$ is empty for all $\chi \in \hat{A}_\theta$; the interesting case concerns the strongly continuous representations that are not uniformly continuous:

**Corollary 4.5.** Let $G$ be a locally compact and second countable abelian group, $m$ a Haar measure on $G$, $X$ a Banach space, and $\theta : G \to \mathcal{L}(X)$ a strongly continuous representation. Then $\theta$ is uniformly continuous if and only if $\Sigma_\theta$ is a non-null set for $m$.

**Proof.** If $\theta$ is uniformly continuous then there exists an open set $U \in \mathcal{V}(1)$ in $G$ such that $\sigma^1(\theta(g)) \subseteq B(0; 1/2)$ for all $g \in U$, and so $U \subseteq \Sigma_\theta$ is a non-null set.

If $\theta$ is not continuous, then by Proposition 2.2 there exists $\chi \in \hat{A}_\theta$ such that $\chi \circ \theta$ is not continuous, that is, by Lemma 4.1, $\chi_1$ is not continuous, so $\Gamma_{\chi_1} \neq \{1\}$ and thus, except for the null set $\Omega_\chi$, $\sigma^1(\theta(g))$ contains the image of $\Gamma_{\chi_1}$ under a rotation. ■

**Remark.** The theorem and its corollary are valid without the hypothesis of second countability of $G$ provided that the space $X$ is separable.
Remark 4.6. If there is $\chi \in \hat{A}_\theta$ such that $\Gamma_{\chi_1} = \mathbb{T}$ then $\{g \in G \mid \sigma^1(\theta(g)) \neq \mathbb{T}\} \subseteq \Omega_\chi$ and thus $\{g \in G \mid \sigma^1(\theta(g)) \neq \mathbb{T}\}$ is a null set.

Recall the following lemma (see [7]):

Lemma 4.7. Let $X$ be a Banach space and $B \in \mathcal{L}(X)$. If $0 \notin \sigma(B)$ then $\sigma^1(B) \neq \mathbb{T}$ if and only if $\sigma^1_k(B) \neq \mathbb{T}$ where $\sigma_k(\cdot)$ is the Kato essential spectrum (the spectrum corresponding to the set of all semi-Fredholm operators).

Corollary 4.8. Let $\theta$ be a strongly continuous representation of an abelian, locally compact, locally connected and second countable group $G$ on a Banach space $X$. Then $\theta$ is uniformly continuous if and only if $\{g \in G \mid \sigma^1_{\epsilon}(\theta(g)) \neq \mathbb{T}\}$ is a non-null set.

Proof. If $\theta$ is uniformly continuous, it suffices to apply Corollary 4.5 for $\sigma^1_k(\theta(g)) \subseteq \sigma^1_{\epsilon}(\theta(g))$.

For the converse, recall that an abelian, locally compact, locally connected and second countable group is isomorphic to $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{D}$ with $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{\aleph_0\}$ and $\mathbb{D}$ discrete (see [6, Proposition 8.34, Proposition 8.43, and Theorem 8.46]). Such a group locally admits division, thus if $\theta$ is not continuous then by Proposition 2.2 there exists $\chi \in \hat{A}_\theta$ such that $\chi \circ \theta$ is not continuous, that is, $\Gamma_{\chi_1} = \mathbb{T}$, and it suffices to apply Remark 4.6 and Lemma 4.7.

Corollary 4.9. If $X$ is a hereditarily indecomposable Banach space then a strongly continuous representation of a locally compact, locally connected and second countable abelian group $G$ on $X$ is automatically uniformly continuous.

Proof. Recall that for all $g \in G$, $\theta(g) = \lambda_g I + S_g$ where $\lambda_g \in \sigma(\theta(g))$ and $S_g$ is a strictly singular operator (see [4]), thus $\lambda_g \neq 0$ and it is easy to check that $\sigma^1_{\epsilon}(\theta(g)) = \{\lambda_g/|\lambda_g|\}$ where $\sigma^1_{\epsilon}(\cdot)$ is the essential spectrum, since $\sigma^1_k(\theta(g)) \subseteq \sigma^1_{\epsilon}(g)$ for all $g \in G$. The result then follows from Corollary 4.8.

Finally, we conclude with a result announced in the introduction:

Corollary 4.10. Let $\theta$ be a strongly continuous representation of a locally compact and second countable abelian group $G$ on a Banach space $X$. Then either $\theta$ is uniformly continuous or there is a null set $M$ in $G$ such that

$$\liminf_{g \to 1, g \in G \setminus M} \rho(\theta(g) - 1) \geq \sqrt{2}.$$ 

Proof. Since $\theta$ is locally bounded, we can apply Lemma 2.1, and for every $\epsilon > 0$ there exists $V_\epsilon \in \mathcal{V}(1)$ such that for all $g \in V_\epsilon$,

$$\sigma(\theta(g)) \subseteq \{z \in \mathbb{C} \mid 1 - \epsilon \leq |z| \leq 1 + \epsilon\}.$$

Assume now that $\theta$ is not continuous. There exists a null set $M$ in $G$ such that for all $g \in G \setminus M$, $\sigma^1(\theta(g))$ contains a regular polygon $\lambda \Gamma_p$. Then if
If \( g \in V_e \setminus M \) there exists \( z_1 \in \sigma^1(\theta(g)) \) such that \( |z_1 - 1| \geq \sqrt{2} \), and thus there exists \( z \in \sigma(\theta(g)) \) such that \( |z - 1| \geq \sqrt{2} - \epsilon \). Hence, \( \rho(\theta(g) - I) \geq \sqrt{2} - \epsilon \) for all \( g \in V_e \). □

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