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Haar measure and continuous representations of locally compact abelian groups

by

JEAN-CHRISTOPHE TOMASI (Bastia)

Abstract. Let $\mathcal{L}(X)$ be the algebra of all bounded operators on a Banach space X, and let $\theta : G \to \mathcal{L}(X)$ be a strongly continuous representation of a locally compact and second countable abelian group G on X. Set $\sigma^1(\theta(g)) := \{\lambda/|\lambda| \mid \lambda \in \sigma(\theta(g))\}$, where $\sigma(\theta(g))$ is the spectrum of $\theta(g)$, and let Σ_θ be the set of all $g \in G$ such that $\sigma^1(\theta(g))$ does not contain any regular polygon of $\mathbb{T}$ (by a *regular polygon* we mean the image under a rotation of a closed subgroup of the unit circle $\mathbb{T}$ different from $\{1\}$). We prove that θ is uniformly continuous if and only if Σ_θ is a non-null set for the Haar measure on G.

1. Introduction. A characterization of uniform continuity for strongly continuous groups was given in [7]. Indeed the authors proved that a strongly continuous one-parameter group $(T(t))_{t\in\mathbb{R}}$ on a Banach space X is uniformly continuous if and only if $\{t \in \mathbb{R} \mid \sigma^1(T(t)) \neq \mathbb{T}\}$ is non-meager, where $\mathbb{T}$ denotes the unit circle of $\mathbb{C}$ and $\sigma^1(T(t)) := \{\lambda/|\lambda| \mid \lambda \in \sigma(T(t))\}$, well defined since $T(t)$ is invertible. The following generalization of this result was obtained in [1]: if G is a second countable and locally compact abelian group then either θ is uniformly continuous or $\Sigma_\theta := \{g \in G \mid \text{there is no } P \in \mathcal{P} \text{ with } P \subseteq \sigma^1(\theta(g))\}$ is meager, where $\mathcal{P}$ is the set of regular polygons of $\mathbb{T}$. So when the representation is not uniformly continuous, the angular distribution of the spectrum of $\theta(g)$ is rather dispersed, except for g in a meager set in G.

In the present work, we are interested in another condition, obtained by replacing *meager set* by *null set*.

EXAMPLE 1.1. Let $(T(t))_{t\in\mathbb{R}}$ be the translation group on $L^2(\mathbb{R})$ defined by $(T(t)f)(x) = f(x+t)$. This one-parameter group is strongly continuous, not uniformly continuous and for all $t \neq 0$, $\sigma(T(t)) = \mathbb{T}$, thus $\Sigma_\theta = \{0\}$ is indeed a null set.

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Following J. Esterle (see [2], [3]) we define a *representation of a topological group G on a Banach algebra A* to be a map $\theta : G \to A$ such that $\theta(1) = I$, where 1 and I respectively denote the unit elements of G and A, and $\theta(uv) = \theta(u)\theta(v)$.

In [2] the author established a *zero-$\sqrt{3}$ law* for representations of locally compact abelian groups: if $\theta : G \to A$ is a locally bounded representation of such a group on a Banach algebra then either θ is uniformly continuous or $\limsup_{g\to 1} \rho(\theta(g) - I) \geq \sqrt{3}$, where ρ denotes the spectral radius.

As a consequence of our results we find, but only in the case of strongly continuous representations of locally compact and second countable abelian groups, that either θ is uniformly continuous or $\liminf_{g\to 1,\, g\in G\setminus M} \rho(\theta(g) - I) \geq \sqrt{2}$ where M is a null set in G.

2. Characterization of uniform continuity. For a locally bounded representation of a locally compact abelian group G, there are some arguments, based on Gelfand–Hille's theorem, Shilov's idempotent theorem and the standard structure theorem for locally compact abelian groups (see [2]) that allow us to go from spectral continuity (that is, $\lim_{g\to 1} \rho(\theta(g) - I) = 0$) to uniform continuity.

Furthermore R. Phillips (see [5]) proved that the continuity for one-parameter groups can be read *through the characters*, in the sense that if $T : \mathbb{R} \to A$ is a locally bounded representation of $\mathbb{R}$ on a commutative Banach algebra A then its uniform continuity is equivalent to the continuity of $t \in \mathbb{R} \mapsto \chi(T(t))$ for all $\chi \in \hat{A}$, where $\hat{A}$ denotes the character space of A.

However, going from the continuity through each character (that is, $\chi \circ T$ continuous for all $\chi \in \hat{A}$) to the uniform condition on $\hat{A}$: $\lim_{t\to 0} \rho(\theta(t) - 1) = 0$ required, in the case of $\mathbb{R}$, an analytical argument difficult to adapt to a general group.

Therefore in order to generalize this result from $\mathbb{R}$ to any locally compact abelian group, we had to use in [1] the Phillips theorem and the standard structure theorem for locally compact abelian groups, and to deal separately with compact groups and euclidean groups $\mathbb{R}^n$.

Here, we present a direct proof of this generalization and also a simplified proof of the Phillips result.

In what follows we denote by $\mathcal{V}(1)$ the family of all neighborhoods of the unit element of G.

LEMMA 2.1. *Let θ be a locally bounded representation of a topological abelian group G on a Banach algebra A. Then for all $\epsilon > 0$ there exists $V_\epsilon \in \mathcal{V}(1)$ such that for all $g \in V_\epsilon$,*

$$\sigma(\theta(g)) \subseteq \{z \in \mathbb{C} \mid 1 - \epsilon \leq |z| \leq 1 + \epsilon\}.$$

Proof. Since θ is locally bounded, there exist $M > 1$ and $V \in \mathcal{V}(1)$ such that for all $g \in V$, $\|\theta(g)\| \le M$. By the continuity of the product, for all $n \ge 1$ there exists $V_n \in \mathcal{V}(1)$ such that for all $g \in V_n$, $\|\theta(g^n)\| \le M$ and $\|\theta(g^{-n})\| \le M$. Since $\sigma(\theta(g^{-n})) = \{1/\lambda \mid \lambda \in \sigma(\theta(g^n))\}$ we obtain

$$\sigma(\theta(g^n)) \subseteq \{z \in \mathbb{C} \mid 1/M \le |z| \le M\},$$

and since $\sigma(\theta(g^n)) = (\sigma(\theta(g)))^n$, we have

$$(1/M)^{1/n} \le |z| \le M^{1/n}$$

for all $g \in V_n$ and $z \in \sigma(\theta(g))$. This yields the desired conclusion. ■

Proposition 2.2. *Let θ be a locally bounded representation of a locally compact abelian group G on a commutative Banach algebra A. The following assertions are equivalent:*

(i) *θ is uniformly continuous.*
(ii) *θ is spectrally continuous.*
(iii) *For all $\chi \in \hat{A}$, $\chi \circ \theta$ is continuous.*

Proof. (i)⇒(ii)⇒(iii): Clear.

(iii)⇒(ii): Let $V \in \mathcal{V}(1)$ be compact and symmetric. Then $H = \bigcup_{n\in\mathbb{N}} V^n$ is a locally compact and σ-compact subgroup of G. We have $H \in \mathcal{V}(1)$, hence it suffices to show that $\theta_H := \theta_{|H}$ is spectrally continuous.

We first show that $\{\chi \circ \theta_H/|\chi \circ \theta_H| \mid \chi \in \hat{A}\}$ is compact in $C(H, \mathbb{T})$ equipped with the topology of compact convergence. Since H is σ-compact, this topology is metrizable, thus it suffices to check that $\{\chi \circ \theta_H/|\chi \circ \theta_H| \mid \chi \in \hat{A}\}$ is sequentially compact. So let $(\chi_n \circ \theta_H/|\chi_n \circ \theta_H|)_{n\in\mathbb{N}}$ be a sequence in $\{\chi \circ \theta_H/|\chi \circ \theta_H| \mid \chi \in \hat{A}\}$. The Gelfand space $\hat{A}$ is compact, and thus $\{\chi \circ \theta_H/|\chi \circ \theta_H| \mid \chi \in \hat{A}\} \subseteq C(H, \mathbb{T})$ is compact for the product topology and so is the set of restrictions to V that we denote $\{\chi \circ \theta_V/|\chi \circ \theta_V|\}$.

By hypothesis we have $\{\chi \circ \theta_V/|\chi \circ \theta_V| \mid \chi \in \hat{A}\} \subseteq C(V, \mathbb{T})$, and thus we can apply Eberlein–Šmulian's theorem (see [9, p. 296]): $\{\chi \circ \theta_V/|\chi \circ \theta_V| \mid \chi \in \hat{A}\}$ is sequentially compact in $\mathbb{T}^V$, therefore we can extract a subsequence $(\chi_{n_k} \circ \theta_V/|\chi_{n_k} \circ \theta_V|)_{k\in\mathbb{N}}$ that converges to an element $\chi \circ \theta_V/|\chi \circ \theta_V|$, that is, for all $g \in V$, $\chi_{n_k}(\theta_V(g))/|\chi_{n_k}(\theta_V(g))| \to \chi(\theta_V(g))/|\chi(\theta_V(g))|$.

But since it concerns restrictions of morphisms, the convergence extends from V to $H = \bigcup_{n\in\mathbb{N}} V^n$, and using the dominated convergence theorem, we find that for all $f \in L^1(H)$, $\hat{f}(\chi_{n_k} \circ \theta_H/|\chi_{n_k} \circ \theta_H|) \to \hat{f}(\chi \circ \theta_H/|\chi \circ \theta_H|)$ (where $L^1(H)$ denotes the L^1-space of H with respect to a Haar measure m and $\hat{f}$ denotes the Fourier transform of f).

Since in the dual group $\hat{H}$ the topology of compact convergence on H coincides with the weak* topology that $\hat{H}$ inherits as a subset of $L^\infty(H)$, we conclude that $\chi_{n_k} \circ \theta_H/|\chi_{n_k} \circ \theta_H| \to \chi \circ \theta_H/|\chi \circ \theta_H|$, which proves the compactness.

Then, by Ascoli's theorem, $\{\chi\circ\theta_H/|\chi\circ\theta_H| \mid \chi\in\hat{A}\}$ is equicontinuous; so for all $\epsilon>0$ there exists $W_\epsilon\in\mathcal{V}(1)$ in H such that for all $h\in W_\epsilon$, $\sup_{\chi\in\hat{A}}\big|\chi\circ\theta_H(h)/|\chi\circ\theta_H(h)|-1\big|<\epsilon$.

Lemma 2.1 yields $V_\epsilon\in\mathcal{V}(1)$ such that for all $h\in V_\epsilon$ and all $\chi\in\hat{A}$,

$$\chi\circ\theta_H(h)\in\{z\in\mathbb{C}\mid 1-\epsilon\le|z|\le 1+\epsilon\},$$

and thus for all $h\in W_\epsilon\cap V_\epsilon$ and all $\chi\in\hat{A}$,

$$\begin{aligned}|\chi\circ\theta_H(h)-1| &\le \big|\chi\circ\theta_H(h)-\chi\circ\theta_H(h)/|\chi\circ\theta_H(h)|\big|\\ &\qquad+\big|\chi\circ\theta_H(h)/|\chi\circ\theta_H(h)|-1\big|\\ &\le 2\epsilon,\end{aligned}$$

that is, $\rho(\theta_H(h)-I)\le 2\epsilon$, and θ_H is spectrally continuous.

(ii)$\Rightarrow$(i): See Theorem 3.3 in [2]. ∎

3. Preliminary results. Let G be a topological group and $\varphi: G\to\mathbb{T}$ a morphism. Define

$$\begin{aligned}\Gamma_\varphi &:= \{\lambda\in\mathbb{T}\mid \text{there is a net } (g_i) \text{ in } G \text{ converging to 1 such that } \varphi(g_i)\to\lambda\}\\ &= \bigcap_{W\in\mathcal{V}(1)}\overline{\varphi(W)}\end{aligned}$$

(see [2]). Then:

- Γ_φ is a closed subgroup of $\mathbb{T}$ (thus $\Gamma_\varphi=\Gamma_k$ the group of kth roots of unity for some $k\ge 1$, or $\Gamma_\varphi=\mathbb{T}$).
- φ is continuous if and only if $\Gamma_\varphi=\{1\}$.
- If the group locally admits division by every $n\ge 1$ (in the sense that for every $n\in\mathbb{N}$ there exist $V\in\mathcal{V}(1)$, a compact subset W of G containing 1 and a map $\psi: V\to W$ such that $\psi(1)=1$ and $\psi^n(u)=u$ for every $u\in V$), then one can easily check that Γ_φ is divisible, thus either $\Gamma_\varphi=\mathbb{T}$ or $\Gamma_\varphi=\{1\}$.

Lemma 3.1. *Let Γ be a subset of $\mathbb{T}$, and V an open subset of $\mathbb{T}$ such that $\lambda V\cap\Gamma\ne\emptyset$ for all $\lambda\in\mathbb{T}$. Then there exists a compact set $K\subseteq V$ such that $\lambda K\cap\Gamma\ne\emptyset$ for all $\lambda\in\mathbb{T}$.*

Proof. Since V is open in $\mathbb{T}$, there exists a sequence $(O_n)_{n\in\mathbb{N}}$ of relatively compact open sets in V with $\overline{O_n}\subseteq O_{n+1}$ for all $n\in\mathbb{N}$ and $V=\bigcup_{n\in\mathbb{N}}O_n$. It suffices to show that there exists an element of the sequence $(\overline{O_n})_{n\in\mathbb{N}}$ intersected by every $\lambda\Gamma$.

If it is not true then for all $n\in\mathbb{N}$ there exists $\lambda_n\in\mathbb{T}$ such that $\lambda_n\Gamma\cap\overline{O_n}=\emptyset$, thus $\lambda_n\Gamma\subseteq\mathbb{T}\setminus\overline{O_n}\subset\mathbb{T}\setminus O_n=:F_n$. As the sequence $(O_n)_{n\in\mathbb{N}}$ is increasing, $(F_n)_{n\in\mathbb{N}}$ is a decreasing sequence of closed sets such that

$$\bigcap_{n\in\mathbb{N}} F_n = \bigcap_{n\in\mathbb{N}} \mathbb{T}\setminus O_n = \mathbb{T}\setminus \bigcup_{n\in\mathbb{N}} O_n = \mathbb{T}\setminus V.$$

Moreover, since $\mathbb{T}$ is compact, we can suppose that $(\lambda_n)_{n\in\mathbb{N}}$ is convergent. Denote by λ its limit, let $\mu\in\Gamma$ and $N\in\mathbb{N}$. For all $k\ge N$, we have $\lambda_k\mu\in\lambda_k\Gamma\subseteq F_k\subseteq F_N$, and so $\lambda\mu\in\overline{F_N}=F_N$. Finally $\lambda\Gamma\subseteq\bigcap_{n\in\mathbb{N}}F_n=\mathbb{T}\setminus V$, which is a contradiction. ∎

Lemma 3.2. *Let G be a locally compact and second countable abelian group and m a Haar measure on G. If $A\subseteq G$ is measurable with $m(A)>0$, then for every $N\ge 1$ there exists $U_1\in\mathcal{V}(1)$ such that for all $(g_1,\dots,g_N)\in U_1^N$,*

$$m\Big(A\cap\bigcap_{i=1}^{N} g_iA\Big)>0.$$

Proof. As G is σ-finite, we can assume that $m(A)<\infty$. Let $\beta\in\,]0,1[$ and $\alpha=\beta/(N+1)$. We know, by regularity of m, that there exist $K\subseteq A\subseteq U$ with K and U respectively compact and open satisfying $m(K)\ge(1-\alpha)m(U)$; since K is compact, there exist $U_1\in\mathcal{V}(1)$ such that $U_1K\subseteq U$. Let us check by finite induction on k that for all $k\in\{1,\dots,N\}$ and for all $(g_1,\dots,g_k)\in U_1^k$,

$$m\Big(K\cap\bigcap_{i=1}^{k} g_iK\Big)\ge(1-(k+1)\alpha)m(U).$$

We have $m(K\cap g_1K)\ge m(K)+m(g_1K)-m(U)$ for $K\cup g_1K\subseteq U$, and since m is translation-invariant, we obtain $m(K\cap g_1K)\ge(1-2\alpha)m(U)$. Suppose that $m(K\cap\bigcap_{i=1}^k g_iK)\ge(1-(k+1)\alpha)m(U)$ for some $1\le k<N$; then

$$\begin{aligned}m\Big(K\cap\bigcap_{i=1}^{k+1} g_iK\Big)&=m\Big(g_1K\cap\Big(K\cap\bigcap_{i=2}^{k+1} g_iK\Big)\Big)\\&\ge m(g_1K)+m\Big(K\cap\bigcap_{i=2}^{k+1} g_iK\Big)-m(U).\end{aligned}$$

Thus, by induction hypothesis and invariance of m,

$$\begin{aligned}m(g_1K)+m\Big(K\cap\bigcap_{i=2}^{k+1} g_iK\Big)-m(U)&\\&\ge m(K)+(1-(k+1)\alpha)m(U)-m(U)\\&\ge(1-(k+2)\alpha)m(U),\end{aligned}$$

which is the expected result.

In particular, for all $(g_1,\dots,g_N)\in U_1^N$,

$$m\Big(A\cap\bigcap_{i=1}^N g_iA\Big)\ge(1-(N+1)\alpha)m(U)$$
$$=(1-\beta)m(U)\ge(1-\beta)m(A)>0. \blacksquare$$

LEMMA 3.3. *Let φ be a morphism from a locally compact abelian group G into $\mathbb{T}$, and m a Haar measure on G. If V is an open subset of $\mathbb{T}$ such that $\lambda V\cap\Gamma_\varphi\ne\emptyset$ for all $\lambda\in\mathbb{T}$, and if $A\subseteq G$ is measurable with $m(A)>0$, then $\varphi(A)\cap V\ne\emptyset$.*

Proof. Suppose that $\varphi(A)\cap V=\emptyset$. Let us prove that there exist a symmetric and open $V_0\in\mathcal{V}(1)$ and an open set V_1 in $\mathbb{T}$ such that for all $\lambda\in\mathbb{T}$,

$$\lambda V_1\cap\Gamma_\varphi\ne\emptyset\quad\text{and}\quad V_0V_1\subseteq V.$$

By Lemma 3.1 there exists a compact set $K\subseteq V$ such that $\lambda K\cap\Gamma_\varphi\ne\emptyset$ for all $\lambda\in\mathbb{T}$; thus if $\pi:\mathbb{T}\times\mathbb{T}\to\mathbb{T}$ denotes the product on $\mathbb{T}\times\mathbb{T}$, the compact set $\{1\}\times K$ is a subset of the open set $\pi^{-1}(V)$ and so there exist a symmetric open unit-neighborhood V_0 and an open set V_1 containing K such that

$$V_0\times V_1\subseteq\pi^{-1}(V)\quad\text{and}\quad V_0V_1\subseteq V.$$

Since $\lambda V_1\cap\Gamma_\varphi\ne\emptyset$ for all $\lambda\in\mathbb{T}$, we can easily deduce that $\mathbb{T}=\bigcup_{\lambda\in\Gamma_\varphi}\lambda V_1$, and then by compactness there exists $N\ge1$ such that $\mathbb{T}=\bigcup_{i=1}^N\lambda_iV_1$ with $\lambda_i\in\Gamma_\varphi$.

By Lemma 3.2 there exists $U_1\in\mathcal{V}(1)$ in G such that for all $(g_1,\dots,g_N)\in U_1^N$, $m(A\cap\bigcap_{i=1}^N g_iA)>0$ and thus $A\cap\bigcap_{i=1}^N g_iA\ne\emptyset$.

Let $i\in\{1,\dots,N\}$ and $\lambda_i\in\Gamma_\varphi$. Since $\Gamma_\varphi\subseteq\overline{\varphi(U_1)}$, there exists $g_i\in U_1$ such that $\lambda_i\varphi(g_i)^{-1}\in V_0$, that is, $\lambda_i\in\varphi(g_i)V_0$, and thus

$$\mathbb{T}=\bigcup_{i=1}^N\lambda_iV_1\subseteq\bigcup_{i=1}^N\varphi(g_i)V_0V_1\subseteq\bigcup_{i=1}^N\varphi(g_i)V,$$

so $\mathbb{T}=\bigcup_{i=1}^N\varphi(g_i)V$ and $G=\varphi^{-1}(\mathbb{T})=\bigcup_{i=1}^N g_i\varphi^{-1}(V)$.

Let $g\in A\cap\bigcap_{i=1}^N g_iA\subseteq G$. There exists $i_0\in\{1,\dots,N\}$ such that $g\in g_{i_0}\varphi^{-1}(V)$, thus $g_{i_0}^{-1}g\in\varphi^{-1}(V)$. Since $g\in g_{i_0}A$, we obtain $g_{i_0}^{-1}g\in A\cap\varphi^{-1}(V)$, which is a contradiction. $\blacksquare$

PROPOSITION 3.4. *Let G be a locally compact abelian group, m a Haar measure on G, $\mathcal{K}(\mathbb{T})$ the space of all compact subsets of $\mathbb{T}$ equipped with the Hausdorff metric and $\omega:G\to\mathcal{K}(\mathbb{T})$ a Borel map. Let $(\varphi_i)_{i\in I}$ be a family of morphisms from G into $\mathbb{T}$ such that $\varphi_i(g)\in\omega(g)$ for all $i\in I$ and $g\in G$. For $i\in I$ set*

$$\Omega_{\varphi_i}:=\{g\in G\mid\forall\lambda\in\mathbb{T},\ \lambda\Gamma_{\varphi_i}\not\subseteq\omega(g)\}.$$

Then the set $\bigcup_{i\in I}\Omega_{\varphi_i}$ has measure zero.

Proof. The proof is in two steps.

STEP 1. Suppose that $\bigcup_{i\in I} \Gamma_{\varphi_i}$ is infinite. Then the family of groups Γ_{φ_i} contains elements of arbitrarily large order, and so for each nonempty open set U of $\mathbb{T}$ there exists $i \in I$ such that $\lambda\Gamma_{\varphi_i} \cap U \neq \emptyset$ for every $\lambda \in \mathbb{T}$.

Let $g \in G$ be such that $\omega(g) \neq \mathbb{T}$. Then the open set $\mathbb{T} \setminus \omega(g)$ is nonempty and thus there exists $i \in I$ such that $\lambda\Gamma_{\varphi_i} \not\subseteq \omega(g)$ for all $\lambda \in \mathbb{T}$, so that $g \in \Omega_{\varphi_i}$. Since the other inclusion is obvious, we obtain $\bigcup_{i\in I} \Omega_{\varphi_i} = \{g \in G \mid \omega(g) \neq \mathbb{T}\}$, which is measurable as the inverse image of an open set in $\mathcal{K}(\mathbb{T})$.

Let $\mathcal{V} = \{V_n \mid n \in \mathbb{N}\}$ be a basis of open subsets on $\mathbb{T}$. Set $A_n = \{g \in G \mid \omega(g) \cap V_n = \emptyset\}$. Then the set $\{g \in G \mid \omega(g) \neq \mathbb{T}\} = \bigcup_{n\in\mathbb{N}} A_n$ is a Borel subset of $\mathbb{T}$.

If $m(\{g \in G \mid \omega(g) \neq \mathbb{T}\}) > 0$ then there exists $n_0 \in \mathbb{N}$ such that $m(A_{n_0}) > 0$; since $\bigcup_{i\in I} \Gamma_{\varphi_i}$ is infinite, there exists $i_0 \in I$ such that $\lambda V_{n_0} \cap \Gamma_{\varphi_{i_0}} \neq \emptyset$ for all $\lambda \in \mathbb{T}$, but $\varphi_{i_0}(g) \in \omega(g)$ for all $g \in G$, hence $\varphi_{i_0}^{-1}(V_{n_0}) \cap A_{n_0} = \emptyset$, which contradicts Lemma 3.3; so $\bigcup_{i\in I} \Omega_{\varphi_i}$ has measure zero.

STEP 2. Suppose that $\bigcup_{i\in I} \Gamma_{\varphi_i}$ is finite, thus $\{\Gamma_{\varphi_i} \mid i \in I\} = \{\Gamma_{p_j}\}_{j=1}^m$. For all $j \in \{1, \ldots, m\}$, define $\Omega_j = \{g \in G \mid \forall\lambda \in \mathbb{T},\ \lambda\Gamma_{p_j} \not\subseteq \omega(g)\}$ and let us check that Ω_j has measure zero.

Let $\mathcal{W} = \{W_n \mid n \in \mathbb{N}\}$ be the (countable) set of finite unions W_n of elements of $\mathcal{V}$ such that $\lambda\Gamma_{p_j} \cap W_n \neq \emptyset$ for all $\lambda \in \mathbb{T}$.

Let $g \in \Omega_j$. For all $\lambda \in \mathbb{T}$, we have $\lambda\Gamma_{p_j} \not\subseteq \omega(g)$, thus $\lambda\Gamma_{p_j} \cap \mathbb{T} \setminus \omega(g) \neq \emptyset$, and by Lemma 3.1, there exists a compact subset $K \subseteq \mathbb{T} \setminus \omega(g)$ such that $\lambda K \cap \Gamma_{p_j} \neq \emptyset$ for all $\lambda \in \mathbb{T}$.

Since K is compact and $\mathbb{T} \setminus \omega(g)$ is open, there exists $W_n \in \mathcal{W}$ such that $K \subseteq W_n \subseteq \mathbb{T} \setminus \omega(g)$, so $g \in B_n := \{g \in G \mid W_n \cap \omega(g) = \emptyset\}$. Therefore $\Omega_j = \bigcup_{n\in\mathbb{N}} B_n$ (the other inclusion is obvious). But for all $n \in \mathbb{N}$, B_n is measurable since ω is Borel, and since $\{C \in \mathcal{K}(\mathbb{T}) \mid C \cap W_n = \emptyset\}$ is a Borel subset of $\mathcal{K}(\mathbb{T})$, Ω_j is measurable too.

If $m(\Omega_j) > 0$ then there exists $n_0 \in \mathbb{N}$ such that $m(B_{n_0}) > 0$; but there exists a morphism φ in the family $(\varphi_i)_{i\in I}$ such that $\Gamma_\varphi = \Gamma_{p_j}$ and since for all $g \in G$, $\varphi(g) \in \omega(g)$, we find that $\varphi^{-1}(W_{n_0}) \cap B_{n_0} = \emptyset$, whereas $\lambda\Gamma_\varphi \cap W_{n_0} \neq \emptyset$ for all $\lambda \in \mathbb{T}$, which contradicts Lemma 3.3.

Accordingly $\bigcup_{i\in I} \Omega_{\varphi_i} = \bigcup_{j=1}^m \Omega_j$ has measure zero. ∎

4. The main result and consequences. Let G be a locally compact abelian group, X a Banach space and $\theta : G \to \mathcal{L}(X)$ a strongly continuous representation of G on X. We are interested in the distribution of the arguments of the elements of the spectrum $\sigma(\theta(g))$ when θ is not uniformly continuous. We write:

- A_θ for the closed subalgebra of $\mathcal{L}(X)$ generated by $\theta(G)$ (so A_θ is commutative),
- $\sigma_{A_\theta}(\theta(g))$ for the spectrum of $\theta(g)$ in A_θ,
- $\hat{A}_\theta$ for the character space of A_θ,
- $K^1 = \{\lambda/|\lambda| \mid \lambda \in K\}$ for $K \subseteq \mathbb{C}^*$.

We have the following results:

LEMMA 4.1.

(i) *For all* $g \in G$, $\sigma^1(\theta(g)) = \sigma^1_{A_\theta}(\theta(g))$.

(ii) *For all* $\chi \in \hat{A}_\theta$, *the map* $g \mapsto |(\chi \circ \theta)(g)|$ *is a continuous morphism from* G *into* $(\mathbb{R}^{+*}, \times)$.

Proof. (i) We have $\sigma(\theta(g)) \subseteq \sigma_{A_\theta}(\theta(g))$, thus $\sigma^1(\theta(g)) \subseteq \sigma^1_{A_\theta}(\theta(g))$. Moreover we know that $\partial\sigma_{A_\theta}(\theta(g)) \subseteq \sigma(\theta(g))$ and since $0 \notin \sigma_{A_\theta}(\theta(g))$ it is clear that $(\partial\sigma_{A_\theta})^1(\theta(g)) = \sigma^1_{A_\theta}(\theta(g))$ (every half-line from the origin that intersects $\sigma_{A_\theta}(\theta(g))$ intersects also its boundary by connectedness), hence $\sigma^1_{A_\theta}(\theta(g)) \subseteq \sigma^1(\theta(g))$.

(ii) θ is locally bounded so there exist $M > 1$ and an open $V \in \mathcal{V}(1)$ such that $\|\theta(g)\| \le M$ for all $g \in V$. Then $|\chi \circ \theta(g^{-1})| \le M$ for all $g \in V$, thus $1/M \le |\chi \circ \theta(g)| \le M$ for all $g \in V$. Therefore $\Gamma_{|\chi\circ\theta|}$ is a bounded multiplicative subgroup of $(\mathbb{R}^{+*}, \times)$, that is, $\Gamma_{|\chi\circ\theta|} = \{1\}$, which shows that $|\chi \circ \theta|$ is continuous. ∎

Recall two useful results:

LEMMA 4.2 (see [7]). *Let* X *be a Banach space,* $T \in \mathcal{L}(X)$, *and* Y *a* T-*invariant closed subspace of* X. *Then* $\rho_\infty(T) \subseteq \rho_\infty(T_{|Y})$ *where* ρ_∞ *denotes the unbounded connected component of the resolvent set* ρ. *If* $0 \in \rho_\infty(T)$ *then* $\sigma^1(T_{|Y}) \subseteq \sigma^1(T)$.

PROPOSITION 4.3 (see [8] or [10]). *If* X *is a separable Banach space, then the map* $T \mapsto \sigma(T)$ (*respectively* $T \mapsto \sigma^1(T)$) *from* $\mathcal{L}(X)$ *into* $\mathcal{K}(\mathbb{C})$ (*respectively* $\mathcal{K}(\mathbb{T})$) *is Borel* (*where* $\mathcal{K}(\mathbb{C})$ *and* $\mathcal{K}(\mathbb{T})$ *are equipped with the Hausdorff topology and* $\mathcal{L}(X)$ *with the strong operator topology*).

For $\chi \in \hat{A}_\theta$, we denote by χ_1 the morphism from G into $\mathbb{T}$ defined by $\chi_1(g) := (\chi \circ \theta)(g)/|(\chi \circ \theta)(g)|$ and we set:

$$\Omega_\chi := \{g \in G \mid \forall \lambda \in \mathbb{T},\ \lambda\Gamma_{\chi_1} \not\subseteq \sigma^1(\theta(g))\},$$
$$\Omega := \bigcup_{\chi \in \hat{A}_\theta} \Omega_\chi,$$
$$\Sigma_\theta := \{g \in G \mid \text{there is no } P \in \mathcal{P} \text{ with } P \subseteq \sigma^1(\theta(g))\},$$

where $\mathcal{P}$ is the set of regular polygons of $\mathbb{T}$.

THEOREM 4.4. *Let G be a locally compact and second countable abelian group, m a Haar measure on G, X a Banach space, and $\theta : G \to \mathcal{L}(X)$ a strongly continuous representation of G on X. Then Ω is a null set for m.*

Proof. Note that $\chi_1(g) \in \sigma^1(\theta(g))$ for all $\chi \in \hat{A}_\theta$.

STEP 1. Suppose that X is separable. By Proposition 4.3 and the strong continuity of θ, the map $g \mapsto \sigma^1(\theta(g))$ from the locally compact abelian group G into $\mathcal{K}(\mathbb{T})$ is Borel, so the result is a consequence of Proposition 3.4.

STEP 2. Suppose that X is not separable. If θ is uniformly continuous then for all $\chi \in \hat{A}$, $\chi \circ \theta$ and χ_1 are continuous by Proposition 2.2 and Lemma 4.1, hence $\Gamma_{\chi_1} = \{1\}$ and $\Omega_\chi = \emptyset$.

If θ is not uniformly continuous, there exist $\delta > 0$ and a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $\lim_n g_n = 1$ and $\|\theta(g_n) - I\| > \delta$. So there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors in X such that $\|\theta(g_n)x_n - x_n\| > \delta$ for all $n \in \mathbb{N}$. Now set $Y := \overline{\operatorname{span}(\bigcup_{n\in\mathbb{N}}\{\theta(g)x_n \mid g \in G\})}$; since G is separable and θ strongly continuous, $\{\theta(g)x_n \mid g \in G\}$ is separable, thus Y is separable, and clearly Y is $(\theta(g))_{g\in G}$-invariant. Using the first step we conclude that $\bigcup_{\chi\in\hat{A}_\theta} \Omega_{\chi,Y}$ has measure zero, where

$$\Omega_{\chi,Y} := \{g \in G \mid \forall\lambda \in \mathbb{T},\ \lambda\Gamma_{\chi_1} \not\subseteq \sigma^1(\theta(g)_{|Y})\}.$$

If $g \in \Omega_\chi$, then $\sigma^1(\theta(g)) \neq \mathbb{T}$, thus $0 \in \rho_\infty(\theta(g))$, and by Lemma 4.2, $\sigma^1(\theta(g)_{/Y}) \subseteq \sigma^1(\theta(g))$, further $g \in \Omega_{\chi,Y}$, that is, $\Omega_\chi \subseteq \Omega_{\chi,Y}$.

Accordingly $\bigcup_{\chi\in\hat{A}_\theta} \Omega_\chi$ is a null set for m. ∎

REMARK. If θ is uniformly continuous, the theorem is uninteresting because Ω_χ is empty for all $\chi \in \hat{A}_\theta$; the interesting case concerns the strongly continuous representations that are not uniformly continuous:

COROLLARY 4.5. *Let G be a locally compact and second countable abelian group, m a Haar measure on G, X a Banach space, and $\theta : G \to \mathcal{L}(X)$ a strongly continuous representation. Then θ is uniformly continuous if and only if Σ_θ is a non-null set for m.*

Proof. If θ is uniformly continuous then there exists an open set $U \in \mathcal{V}(1)$ in G such that $\sigma^1(\theta(g)) \subseteq B(0; 1/2)$ for all $g \in U$, and so $U \subseteq \Sigma_\theta$ is a non-null set.

If θ is not continuous, then by Proposition 2.2 there exists $\chi \in \hat{A}_\theta$ such that $\chi \circ \theta$ is not continuous, that is, by Lemma 4.1, χ_1 is not continuous, so $\Gamma_{\chi_1} \neq \{1\}$ and thus, except for the null set Ω_χ, $\sigma^1(\theta(g))$ contains the image of Γ_{χ_1} under a rotation. ∎

REMARK. The theorem and its corollary are valid without the hypothesis of second countability of G provided that the space X is separable.

REMARK 4.6. If there is $\chi \in \hat{A}_\theta$ such that $\Gamma_{\chi_1} = \mathbb{T}$ then $\{g \in G \mid \sigma^1(\theta(g)) \neq \mathbb{T}\} \subseteq \Omega_\chi$ and thus $\{g \in G \mid \sigma^1(\theta(g)) \neq \mathbb{T}\}$ is a null set.

Recall the following lemma (see [7]):

LEMMA 4.7. *Let X be a Banach space and $B \in \mathcal{L}(X)$. If $0 \notin \sigma(B)$ then $\sigma^1(B) \neq \mathbb{T}$ if and only if $\sigma_k^1(B) \neq \mathbb{T}$ where $\sigma_k(\cdot)$ is the Kato essential spectrum* (*the spectrum corresponding to the set of all semi-Fredholm operators*).

COROLLARY 4.8. *Let θ be a strongly continuous representation of an abelian, locally compact, locally connected and second countable group G on a Banach space X. Then θ is uniformly continuous if and only if $\{g \in G \mid \sigma_k^1(\theta(g)) \neq \mathbb{T}\}$ is a non-null set.*

Proof. If θ is uniformly continuous, it suffices to apply Corollary 4.5 for $\sigma_k^1(\theta(g)) \subseteq \sigma^1(\theta(g))$.

For the converse, recall that an abelian, locally compact, locally connected and second countable group is isomorphic to $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{D}$ with $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{\aleph_0\}$ and $\mathbb{D}$ discrete (see [6, Proposition 8.34, Proposition 8.43, and Theorem 8.46]). Such a group locally admits division, thus if θ is not continuous then by Proposition 2.2 there exists $\chi \in \hat{A}_\theta$ such that $\chi \circ \theta$ is not continuous, that is, $\Gamma_{\chi_1} = \mathbb{T}$, and it suffices to apply Remark 4.6 and Lemma 4.7. ∎

COROLLARY 4.9. *If X is a hereditarily indecomposable Banach space then a strongly continuous representation of a locally compact, locally connected and second countable abelian group G on X is automatically uniformly continuous.*

Proof. Recall that for all $g \in G$, $\theta(g) = \lambda_g I + S_g$ where $\lambda_g \in \sigma(\theta(g))$ and S_g is a strictly singular operator (see [4]), thus $\lambda_g \neq 0$ and it is easy to check that $\sigma_e^1(\theta(g)) = \{\lambda_g/|\lambda_g|\}$ where $\sigma_e^1(.)$ is the essential spectrum, since $\sigma_k^1(\theta(g)) \subseteq \sigma_e^1(g)$ for all $g \in G$. The result then follows from Corollary 4.8. ∎

Finally, we conclude with a result announced in the introduction:

COROLLARY 4.10. *Let θ be a strongly continuous representation of a locally compact and second countable abelian group G on a Banach space X. Then either θ is uniformly continuous or there is a null set M in G such that*

$$\liminf_{g \to 1,\, g \in G \setminus M} \rho(\theta(g) - 1) \geq \sqrt{2}.$$

Proof. Since θ is locally bounded, we can apply Lemma 2.1, and for every $\epsilon > 0$ there exists $V_\epsilon \in \mathcal{V}(1)$ such that for all $g \in V_\epsilon$,

$$\sigma(\theta(g)) \subseteq \{z \in \mathbb{C} \mid 1 - \epsilon \leq |z| \leq 1 + \epsilon\}.$$

Assume now that θ is not continuous. There exists a null set M in G such that for all $g \in G \setminus M$, $\sigma^1(\theta(g))$ contains a regular polygon $\lambda \Gamma_p$. Then if

$g \in V_\epsilon \setminus M$ there exists $z_1 \in \sigma^1(\theta(g))$ such that $|z_1 - 1| \geq \sqrt{2}$, and thus there exists $z \in \sigma(\theta(g))$ such that $|z - 1| \geq \sqrt{2} - \epsilon$. Hence, $\rho(\theta(g) - I) \geq \sqrt{2} - \epsilon$ for all $g \in V_\epsilon$. ■

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Jean-Christophe Tomasi
IUFM Institut Universitaire de Formation des Maîtres
Ancien Collège de Montesoro
Avenue Paul Giacobbi
20600 Bastia, France
E-mail: jean-christophe.tomasi@wanadoo.fr

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