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Hausdorff and Fourier dimension

by

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Abstract. There is no constraint on the relation between the Fourier and Hausdorff dimension of a set beyond the condition that the Fourier dimension must not exceed the Hausdorff dimension.

1. Introduction. Throughout this paper we work on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ but similar results hold on $\mathbb{T}^n$ and $\mathbb{R}^n$. All measures will be Borel measures and $|I|$ will denote the length of an interval I.

Definition 1.1. The *Hausdorff dimension* of a set $E \subseteq \mathbb{T}$ is the infimum of the set of real numbers α with the following property. Given any $\epsilon > 0$, we can find a countable collection $\mathcal{I}$ of closed intervals such that

$$\bigcup_{I\in\mathcal{I}} I \supseteq E \quad \text{and} \quad \sum_{I\in\mathcal{I}} |I|^\alpha \le \epsilon.$$

It is easy to see that $0 \le \alpha \le 1$.

Salem proved the following result (see, for example, Section 3 of Chapter 10 in [Kah] or the original paper [S]).

Theorem 1.2. *If μ is a probability measure and $|n|^{\alpha/2}\hat{\mu}(n) \to 0$ as $|n| \to \infty$ then* $\operatorname{supp}\mu$ *has Hausdorff dimension at least α.*

Definition 1.3. The *Fourier dimension* of a closed set E in $\mathbb{T}$ is the supremum of the set of real numbers β such that there exists a probability measure μ with $\operatorname{supp}\mu \subseteq E$ and $|n|^{\beta/2}\hat{\mu}(n) \to 0$ as $|n| \to \infty$.

It is easy to see that $\beta \ge 0$.

Salem's theorem tells us that the Fourier dimension cannot exceed the Hausdorff dimension. It is relatively easy to find examples of sets with Fourier dimension 0 and specified Hausdorff dimension. (For example, in [Kau3], Kaufman has shown that there are Kronecker sets of every possible Hausdorff dimension, and a Kronecker set automatically has Fourier dimension 0.) In [S], Salem showed that there exist sets of every possible

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Hausdorff dimension with their Hausdorff dimension equal to their Fourier dimension. Such sets are now called *Salem sets* and several constructions of such sets have been found (see [Kah]). An interesting application of Salem sets was discovered by Mockenhaupt [M1] who used them to extend work by Stein and Tomas from multidimensional spaces to one dimension. (For further applications see [M2] and [P].)

If we take a Salem set $E_1 \subseteq [0,1/4]$ of Hausdorff dimension β and a closed set $E_2 \subseteq [1/2,3/4]$ with Fourier dimension 0 and Hausdorff dimension α where $1 \geq \alpha \geq \beta \geq 0$ then, automatically,

$$\operatorname{Fourier\,dim}(E_1 \cup E_2) = \max(\operatorname{Fourier\,dim}(E_1), \operatorname{Fourier\,dim}(E_2)) = \beta$$

and

$$\operatorname{Haus\,dim}(E_1 \cup E_2) = \max(\operatorname{Haus\,dim}(E_1), \operatorname{Haus\,dim}(E_2)) = \alpha,$$

but this example is not very enlightening since the Fourier dimension reflects properties of E_1 and the Hausdorff dimension properties of E_2.

We shall give an example of a set with specified Fourier and Hausdorff dimensions which is not subject to the objections raised above.

THEOREM 1.4. *Given $1 \geq \alpha > \beta > 0$, there exists a probability measure μ such that $|n|^{\beta/2}\hat{\mu}(n) \to 0$ as $|n| \to \infty$ and $\operatorname{supp}\mu$ has Hausdorff dimension α and Fourier dimension β.*

The proof I shall give works, with slight modifications, in the cases $\alpha = \beta$ and $\beta = 0$ but simpler proofs already exist in these two cases.

My original proof was probabilistic, but I have since had the pleasure of reading the PhD thesis of Papadimitropoulos in which he explains a non-probabilistic construction of Salem sets due to Kaufman [Kau2]. (Kaufman's construction was inspired by a paper of Jarník [J] and further developed by Bluhm [B]. Part of Papadimitropoulos's thesis is published as [P].) Kaufman's ideas allow a much neater development and I have rewritten this paper to take advantage of this fact.

2. First remarks. We shall prove Theorem 1.4 in the following form.

THEOREM 2.1. *Given $1 \geq \alpha > \beta > 0$, there exists a probability measure μ with support E having the following properties:*

(i) *E has Hausdorff dimension α.*

(ii) *If σ is a non-zero measure with $\operatorname{supp}\sigma \subseteq E$ and $\gamma < \beta$ then*

$$\limsup_{|n|\to\infty} |n|^{\gamma/2}|\hat{\sigma}(n)| = \infty.$$

(iii) *$|n|^{\beta/2}\hat{\mu}(n) \to 0$ as $|n| \to \infty$.*

We obtain Theorem 2.1 from a more specific result.

THEOREM 2.2. *Given $1 \geq \alpha > \beta > 0$, there exists a non-zero positive measure μ with support E having the following properties:*

(i) *There exists a constant C such that*
$$C|I|^{\alpha} \geq \mu(I) \quad \text{for every interval } I.$$
(ii) *Given $\epsilon > 0$ and $\alpha > \gamma > \beta$, we can find intervals $I_1, \ldots, I_M$ with*
$$\bigcup_{m=1}^{M} I_m \supseteq E \quad \text{and} \quad |I_1| = \cdots = |I_M| \leq \epsilon M^{-1/\gamma}.$$
(iii) $|n|^{\beta/2}\hat{\mu}(n) \to 0$ *as* $|n| \to \infty$.

In order to introduce some notation required in the proof of Theorem 2.1 from Theorem 2.2 and elsewhere, we state the following standard lemma whose proof is left to the reader.

LEMMA 2.3. *Let $K : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function with the following properties:*

(i′) $K(x) \geq 0$ *for all* $x \in \mathbb{R}$.
(ii′) $\int_{\mathbb{R}} K(x)\, dx = 1$.
(iii′) $K(x) = 0$ *for* $|x| \geq 1/4$.

If N is a positive integer and we define $K_N : \mathbb{T} \to \mathbb{R}$ by
$$K_N(t) = \begin{cases} NK(Nt) & \text{if } |t| \leq 1/(4N), \\ 0 & \text{otherwise,} \end{cases}$$
then K_N is an infinitely differentiable function having the following properties:

(i) $K_N(t) \geq 0$ *for all* $t \in \mathbb{T}$.
(ii) $\int_{\mathbb{T}} K_N(t)\, dt = 1$.
(iii) $K_N(t) = 0$ *for* $|t| \geq 1/(4N)$.
(iv) $|\hat{K}_N(r)| \leq 1$ *for all* r.
(v) *There exists a constant A independent of N such that* $|\hat{K}_N(r)| \leq A(N/r)^2$ *for all* $r \neq 0$.
(vi) *There exists a constant B independent of N such that* $\|K_N\|_{\infty} \leq BN$.

Proof of Theorem 2.1 from Theorem 2.2. First we show that E has Hausdorff dimension exactly α. The proof is entirely standard. Condition (ii) of Theorem 2.2 shows, directly from the definition, that E has Hausdorff dimension at most α. On the other hand, condition (i) of Theorem 2.2 tells us that if $\mathcal{I}$ is a countable collection of closed intervals such that $\bigcup_{I \in \mathcal{I}} I \supseteq E$, then
$$C \sum_{I \in \mathcal{I}} |I|^{\alpha} \geq \sum_{I \in \mathcal{I}} \mu(I) \geq \mu(E) > 0,$$
so E cannot have Hausdorff dimension less than α.

Next we prove (ii). Without loss of generality, we may suppose that $\|\sigma\| = 1$. We shall suppose that $|n|^{\gamma/2}|\hat{\sigma}(n)| \leq B$ for all $n \neq 0$ and derive a contradiction.

Observe first that $\sigma * K_q \to \sigma$ weakly as $m \to \infty$. In particular, we can find a q_0 such that

$$\|\sigma * K_q\|_1 \geq 1/2 \quad \text{for all } q \geq q_0.$$

Let $\epsilon > 0$. By condition (ii) of Theorem 2.2, we can find a set of intervals $I_1, \dots, I_M$ with

$$\bigcup_{m=1}^{M} I_m \supseteq E \quad \text{and} \quad |I_1| = \dots = |I_M| \leq \epsilon M^{-1/\gamma}.$$

We take ϵ small enough to ensure that $M \geq q_0$ and choose N to be the smallest integer with $\epsilon M^{-1/\gamma} \geq N^{-1}$. Setting

$$L_N = \sigma * K_N,$$

we know that L_n is an infinitely differentiable function with

$$\|L_N\|_1 = \int_{\mathbb{T}} |L_N(t)|\, dt \geq \frac{1}{2}.$$

We also know that

$$\operatorname{supp} L_N \subseteq \operatorname{supp} K_N + \operatorname{supp} \sigma \subseteq \bigcup_{m=1}^{M} (I_m + \operatorname{supp} K_N) \subseteq \bigcup_{m=1}^{M} J_m$$

where $J_1, \dots, J_M$ are closed intervals with

$$|J_1| = \dots = |J_M| \leq 2N^{-1}.$$

Thus, using Schwarz's inequality,

$$\begin{aligned}\frac{1}{4} &\leq \|L_N\|_1^2 = \|L_N \mathbb{I}_{\operatorname{supp} L_N}\|_1^2 \leq \|L_N\|_2^2 \|\mathbb{I}_{\operatorname{supp} L_N}\|_2^2 = \|L_N\|_2^2 \int_{\operatorname{supp} L_N} 1\, dt \\ &\leq \|L_N\|_2^2 \cdot (2MN^{-1}) \leq 4\|L_N\|_2^2 \epsilon^{\gamma} N^{\gamma - 1}\end{aligned}$$

and so

$$\int_{\mathbb{T}} |L_N(t)|^2\, dt = \|L_N\|_2^2 \geq \frac{\epsilon^{-\gamma}}{16} N^{1-\gamma}.$$

On the other hand, we know that

$$|\hat{L}_N(r)| = |\hat{K}_N(r)|\, |\hat{\sigma}(r)| \leq \begin{cases} 1 & \text{for } r = 0, \\ B|r|^{-\gamma/2} & \text{for } |r| \leq N, \\ ABN^2|r|^{-(\gamma+4)/2} & \text{for } |r| \geq N+1. \end{cases}$$

Thus, using Parseval's equality,

$$\begin{aligned}\|L_N\|_2^2 &= \sum_{r=-\infty}^{\infty} |\hat{L}_N(r)|^2 = |\hat{L}_N(0)|^2 + \sum_{1\le|r|\le N} |\hat{L}_N(r)|^2 + \sum_{|r|\ge N+1} |\hat{L}_N(r)|^2 \\ &\le 1 + \sum_{1\le|r|\le N} B^2|r|^{-\gamma} + \sum_{|r|\ge N+1} A^2B^2N^4|r|^{-4-\gamma} \\ &\le 1 + \frac{4B^2}{1-\gamma}N^{1-\gamma} + A^2B^2N^{1-\gamma} = 1 + B^2\left(\frac{4}{1-\gamma} + A^2\right)N^{1-\gamma}\end{aligned}$$

and, combining the results of the last two paragraphs, we get

$$\frac{\epsilon^{-\gamma}}{16}N^{1-\gamma} \le 1 + B^2\left(\frac{4}{1-\gamma} + A^2\right)N^{1-\gamma}.$$

This inequality fails for ϵ sufficiently small, so (iii) follows by reductio ad absurdum. ∎

3. Baire category. It is notationally simpler to obtain the proof of Theorem 2.2 by Baire category methods than by a direct inductive construction. For this purpose, we look at a metric space of a type that I have considered in various other papers (for example [Kö]).

LEMMA 3.1.

(i) *Consider the space $\mathcal{F}$ of non-empty closed subsets of $\mathbb{T}$. If we set*

$$d_{\mathcal{F}}(E,F) = \sup_{e\in E}\inf_{f\in F}|e-f| + \sup_{f\in F}\inf_{e\in E}|e-f|,$$

then $(\mathcal{F}, d_{\mathcal{F}})$ is a complete metric space.

(ii) *Let $1 > \beta > 0$. Consider the space $\mathcal{E}$ consisting of ordered pairs (E,μ) where $E \in \mathcal{F}$ and μ is a positive measure with* $\operatorname{supp}\mu \subseteq E$ *and $|r|^\beta\hat{\mu}(r) \to 0$ as $|r| \to \infty$. If we take*

$$d_{\mathcal{E}}((E,\mu),(F,\sigma)) = d_{\mathcal{F}}(E,F) + \|\mu - \sigma\| + \sup_{r\neq 0}|r|^\beta|\hat{\mu}(r) - \hat{\sigma}(r)|,$$

then $(\mathcal{E}, d_{\mathcal{E}})$ is a non-empty complete metric space.

(iii) *We continue with the notation and hypotheses of part* (ii). *Suppose $1 \ge \alpha > 0$. Let $C > 0$ and let $\mathcal{G} = \mathcal{G}(\alpha,\beta)$ be the set of $(E,\mu) \in \mathcal{E}$ such that*

$$C|I|^\alpha \ge \mu(I) \quad \text{for every interval } I.$$

Then $\mathcal{G}$ is a closed subset of $\mathcal{E}$ (with the metric $d_{\mathcal{E}}$). Provided that C is large enough, $\mathcal{G}$ contains the point $(\mathbb{T}, 2\tau)$, where τ is the Lebesgue measure of total mass 1.

Proof. (i) This metric is called the Hausdorff metric and is discussed, with proofs, in [Ku] (see Chapter II §21 VII and Chapter III §33 IV).

(ii) Use weak compactness. Observe that, writing m for the Haar measure, we have $(\mathbb{T}, m) \in \mathcal{E}$.

(iii) If I is a fixed interval, the set $\mathcal{G}_I$ of $(E,\mu) \in \mathcal{E}$ such that

$$C|I|^\alpha \geq \mu(I)$$

is closed. Since $\mathcal{G}$ is the intersection of such sets, it must be closed.

If we choose $C = 2$, then $(\mathbb{T}, 2\tau) \in \mathcal{G}$. ∎

In what follows we shall suppose $1 \geq \alpha > \beta > 0$ and consider $\mathcal{G} = \mathcal{G}(\alpha,\beta)$ defined as in Lemma 3.1 with C sufficiently large that $(\mathbb{T}, 2\tau) \in \mathcal{G}(\alpha,\beta)$. We write $d_{\mathcal{G}} = d_{\mathcal{G}(\alpha,\beta)}$ for the restriction of the metric $d_{\mathcal{E}}$ to $\mathcal{G}(\alpha,\beta)$. Lemma 3.1 tells us that $(\mathcal{G}, d_{\mathcal{G}})$ is a non-empty complete metric space and so we may apply Baire category methods. Our Baire category version of Theorem 2.2 runs as follows.

Theorem 3.2. *Suppose that $1 \geq \alpha > \beta > 0$. Quasi-all points (F,σ) in the space $(\mathcal{G}, d_{\mathcal{G}})$ have the property that, given $\epsilon > 0$ and $\alpha > \gamma > \beta$, we can find intervals $I_1, \ldots, I_M$ with*

$$\bigcup_{m=1}^{M} I_m \supseteq F \quad \textit{and} \quad |I_1| = \cdots = |I_M| < \epsilon M^{-1/\gamma}.$$

Proof of Theorem 2.2 from Theorem 3.2. If $\epsilon > 0$, and $\alpha > \gamma > \beta$ are given, Baire's category theorem tells us that we can find $(F,\sigma) \in \mathcal{G}$ and intervals $I_1, \ldots, I_M$ with

$$\bigcup_{m=1}^{M} I_m \supseteq F \quad \text{and} \quad |I_1| = \cdots = |I_M| < \epsilon M^{-1/\gamma}$$

and

$$d_{\mathcal{G}}((\mathbb{T}, 2\tau), (F,\sigma)) < 1.$$

We note that $\|2\tau - \sigma\| < 1$ and so $\|\sigma\| > 1$. If we set $E = \operatorname{supp}\sigma$ and $\mu = \|\sigma\|^{-1}\sigma$, the conditions of Theorem 2.2 can be read off directly. (Note that we do not claim that $E = F$.) ∎

Theorem 3.2 follows easily from the following simpler result.

Lemma 3.3. *Suppose that $1 \geq \alpha > \gamma > \beta > 0$ and $\epsilon > 0$. The set $\mathcal{A}_{\gamma,\epsilon}$ consisting of all $(F,\sigma) \in \mathcal{G}$ such that we can find intervals $I_1, \ldots, I_M$ with*

$$\bigcup_{m=1}^{M} I_m \supseteq F \quad \textit{and} \quad |I_1| = \cdots = |I_M| < \epsilon M^{-1/\gamma}$$

is dense in $(\mathcal{G}, d_{\mathcal{G}})$.

Proof of Theorem 3.2 from Lemma 3.3. Choose N sufficiently large that $\alpha-\beta>2/N$. It is easy to see that $\mathcal{A}_{\gamma,\epsilon}$ is open and so

$$\mathcal{C}=\bigcap_{n=N}^{\infty}\mathcal{A}_{\alpha-1/n,1/n}\cap\bigcap_{n=N}^{\infty}\mathcal{A}_{\beta+1/n,1/n}$$

is the complement of a set of first category. Since every element of $\mathcal{C}$ satisfies the conditions of Theorem 3.2, we are done. ■

The task of proving Lemma 3.3 is made easier by the following observation.

LEMMA 3.4. *Suppose that* $1\geq\alpha>\beta>0$. *Given any* $\eta>0$ *and any* $(F,\sigma)\in\mathcal{G}$, *we can find an* $(E,\mu)\in\mathcal{G}$ *with* $d_{\mathcal{G}}((F,\sigma),(E,\mu))<\eta$ *and a* $\delta>0$ *such that* $d\mu(t)=f(t)\,dt$ *for some infinitely differentiable function* $f:\mathbb{T}\to\mathbb{R}$ *and*

$$(C-\delta)|I|^{\alpha}\geq\mu(I)\quad \textit{for every interval } I.$$

Proof. Set $\kappa=(1-\eta)\sigma$. Provided that $\eta>0$ is small enough we have $(F,\kappa)\in\mathcal{G}$ and $d_{\mathcal{G}}((F,\kappa),(F,\sigma))<\eta/2$. Further, provided we choose $\delta>0$ sufficiently small that $(1-\eta)C>C-\delta$, we have

$$(C-\delta)|I|^{\alpha}\geq\kappa(I)\quad\text{for every interval } I.$$

Now set $E=F+[-N^{-1}/4,N^{-1}/4]$ and $\mu=\kappa*K_N$. Automatically we have $(E,\mu)\in\mathcal{G}$, $d\mu(t)=f(t)\,dt$ for some infinitely differentiable function f, and

$$(C-\delta)|I|^{\alpha}\geq\mu(I)\quad\text{for every interval } I.$$

Finally, provided we take N sufficiently large, $d_{\mathcal{G}}((F,\kappa),(E,\mu))<\eta/2$ so that $d_{\mathcal{G}}((F,\sigma),(E,\mu))<\eta$ and we are done. ■

Our task thus reduces to proving the following lemmas.

LEMMA 3.5. *Let* $\alpha>\gamma>\beta$. *If* $(E,\mu)\in\mathcal{G}$ *is such that* $d\mu(t)=f(t)\,dt$ *for some infinitely differentiable* f *and there exists a* $\delta>0$ *such that*

$$(C-\delta)|I|^{\alpha}\geq\mu(I)\quad\textit{for every interval } I,$$

then, given any $\epsilon>0$, *we can find an* $(F,\sigma)\in\mathcal{G}$ *with* $d_{\mathcal{G}}((E,\mu),(F,\sigma))<\epsilon$ *and intervals* $I_1,\dots,I_M$ *with*

$$\bigcup_{m=1}^{M}I_m\supseteq F\quad\textit{and}\quad |I_1|=\dots=|I_M|<\epsilon M^{-1/\gamma}.$$

4. Kaufman's non-probabilistic method.

The standard method of producing measures of the type used in this paper is probabilistic and has been beautifully exploited by Kaufman [Kau1]. His paper [Kau2] gives a non-probabilistic method which we follow in this section.

We write $\mathcal{P}(N)$ for the set of primes p with $N+1 \le p \le 2N$ and $\operatorname{card} X$ for the number of elements in a finite set X. The prime number theorem tells us that $(N^{-1}\log N)\operatorname{card}\mathcal{P}(N) \to 1$ as $n \to \infty$, but we shall only need the simpler result of Chebyshev.

Lemma 4.1. *There exist constants $A_1 > 1 > A_2 > 0$ such that*

$$A_1 \frac{N}{\log N} \ge \operatorname{card}\mathcal{P}(N) \ge A_2 \frac{N}{\log N}$$

for all $N \ge 2$.

We combine this with a simple observation.

Lemma 4.2. *If $m \ge 2$ and*

$$\sigma_m = \sum_{j=1}^{m-1} \delta_{j/m},$$

then σ_m is a positive measure of mass $m-1$ and

$$\hat{\sigma}_m(r) = \begin{cases} m-1 & \text{if } r \equiv 0 \pmod m, \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Observe that

$$\hat{\sigma}_m(r) = \sum_{j=1}^{m-1} \hat{\delta}_{j/m}(r) = \sum_{j=1}^{m-1} \exp(2\pi i j/m) = -1 + \sum_{j=0}^{m-1} \exp(2\pi i j/m). \blacksquare$$

If $N \ge 2$, we set

$$q(N) = \sum_{p\in\mathcal{P}(N)} (p-1) \quad \text{and} \quad \tau_N = q(N)^{-1} \sum_{p\in\mathcal{P}(N)} \sigma_p.$$

The next lemma gives the key properties of τ_N.

Lemma 4.3.

(i) *If $p, q \in \mathcal{P}(N)$ and $p \ne q$ then*

$$\{u/p : 1 \le u \le p-1\} \cap \{v/q : 1 \le v \le q-1\} = \emptyset.$$

(ii) *The measure τ_N is a probability measure of the form*

$$\tau_N = \frac{1}{\operatorname{card} E(N)} \sum_{e\in E(N)} \delta_e$$

with $E(N)$ a finite set with the property that

$$e, f \in E(N) \text{ and } e \ne f \ \Rightarrow\ |e-f| \ge N^{-2}.$$

(iii) *$|\tau_N(I)| \le |I| + 3/N$ for all intervals I.*

(iv) $2A_1N^2(\log N)^{-1} \geq \operatorname{card} E(N) \geq A_2N^2(\log N)^{-1}$ *for some constants* $A_1 > 1 > A_2 > 0$ *independent of* N.

(v) *There exists a constant* $A_3 > 0$ *such that*

$$|\hat{\tau}_N(r)| \leq A_3kN^{-1}\log N$$

for all $1 \leq |r| \leq N^k$.

Proof. (i) & (ii) If $p \in \mathcal{P}(N)$ and u and v are integers such that $u-v \not\equiv 0 \pmod p$, then

$$\left|\frac{u}{p} - \frac{v}{q}\right| \geq \frac{1}{p} > \frac{1}{N^2}.$$

If $p, q \in \mathcal{P}(N)$, $p \neq q$ and and u and v are integers such that $uq - vp \not\equiv 0 \pmod{pq}$, then

$$\left|\frac{u}{p} - \frac{v}{q}\right| \geq \frac{1}{p} > \frac{1}{N^2}.$$

(iii) Observe that

$$\sigma_p(I) = \operatorname{card}\{u/p \in I : 1 \leq u \leq p-1\} \leq p|I| + 2$$

and so

$$\begin{aligned}\tau_N(I) = q(N)^{-1} \sum_{p\in\mathcal{P}(N)} \sigma_p(I) &\leq q(N)^{-1} \sum_{p\in\mathcal{P}(N)} p|I| + 2 \\ &= q(N)^{-1}\big((q(N) + \operatorname{card}\mathcal{P}(N))|I| + 2\operatorname{card}\mathcal{P}(N)\big) \\ &\leq q(N)^{-1}(q(N)|I| + 3\operatorname{card}\mathcal{P}(N)) \\ &\leq q(N)^{-1}(q(N)|I| + 3q(N)N^{-1}) = |I| + 3/N.\end{aligned}$$

(iv) Immediate from Lemma 4.1.

(v) Suppose that $1 \leq |r| \leq N^k$. It follows that r is divisible by at most k primes $p \in \mathcal{P}(N)$. Thus

$$\begin{aligned}|\hat{\tau}_N(r)| = \Big|q(N)^{-1} \sum_{p\in\mathcal{P}(N)} \hat{\sigma}_p(r)\Big| &\leq q(N)^{-1} \sum_{p\in\mathcal{P}(N)} |\hat{\sigma}_p(r)| \\ &= q(N)^{-1} \sum_{p\in\mathcal{P}(N),\, p\mid r} |\hat{\sigma}_p(r)| + q(N)^{-1} \sum_{p\in\mathcal{P}(N),\, p\nmid r} |\hat{\sigma}_p(r)| \\ &\leq q(N)^{-1}(2Nk + \operatorname{card}\mathcal{P}(N)) \leq 2q(N)^{-1}N(k+1).\end{aligned}$$

Using part (i) and the Chebyshev estimate of Lemma 4.1, we have

$$q(N) \geq A_2N(\log N)^{-1} \cdot N = A_2N^2(\log N)^{-1}$$

and so, choosing A_3 appropriately,

$$|\hat{\tau}_N(r)| \leq A_3kN^{-1}\log N$$

as required. ∎

5. Completion of the proof. We now smooth the measure τ_N.

LEMMA 5.1. *Suppose that* $1 > \alpha > \gamma > \beta > 0$. *Given* $\eta > 0$, $\theta > 0$ *and* R *we can find and* $M = M(\alpha, \beta, \gamma, \eta, R)$, $\kappa = \kappa(\alpha, \beta, \gamma, \eta, R, \theta) > 0$ *with* $M \geq R$ *and* $\kappa < \theta$ *together with a positive infinitely differentiable function* $g_\eta : \mathbb{T} \to \mathbb{R}$ *with* $\int_{\mathbb{T}} g_\eta(t)\, dt = 1$ *having the following properties:*

(i) $|\hat{g}_\eta(r)| \leq \eta |r|^{-\beta/2}$ *for* $r \neq 0$.
(ii) $\int_I g_\eta(t)\, dt \leq (1+\eta)|I|$ *for* $|I| \geq \kappa/2$.
(iii) $\int_I g_\eta(t)\, dt \leq \eta |I|^\alpha$ *for* $|I| \leq \kappa$.
(iv) *We can find intervals* $I_1, \ldots, I_M$ *with*

$$\bigcup_{m=1}^{M} I_m \supseteq \operatorname{supp} g_\eta \quad \textit{and} \quad |I_1| = \cdots = |I_M| < \eta M^{-1/\gamma}.$$

Proof. Consider τ_N as in Lemma 4.3. We choose M to be the number of points in the support of τ_N. Provided that N is sufficiently large, we can find an integer P with $\frac{1}{4}\eta M^{-1/\gamma} \leq P^{-1} \leq \frac{1}{2}\eta M^{-1/\gamma}$. We set

$$g_\eta = \tau_N * K_P$$

where K_P is defined as in Lemma 2.3. We claim that, provided only that N is large enough, g_η and M will have the required properties.

Observe that g_η is automatically infinitely differentiable and positive. If we take the I_j to be intervals of the form $[u - \eta M^{-1/\gamma}, u + \eta M^{-1/\gamma}]$ with $u \in \operatorname{supp} \tau_N$, then condition (iii) of Lemma 2.3 tells us that

$$\operatorname{supp} g_\eta \subseteq \bigcup_{m=1}^{M} I_m,$$

so condition (iv) follows.

We observe that conditions (ii) and (iv) of Lemma 4.3 tell us that, provided N is large enough,

$$[u - \eta M^{-1/\gamma}, u + \eta M^{-1/\gamma}] \cap [v - \eta M^{-1/\gamma}, v + \eta M^{-1/\gamma}] = \emptyset$$

and so

$$\operatorname{supp}(\delta_u * K_p) \cap \operatorname{supp}(\delta_v * K_p) = \emptyset$$

whenever $u, v \in \operatorname{supp} \tau_N$ and $u \neq v$.

We now set $\kappa = 10N^{-1}\eta$ and observe that, using Lemma 4.3(iii), we have

$$\int_I g_\eta(t)\, dt \leq \tau_N(I) + 2M^{-1} \leq |I| + 3N^{-1} + 2M^{-1} \leq |I| + 5N^{-1} \leq (1+\eta)|I|$$

whenever I is an interval with $|I| \geq \kappa/2$. Thus condition (ii) holds.

Our proof of condition (iii) splits into three parts depending on the length of the interval I. First, suppose I is an interval with $4^{-1}N^{-2} \leq |I| \leq \kappa$.

Lemma 4.3(ii) tells us that

$$\operatorname{card}(I \cap E_N) \le N^2 |I|$$

and so

$$\int_I g_\eta(t)\,dt \le (N^2|I| + 2)M^{-1} \le 20M^{-1}N^2|I|.$$

By Lemma 4.3(v), it follows that

$$\int_I g_\eta(t)\,dt \le 4A_2^{-1}(\log N)|I|.$$

Since $|I| \ge 4N^{-2}$, it follows that

$$\int_I g_\eta(t)\,dt \le \eta |I|^\alpha,$$

provided only that N is large enough, independent of the I chosen.

If I is an interval with $M^{-1/\gamma} \le |I| \le 4^{-1}N^{-2}$, then

$$\int_I g_\eta(t)\,dt \le M^{-1} \le |I|^\gamma.$$

Since $\alpha > \gamma$, it follows that

$$\int_I g_\eta(t)\,dt \le \eta |I|^\alpha,$$

provided only that N is large enough, independent of the I chosen.

If I is an interval with $|I| \le M^{-1/\gamma}$, we argue as follows. Using Lemma 2.3, we see that

$$\|g_\eta\|_\infty = M^{-1}\|K_p\|_\infty \le B\eta^{-1}M^{-1+1/\gamma}$$

for some constant B. In particular,

$$\int_I g_\eta(t)\,dt \le B\eta^{-1}M^{-1+1/\gamma}|I| \le B\eta^{-1}|I|^{\gamma-1}|I| = B\eta^{-1}|I|^\gamma.$$

Since $\alpha > \gamma$, it follows that

$$\int_I g_\eta(t)\,dt \le \eta |I|^\alpha,$$

provided only that N is large enough, independent of the I chosen. Together with the previous two paragraphs, this shows that condition (iii) holds.

To obtain (i), take k to be the integer with $8\beta^{-1} + 1 \ge k > 8\beta^{-1}$. By Lemma 4.3(v),

$$|\hat{g}_\eta(r)| = |\hat{\tau}_N(r)|\,|\hat{K}_P(r)| \le |\hat{\tau}_N(r)| \le A_3 k N^{-1} \log N$$

for $0 \ne |r| \le N^k$. By Lemma 2.3,

$$|\hat{g}_\eta(r)| = |\hat{\tau}_N(r)|\,|\hat{K}_P(r)| \le |\hat{K}_P(r)| \le ANr^{-2}$$

for all $r \neq 0$ and so, in particular, for $N^k \leq r$. Using Lemma 4.3(iii), we see that (i) holds, provided only that N is large enough. ∎

We now prove Lemma 3.5 and so complete the proof of Theorem 1.4.

Proof of Lemma 3.5. Suppose, as stated, that $(E,\mu) \in \mathcal{G}$ is such that $d\mu(t) = f(t)\,dt$ for some infinitely differentiable f and there exists a $\delta > 0$ such that

$$(C-\delta)|I|^\alpha \geq \mu(I) \quad \text{for every interval } I.$$

We shall show that if g_η is defined as in Lemma 5.1 then, provided that η and θ are small enough (with θ depending on η), taking $d\sigma(t) = g_\eta(t)d\mu(t)$ and $F = E \cap \operatorname{supp} g_M$ we have $(F,\sigma) \in \mathcal{G}$ with $d_{\mathcal{G}}((E,\mu),(F,\sigma)) < \epsilon$. Note that, since $F \subseteq \operatorname{supp} g_\eta$, condition (iv) of Lemma 5.1 guarantees the existence of intervals $I_1, \dots, I_M$ with

$$\bigcup_{m=1}^{M} I_m \supseteq F \quad \text{and} \quad |I_1| = \cdots = |I_M| < \epsilon M^{-1/\gamma}$$

whenever $\eta \leq \epsilon$.

Set $\|f\|_* = \|f\|_\infty + \|f''\|_\infty$. Automatically $|\hat{f}(0)| \leq \|f\|_\infty \leq \|f\|_*$ and, integrating by parts twice, we have

$$|\hat{f}(u)| \leq (2\pi)^{-2}u^{-2} \leq \|f''\|_\infty u^{-2} \leq \|f\|_* u^{-2}$$

for all $u \neq 0$. If $r \neq 0$, condition (i) of Lemma 5.1 yields

$$\begin{aligned}
|\hat{\mu}(r) - \hat{\sigma}(r)| &= |\widehat{g_\eta f}(r) - \hat{f}(r)| \\
&= \Big|\sum_{u\neq r} \hat{f}(u)\hat{g}_\eta(r-u)\Big| \leq \sum_{u \neq r} |\hat{f}(u)|\,|\hat{g}_\eta(r-u)| \\
&\leq \eta\|f\|_*\Big(|r|^{-\beta/2} + \sum_{u\neq 0,r} \frac{|r-u|^{-\beta/2}}{u^2}\Big) \\
&= \eta\|f\|_*\Big(|r|^{-\beta/2} + \sum_{1\leq|u|\leq|r|/2} \frac{|r-u|^{-\beta/2}}{u^2} + \sum_{|u|>r/2,\,u\neq r} \frac{|r-u|^{-\beta/2}}{u^2}\Big) \\
&\leq \eta\|f\|_*\Big(|r|^{-\beta/2} + \sum_{1\leq|u|\leq|r|/2} \frac{2|r|^{-\beta/2}}{u^2} + \sum_{|u|>|r|/2} \frac{1}{u^2}\Big) \\
&\leq \eta\|f\|_*\Big(17|r|^{-\beta/2} + \frac{8}{|r|}\Big) \leq 25\eta\|f\|_*|r|^{-\beta/2} \leq \frac{\epsilon}{4}|r|^{-\beta/2},
\end{aligned}$$

provided η is small enough (independent of r). A similar calculation shows that $|\hat{\mu}(0) - \hat{\sigma}(0)| \leq \epsilon/4$, provided that η is sufficiently small.

Condition (ii) of Lemma 5.1 tells us that

$$d_{\mathcal{E}}(E,F) < \epsilon/2$$

provided only that η is sufficiently small and so

$$d_{\mathcal{E}}((E,\mu),(F,\sigma)) < \epsilon.$$

Our task thus reduces to showing that, if η is sufficiently small and M sufficiently large, then

$$C|I|^{\alpha} \geq \sigma(I) \quad \text{for all intervals } I.$$

To this end, observe that, by choosing θ sufficiently small in Lemma 5.1, we can ensure that $|f(s) - f(t)| \leq \eta$ whenever $|s-t| \leq \kappa$. Any interval I of length at least κ can be written as the union of a collection $\mathcal{J}$ of intervals J with $\kappa/2 \leq |J| \leq \kappa$ intersecting only at end points. Thus, using condition (ii) of Lemma 5.1,

$$\begin{aligned}
\sigma(I) &= \int_I f(t) g_M(t)\,dt = \sum_{J\in\mathcal{J}} \int_J f(t) g_M(t)\,dt \\
&\leq \sum_{J\in\mathcal{J}} \int_J \biggl(\frac{1}{|J|}\int_J f(s)\,ds + \eta\biggr) g_M(t)\,dt \\
&\leq \sum_{J\in\mathcal{J}} (1+\eta)|J| \cdot \biggl(\frac{1}{|J|}\int_J f(s)\,ds + \eta\biggr) \\
&= \sum_{J\in\mathcal{J}} (1+\eta)\biggl(\int_J f(s)\,ds + \eta|J|\biggr) \\
&= (1+\eta)\int_I f(s)\,ds + \eta(1+\eta)|I| = (1+\eta)\mu(I) + \eta(1+\eta)|I| \\
&\leq (1+\eta)(C-\delta)|I|^{\alpha} + \eta(1+\eta)|I| \leq C|I|^{\alpha}
\end{aligned}$$

provided only that η is small enough independent of the choice of I.

Finally, if I is an interval of length less than κ, condition (iii) of Lemma 5.1 yields

$$\sigma(I) = \int_I f(t) g_M(t)\,dt \leq \|f\|_\infty \int_I g_M(t)\,dt \leq \eta \|f\|_\infty |I|^{\alpha} \leq C|I|^{\alpha}$$

provided only that η is small enough independent of the choice of I. We have shown that $(F,\sigma) \in \mathcal{G}$ and this concludes the proof and the paper. ∎

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