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A "hidden" characterization of polyhedral convex sets

by

Taras Banakh (Lviv and Kielce) and Ivan Hetman (Lviv)

Abstract. We prove that a closed convex subset C of a complete linear metric space X is polyhedral in its closed linear hull if and only if no infinite subset $A \subset X \setminus C$ can be hidden behind C in the sense that $[x, y] \cap C \neq \emptyset$ for any distinct $x, y \in A$.

1. Introduction. A convex subset C of a real linear topological space L is called *polyhedral in* L if it can be written as a finite intersection $C = \bigcap_{i=1}^{n} f_i^{-1}((-\infty, a_i])$ of closed half-spaces determined by some continuous linear functionals $f_1, \ldots, f_n : L \to \mathbb{R}$ and some real numbers $a_1, \ldots, a_n$ (see [1]).

This notion also has an algebraic version. We shall say that a convex subset C of a linear space L is *polyhedric* in a convex set $D \supset C$ of L if $C = \bigcap_{i=1}^{n} H_i$ for some convex subsets $H_1, \ldots, H_n \subset D$ having convex complements $D \setminus H_i$, $i \le n$.

In this paper polyhedral sets will be characterized with the help of a combinatorial notion of a hidden set.

We say that a subset A of a linear space L is *hidden behind a set* $C \subset L$ if $A \subset L \setminus C$ and for any distinct points $a, b \in A$ the closed segment $[a, b] = \{ta + (1-t)b : t \in [0, 1]\}$ meets C. In this case we shall also say that the set C *hides* the set A.

The main result of this paper is the following "hidden" characterization of closed polyhedral convex sets in complete linear metric spaces. This characterization has been applied in the paper [2] devoted to recognizing the topological type of connected components of the hyperspace of closed convex subsets of a Banach space. Another characterization of polyhedral convex sets can be found in [10].

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Theorem 1.1. *For a closed convex subset C of a complete linear metric space X the following conditions are equivalent:*

(1) *C is polyhedral in its closed linear hull $\overline{\mathrm{lin}}(C)$;*
(2) *C is polyhedric in its affine hull $\mathrm{aff}(C)$;*
(3) *C hides no infinite subset $A \subset X \setminus C$.*

The proof of this theorem is rather long and will be presented in Section 3. Now let us show that the assumption of the completeness of the linear space X in Theorem 1.1 is essential. A counterexample will be constructed in the (non-complete) normed space

$$c_{00} = \{(x_n)_{n\in\omega} \in \mathbb{R}^\omega : \exists n \in \omega\ \forall m \geq n\ \ x_m = 0\}$$

endowed with the sup-norm $\|x\| = \sup_{n\in\omega} |x_n|$, where $x = (x_n)_{n\in\omega} \in c_{00}$.

Example 1.2. *The standard infinite-dimensional simplex*

$$\Delta = \Big\{(x_n)_{n\in\omega} \in c_{00} \cap [0,1]^\omega : \sum_{n\in\omega} x_n = 1\Big\} \subset c_{00}$$

hides no infinite subset of $c_{00} \setminus \Delta$ but is not polyhedral in c_{00}.

Proof. First we show that the simplex Δ is not polyhedral in c_{00}. Assuming the opposite, we would find linear functionals $f_1, \dots, f_n : c_{00} \to \mathbb{R}$ and real numbers $a_1, \dots, a_n$ such that $\Delta = \bigcap_{i=1}^n f_i^{-1}((-\infty, a_i])$. Consider the linear subspace $X_0 = \bigcap_{i=1}^n f_i^{-1}(0)$ that has finite codimension in c_{00}. It follows that for each $x_0 \in \Delta$, we get $x_0 + X_0 \subset \bigcap_{i=1}^n f_i^{-1}((-\infty, a_i]) = \Delta$, which implies that the set Δ is unbounded. This contradiction shows that Δ is not polyhedral in c_{00}.

Now assume that some infinite subset $A \subset c_{00} \setminus \Delta$ can be hidden behind the simplex Δ. Decompose the space c_{00} into the union $c_{00} = \Sigma_< \cup \Sigma_1 \cup \Sigma_>$ of the sets

$$\Sigma_< = \Big\{(x_n)_{n\in\omega} \in c_{00} : \sum_{n\in\omega} x_n < 1\Big\},$$

$$\Sigma_1 = \Big\{(x_n)_{n\in\omega} \in c_{00} : \sum_{n\in\omega} x_n = 1\Big\},$$

$$\Sigma_> = \Big\{(x_n)_{n\in\omega} \in c_{00} : \sum_{n\in\omega} x_n > 1\Big\}.$$

Observe that for any two points $x, y \in \Sigma_<$ the segment $[x,y]$ does not intersect Δ. Consequently, $|A \cap \Sigma_<| \leq 1$. For the same reason, $|A \cap \Sigma_>| \leq 1$. So, we lose no generality assuming that $A \subset \Sigma_1 \setminus \Delta$. For each $a \in A$ let

$$\mathrm{supp}_+(a) = \{n \in \omega : x_n > 0\} \quad \text{and} \quad \mathrm{supp}_-(a) = \{n \in \omega : x_n < 0\}.$$

It is easy to see that each point $a \in \Sigma_1 \setminus \Delta$ has non-empty negative support $\mathrm{supp}_-(a)$.

Fix any point $b \in A$. We claim that $\mathrm{supp}_-(a) \subset \mathrm{supp}_+(b)$ for any $a \in A \setminus \{b\}$. In the opposite case the set $\mathrm{supp}_-(a) \setminus \mathrm{supp}_+(b)$ contains some $k \in \omega$ and then $[a, b[\subset \{(x_n)_{n\in\omega} \in c_{00} : x_k < 0\} \setminus \Delta$, which is impossible as $\{a, b\}$ is hidden behind Δ. Since (the power-set of) $\mathrm{supp}_+(b)$ is finite and $A \setminus \{b\}$ is infinite, the Pigeonhole Principle yields distinct points $a, a' \in A$ such that $\mathrm{supp}_-(a) = \mathrm{supp}_-(a') \subset \mathrm{supp}_+(b)$. Now we see that for any $k \in \mathrm{supp}_-(a) = \mathrm{supp}_-(a')$, we get $[a, a'] \subset \{(x_n)_{n\in\omega} \in c_{00} : x_n < 0\} \subset c_{00} \setminus \Delta$, which contradicts the choice of A as a set hidden behind Δ. ■

2. Preliminaries. In this section we prove some lemmas which will be used in the proof of Theorem 1.1.

LEMMA 2.1. *Let $T : X \to Y$ be a continuous linear operator between linear topological spaces. If a convex subset $D \subset Y$ is polyhedral in its closed linear hull $\overline{\mathrm{lin}}(D)$, then $C = T^{-1}(D)$ is polyhedral in $\overline{\mathrm{lin}}(C)$.*

Proof. Write the polyhedral set D as a finite intersection

$$D = \bigcap_{i=1}^{n} f_i^{-1}((-\infty, a_i])$$

of closed half-spaces defined by continuous linear functionals $f_1, \ldots, f_n : \overline{\mathrm{lin}}(D) \to \mathbb{R}$ and real numbers $a_1, \ldots, a_n$. The continuity of T implies that $T(\overline{\mathrm{lin}}(C)) \subset \overline{\mathrm{lin}}(D)$. Consequently, for every $i \le n$ the continuous linear functional $g_i = f_i \circ T : \overline{\mathrm{lin}}(C) \to \mathbb{R}$ is well-defined. Since $C = T^{-1}(D) = \bigcap_{i=1}^{n} g_i^{-1}((-\infty, a_i])$, the set C is polyhedral in $\overline{\mathrm{lin}}(C)$. ■

An operator $A : X \to Y$ between linear spaces is called *affine* if

$$A(tx + (1-t)y) = tA(x) + (1-t)A(y) \quad \text{for any } x, y \in X \text{ and } t \in \mathbb{R}.$$

It is well-known that an operator $A : X \to Y$ is affine if and only if the operator $B : X \to Y$, $B : x \mapsto A(x) - A(0)$, is linear. The following lemma trivially follows from the definition of a hidden set.

LEMMA 2.2. *Let $T : X \to Y$ be an affine operator between linear topological spaces, $D \subset Y$ be a convex set, $C = T^{-1}(D)$, and $A \subset X \setminus C$ be a subset such that $T|A$ is injective. Then C hides A if and only if $D = T(C)$ hides $T(A)$.*

Let us recall that a convex subset C of a linear topological space X is called a *convex body* in X if C has non-empty interior in X.

LEMMA 2.3. *Let C be an infinite-dimensional closed convex subset of a complete linear metric space Y. If C is infinite-dimensional, then there is an injective continuous affine operator $T : l_2 \to Y$ such that $T^{-1}(C)$ is a closed convex body in l_2.*

Proof. By [7, 1.2.2], the topology of Y is generated by a complete invariant metric d such that the F-norm $\|y\| = d(y,0)$ has the property $\|ty\| \le \|y\|$ for all $y \in Y$ and $t \in [-1,1]$.

We lose no generality assuming that $0 \in C$. In this case for any points $y_n \in C$, $n \in \omega$, and any non-negative real numbers t_n, $n \in \omega$, with $\sum_{n\in\omega} t_n \le 1$ we get $\sum_{n\in\omega} t_n y_n \in C$ whenever the series $\sum_{n=0}^{\infty} t_n y_n$ converges in Y.

The set C is infinite-dimensional and hence contains a linearly independent sequence $(y_n)_{n=1}^{\infty}$. Multiplying each y_n by a small positive real number, we can additionally assume that $\|y_n\| \le 2^{-n}$. It follows that the series $\sum_{n=1}^{\infty}(1/4^n)y_n$ converges in Y and its sum $s_0 = \sum_{n=1}^{\infty}(1/4^n)y_n$ belongs to the closed convex set C as $\sum_{n=1}^{\infty} 1/4^n = 1/3 \le 1$.

Let l_2^f be the linear hull of the standard orthonormal basis $(e_n)_{n\in\omega}$ in the separable Hilbert space l_2. Define a linear operator $S : l_2^f \to Y$ letting $S(e_n) = (1/4^n)y_n$ for every $n \in \mathbb{N}$. The convergence of the series $\sum_{n=1}^{\infty} \|y_n\|$ implies that the operator S is continuous and hence can be extended to a continuous linear operator $\bar{S} : l_2 \to Y$. Let $B_1 = \{x \in l_2 : \|x\| < 1\}$. We claim that $\bar{S}(B_1) + s_0 \subset C$. Indeed, for every $x = (x_n)_{n=1}^{\infty} \in B_1$ and every $n \in \mathbb{N}$ we get $|x_n| \le 1$ and hence

$$\frac{1}{4^n} + \frac{x_n}{4^n} \ge \frac{1}{4^n} - \frac{1}{4^n} = 0.$$

Taking into account that

$$\sum_{n=1}^{\infty}\left(\frac{1}{4^n} + \frac{x_n}{4^n}\right) \le \sum_{n=1}^{\infty}\frac{2}{4^n} = \frac{2}{3} < 1$$

and $0 \in C$, we conclude that

$$s_0 + \bar{S}(x) = \sum_{n=1}^{\infty}\frac{1}{4^n}y_n + \sum_{n=1}^{\infty}\frac{x_n}{4^n}y_n \in C.$$

Let $H = \bar{S}^{-1}(0)$ and $H^{\perp}$ be the orthogonal complement of H in l_2. It follows that the affine operator $T : H^{\perp} \to Y$, $T : x \mapsto \bar{S}(x) + s_0$, is injective and $T^{-1}(C)$ contains the unit ball $B_1 \cap H^{\perp}$ of the Hilbert space $H^{\perp}$. So, $T^{-1}(C)$ is a closed convex body in $H^{\perp}$. Since $\bar{S}(l_2) = \bar{S}(K^{\perp}) \supset \{y_n\}_{n\in\omega}$, the Hilbert space $H^{\perp}$ is infinite-dimensional and hence can be identified with l_2. ∎

The following lemma is the most important and technically difficult ingredient of the proof of Theorem 1.1.

LEMMA 2.4. *If a closed convex body $\bar{C}$ in a separable Hilbert space X is not polyhedral, then $\bar{C}$ hides some infinite subset $A \subset X \setminus \bar{C}$.*

Proof. Let $\langle\cdot,\cdot\rangle$ denote the inner product of X. Each element $y \in X$ determines a functional $y^* : x \mapsto \langle x, y\rangle$ on X. By the Riesz Representation Theorem [3, 3.4], the operator $y \mapsto y^*$ is a linear isometry between X and its dual Hilbert space X^*. Let $S = \{x \in X : \|x\| = 1\}$ and $S^* = \{x^* \in X^* : \|x^*\| = 1\}$.

Let C be the interior of the convex body $\bar{C}$ and $\partial C = \bar{C} \setminus C$ be the boundary of $\bar{C}$ in X. A functional $x^* \in S^*$ is said to *support* $\bar{C}$ at a point $x \in \partial C$ if $x^*(x) = \sup x^*(C)$. The Hahn–Banach Theorem guarantees that for each point $x \in \partial C$ there is a supporting functional $x^* \in S^*$ of C at x. If such a supporting functional is unique, then the point x is called *smooth*. By the classical Mazur's Theorem [6, 1.20], the set Σ of smooth points is a dense G_δ in the boundary ∂C of C. By $\sigma : \Sigma \to S^*$ we shall denote the function assigning to each smooth point $x \in \Sigma$ the unique supporting functional $\sigma_x \in S^*$ of C at x. Let us observe that the function σ has closed graph

$$\begin{aligned}\Gamma &:= \{(x, \sigma_x) : x \in \Sigma\}\\ &= (\Sigma \times S^*) \cap \{(x, x^*) \in C \times S^* : x^*(x) \geq \sup x^*(C)\}\end{aligned}$$

in the Polish space $\Sigma \times S^*$. Let $\mathrm{pr}_1 : \Gamma \to \Sigma$ and $\mathrm{pr}_2 : \Gamma \to S^*$ be the projections on the respective factors. Observe that pr_1 is a bijective and continuous map between Polish spaces. By the Luzin–Suslin Theorem [5, 15.1], it is a Borel isomorphism, which implies that the map $\sigma = \mathrm{pr}_2 \circ \mathrm{pr}_1^{-1} : \Sigma \to S^*$ is Borel measurable. By Theorem 8.38 of [5], there is a dense G_δ-subset $G \subset \Sigma$ such that the restriction $\sigma|G$ is continuous.

Claim 2.5. *The image $\sigma(G)$ is infinite.*

Proof. Assume that $\sigma(G)$ is finite and find functionals $f_1, \dots, f_n \in S^*$ such that $\sigma(G) = \{f_1, \dots, f_n\}$. Since the set $\bar{C} \subsetneq \bigcap_{i=1}^n f_i^{-1}((-\infty, \max f_i(\bar{C})])$ is not polyhedral, there is a point $x \in X \setminus \bar{C}$ such that $f_i(x) \leq \max f_i(\bar{C})$ for all $i \leq n$. Fix any point $x_0 \in C$. Since C is open, $f_i(x_0) < \max f_i(\bar{C})$ for all $i \leq n$. Since $x \notin \bar{C}$, the segment $[x, x_0]$ meets ∂C at some point $y = (1-t)x + tx_0$ where $t \in (0,1)$. Then $f_i(y) = (1-t)f_i(x) + tf_i(x_0) < \max f_i(\bar{C})$. It follows that the set $U = \bigcap_{i=1}^n f_i^{-1}(-\infty, \max f_i(\bar{C}))$ is an open neighborhood of y in X. Since the set G in dense in ∂C, there is a point $z \in G \cap U$. Consider the unique supporting functional σ_z of C at z. The inclusion $z \in U$ implies that $\sigma_z \in \sigma(G) \setminus \{f_1, \dots, f_n\}$, which is the desired contradiction. ■

Depending on the cardinality of the set $\sigma(G)$ we divide the further proof of Lemma 2.4 into two Lemmas 2.6 and 2.8.

Lemma 2.6. *If $\sigma(G)$ is uncountable, then the set C hides some infinite subset of X.*

Proof. The continuous map $\sigma|G$ induces a closed equivalence relation $E = \{(x,y) \in G \times G : \sigma(x) = \sigma(y)\}$ on the Polish space G. Since this equivalence relation has uncountably many equivalence classes, Silver's Theorem [8] yields a topological copy $K \subset G$ of the Cantor cube $\{0,1\}^\omega$ such that K has at most one-point intersection with each equivalence class. This is equivalent to saying that the restriction $\sigma|K$ is injective. The existence of such a Cantor set K can also be derived from Feng's Theorem [4] saying that the Open Coloring Axiom holds for analytic spaces.

For any $x \in K$ let $y_x \in S$ be the unique vector such that $\sigma_x(z) = \langle z, y_x\rangle$ for all $z \in X$. For $\varepsilon \in [0,1]$ consider the open subset $\Lambda(x,\varepsilon) = \{z \in K : [x + \varepsilon y_x, z] \cap C \neq \emptyset\}$ of the Cantor set K.

CLAIM 2.7. *For any $x \in K$ the sets $\Lambda(x,\varepsilon)$ have the following properties:*

(1) $\Lambda(x,\varepsilon) \supset \Lambda(x,\delta)$ *for any* $0 < \varepsilon \le \delta \le 1$;
(2) $\bigcup_{\varepsilon\in(0,1]} \Lambda(x,\varepsilon) = K \setminus \{x\}$.

Proof. (1) Fix $0 < \varepsilon \le \delta \le 1$ and $z \in \Lambda(x,\delta)$. By the definition of the set $\Lambda(x,\delta)$, the segment $[x + \delta y_x, z]$ meets the open convex set C at some point c. Since the points x, z belong to the convex set $\bar{C}$ and the point c belongs to its interior C, the triangle

$$\Delta = \{t_c c + t_x x + t_z z : t_c > 0,\ t_x, t_z \ge 0,\ t_c + t_x + t_y = 1\}$$

lies in C. Since the segment $[x + \varepsilon y_x, z]$ intersects this triangle, it has nonempty intersection with C.

(2) Take any point $z \in K \setminus \{x\}$. Since $\sigma|K$ is injective, the supporting functionals σ_x and σ_y are distinct. Then the open segment $]x,z[\ = [x,z] \setminus \{x,z\}$ lies in C. In the opposite case, $[x,z] \subset \partial C$ and for the midpoint $\frac{1}{2}x + \frac{1}{2}z$ there would exist a supporting functional x^*, which would be supporting at each point of the segment $[x,y]$. This is impossible as the points x, z are smooth and have unique and distinct supporting functionals. This contradiction proves that $[x,z]$ meets C. Then for some $\varepsilon > 0$ the segment $[x + \varepsilon y_x, z]$ also meets C, which implies that $z \in \Lambda(x,\varepsilon)$. ■

Being homeomorphic to the Cantor cube, the space K carries an atomless σ-additive Borel probability measure μ. Fix any $x_0 \in K$. Using Claim 2.7(2), find $\varepsilon_0 \in (0,1]$ such that $\mu(\Lambda(x_0,\varepsilon_0)) > 1 - 2^{-1}$. Next proceed by induction and construct a sequence $(x_n)_{n\in\omega}$ of points and a sequence $(\varepsilon_n)_{n\in\omega}$ of positive real numbers such that for every $n \in \mathbb{N}$:

(1) $x_n \in \bigcap_{k<n} \Lambda(x_k,\varepsilon_k)$;
(2) $\mu(\Lambda(x_n,\varepsilon_n)) > 1 - 2^{-n-1}$;
(3) $[x_k + \varepsilon_k y_{x_k}, x_n + \varepsilon_n y_{x_n}] \cap C \neq \emptyset$ for all $k < n$;
(4) $x_n + \varepsilon_n y_{x_n} \notin \bar{C}$.

Assume that for some n, the points x_k, $k<n$, and real numbers ε_k, $k<n$, have been constructed. Consider the intersection $\bigcap_{k<n} \Lambda(x_k,\varepsilon_k)$ and observe that it has positive measure:

$$\begin{aligned}\mu\Big(\bigcap_{k<n} \Lambda(x_k,\varepsilon_k)\Big) &= 1-\mu\Big(K\setminus \bigcap_{k<n} \Lambda(x_k,\varepsilon_k)\Big)\\ &= 1-\mu\Big(\bigcup_{k<n} K\setminus \Lambda(x_k,\varepsilon_k)\Big) \ge 1-\sum_{k<n}\mu(K\setminus \Lambda(x_k,\varepsilon_k))\\ &= 1-\sum_{k<n}(1-\mu(\Lambda(x_k,\varepsilon_k)) > 1-\sum_{k<n} 2^{-k-1} > 0.\end{aligned}$$

So, this intersection is not empty and we can select a point x_n satisfying (1). For every $k<n$ the definition of $\Lambda(x_k,\varepsilon_k)$ ensures that the segment $[x_k+\varepsilon_k y_{x_k}, x_n]$ meets the interior C of the convex set $\bar{C}$. Consequently, there is $\varepsilon_n'>0$ such that for every $\varepsilon_n\le\varepsilon_n'$ and every $k<n$ the segment $[x_k+\varepsilon_k y_{x_k}, x_n+\varepsilon_n y_{x_n}]$ still meets the open set C. Finally, using Claim 2.7(2), choose $\varepsilon_n\in(0,\varepsilon_n']$ such that $\mu(\Lambda(x_n,\varepsilon_n'))>1-2^{-n-1}$. Observe that

$$\sigma_{x_n}(x_n+\varepsilon_n y_{x_n}) = \sigma_{x_n}(x_n)+\varepsilon_n\sigma_{x_n}(y_{x_n}) \ge \max\sigma_{x_n}(\bar{C})+\varepsilon_n$$

and hence $x_n+\varepsilon_n y_{x_n}\notin\bar{C}$. This completes the inductive step.

The conditions (3) and (4) of the inductive construction guarantee that $A=\{x_n+\varepsilon_n y_{x_n}\}_{n\in\omega}$ is the required infinite set, hidden behind the convex set $\bar{C}$. ∎

LEMMA 2.8. *If $\sigma(G)$ is countable, then $\bar{C}$ hides some infinite subset of X.*

Proof. Denote by F the set of $f\in\sigma(G)$ for which $f^{-1}(\sup f(C))\cap C$ has non-empty interior in ∂C.

CLAIM 2.9. *The set F is infinite.*

Proof. Assume that F is finite, say $F=\{f_1,\dots,f_n\}$ for some $f_1,\dots,f_n\in S^*$. Since $\bar{C}$ is not polyhedral,

$$\bar{C}\ne\bigcap_{i=1}^{n} f_i^{-1}((-\infty,\max f_i(\bar{C})]).$$

Repeating the argument from Claim 2.5, we can find $y\in\partial C$ such that $f_i(y)<\max f_i(\bar{C})$ for all $i\le n$. Then $U=\bigcap_{i=1}^n f_i^{-1}((-\infty,\max f_i(\bar{C}))$ is an open neighborhood of y in X. Since $G\cap U\subset\bigcup_{f\in\sigma(G)} f^{-1}(\max f(\bar{C}))$, the Baire Theorem implies that for some $f\in\sigma(G)$ the set $f^{-1}(\max f(\bar{C}))\cap G\cap U$ has non-empty interior in $G\cap U$. Since $G\cap U$ is dense in $U\cap\partial C$, the set $f^{-1}(\max f(\bar{C}))\cap U$ has non-empty interior in $U\cap\partial C$ and in ∂C. Consequently, $f\in F$. Since $f^{-1}(\max f(\bar{C}))\cap U\ne\emptyset$, we conclude that $f\in F\setminus\{f_1,\dots,f_n\}$, which is the desired contradiction. ∎

By Claim 2.9, the set $F \subset \sigma(G) \subset S^*$ is infinite and hence contains an infinite discrete subspace $\{f_n\}_{n\in\omega}$. By the definition of F, for every $n \in \omega$ we can choose $x_n \in \partial C$ and a positive real number ε_n such that $\partial C \cap \bar{B}(x_n, \varepsilon_n) \subset f_n^{-1}(\max f_n(\bar{C}))$. Here $\bar{B}(x_n, \varepsilon_n) = \{x \in X : \|x - x_n\| \le \varepsilon\}$. Moreover, since the subspace $\{f_n\}_{n\in\omega}$ of S^* is discrete, we can additionally assume that $\bar{B}(f_n, \varepsilon_n) \cap \bar{B}(f_m, \varepsilon_m) = \emptyset$ for any distinct $n, m \in \omega$. For every $n \in \omega$ let $y_n \in S$ be the unique point such that $f_n(z) = \langle z, y_n\rangle$ for all $z \in X$. The Riesz Representation Theorem guarantees that

$$\|y_n - y_m\| = \|f_n - f_m\| \ge \varepsilon_n + \varepsilon_m \quad \text{for all } n \ne m.$$

We shall need the following elementary (but not trivial) geometric fact.

Claim 2.10. *For any distinct $n, m \in \omega$ and a positive $\delta_n \le \frac{1}{3}\varepsilon_n^2$ the segment $[x_n + \delta_n y_n, x_m]$ meets the open convex set C.*

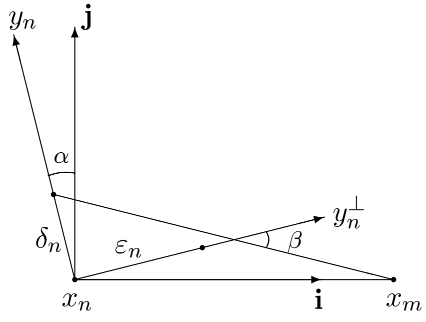
Proof. Assume for contradiction that $[x_n + \delta_n y_n, x_m] \cap C = \emptyset$. Taking into account that $f_n^{-1}(f_n(x_n)) \cap \bar{B}(x_n, \varepsilon_n) \subset \bar{C}$, we conclude that $\|x_n - x_m\| \ge \varepsilon_n$. Now consider the unit vector

$$\mathbf{i} = \frac{x_m - x_n}{\|x_m - x_n\|}.$$

Since $\langle x_m, y_n\rangle = f_n(x_m) \le \max f_n(\bar{C}) = f_n(x_n) = \langle x_n, y_n\rangle$, we get $\langle x_m - x_n, y_n\rangle \le 0$, which means that the angle between the vectors y_n and $\mathbf{i}$ is obtuse. Since $[x_n + \delta_n y_n, x_m] \cap C = \emptyset$, the unit vector y_n is not equal to $-\mathbf{i}$ and hence the unit vector

$$\mathbf{j} = \frac{y_n - \langle \mathbf{i}, y_n\rangle \cdot \mathbf{i}}{\|y_n - \langle \mathbf{i}, y_n\rangle \cdot \mathbf{i}\|}$$

is well-defined. Let α be the angle between the vectors y_n and $\mathbf{j}$. It follows that $y_n = -\sin(\alpha)\,\mathbf{i} + \cos(\alpha)\,\mathbf{j}$. Consider the vector $y_n^\perp = \cos(\alpha)\mathbf{i} + \sin(\alpha)\mathbf{j}$, which is orthogonal to y_n. Looking at the picture below, we can see that α is less than the angle β between $y_n^\perp$ and $x_m - (x_n + \delta_n y_n)$.



Since $x_n + \varepsilon_n y_n^\perp \in f_n^{-1}(f_n(x_n)) \cap \bar{B}(x_n, \varepsilon_n) \subset \bar{C}$ and $[x_n + \delta_n y_n, x_m] \cap C = \emptyset$, the angle β is less than $\arctan(\delta_n/\varepsilon_n)$. Then

$$\|y_n - \mathbf{j}\| = 2\sin(\alpha/2) \le \alpha \le \beta \le \arctan(\delta_n/\varepsilon_n) \le \delta_n/\varepsilon_n \le \varepsilon_n/3.$$

Next, we evaluate the distance $\|y_m - \mathbf{j}\|$. It is clear that $\|y_m - \mathbf{j}\| = 2\sin(\gamma/2)$ where γ is the angle between y_m and $\mathbf{j}$.

Let us consider separately two possible cases.

1) The vector y_m lies in the plane spanned by $\mathbf{i}$ and $\mathbf{j}$. Since $f_m(x) = \langle x, y_m\rangle$ is a supporting functional of C at x_m, we get $\langle x_n - x_m, y_m\rangle \le 0$ and hence $\langle \mathbf{i}, y_m\rangle \ge 0$.

On the other hand, $[x_n + \delta_n y_n, x_m] \cap C = \emptyset$ and $f_m^{-1}(f_m(x_m)) \cap B(x_m, \varepsilon_m) \subset \bar{C}$ imply that $\langle x_n + \delta_n y_n - x_m, y_m\rangle \ge 0$ and $\langle x_n + \delta_n \mathbf{j} - x_m, y_m\rangle \ge 0$. Consequently, $\gamma < \pi/2$ and $y_m = \sin(\gamma)\mathbf{i} + \cos(\gamma)\mathbf{j}$. It follows that

$$-\|x_n - x_m\| \sin(\gamma) + \delta_n \cos(\gamma) = \langle x_n - x_m + \delta_n \mathbf{j}, y_m\rangle \ge 0$$

and hence $\tan(\gamma) \le \delta_n / \|x_n - x_m\| \le \delta_n/\varepsilon_n$ and

$$\|y_m - \mathbf{j}\| = 2\sin(\gamma/2) \le \gamma \le \tan(\gamma) \le \delta_n/\varepsilon_n \le \varepsilon_n/3.$$

Then

$$\|f_n - f_m\| = \|y_n - y_m\| \le \|y_n - \mathbf{j}\| + \|\mathbf{j} - y_m\| \le 2\varepsilon_n/3 < \varepsilon_n + \varepsilon_m,$$

which contradicts the choice of the sequence (ε_k).

2) The vectors $\mathbf{i}, \mathbf{j}, y_m$ are linearly independent. Let $\mathbf{k}$ be a unit vector in X such that $\mathbf{k}$ is orthogonal to $\mathbf{i}$ and $\mathbf{j}$ and $y_m = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ for some real numbers a, b, c. It follows that $x_n \pm \varepsilon_n \mathbf{k} \in f_n^{-1}(f_n(x_n)) \cap \bar{B}(x_n, \varepsilon_n) \subset \bar{C}$. Since f_m is a supporting functional of C at x_m, we get $0 \ge \langle x_n \pm \varepsilon_n \mathbf{k} - x_m, y_m\rangle = -\|x_n - x_m\| a \pm \varepsilon_n c$, which implies

$$|c| \le \frac{\|x_n - x_m\|}{\varepsilon_n} a.$$

On the other hand, $[x_n + \delta_n \mathbf{j}, x_m] \cap C = \emptyset$ implies $0 \le \langle x_n + \delta_n \mathbf{j} - x_m, y_m\rangle = -\|x_n - x_m\| a + \delta_n b$ and

$$\frac{a}{b} \le \frac{\delta_n}{\|x_n - x_m\|}.$$

Now we see that

$$\begin{aligned}\|y_m - \mathbf{j}\| &= 2\sin(\gamma/2) \le \tan(\gamma) = \frac{\sqrt{a^2 + c^2}}{|b|} \le \frac{a}{b}\sqrt{1 + \frac{\|x_n - x_m\|^2}{\varepsilon_n^2}} \\ &\le \frac{\delta_n}{\|x_n - x_m\|}\sqrt{1 + \frac{\|x_n - x_m\|^2}{\varepsilon_n^2}} \le \delta_n \sqrt{\frac{1}{\|x_n - x_m\|^2} + \frac{1}{\varepsilon_n^2}} \\ &\le \delta_n \sqrt{\frac{1}{\varepsilon_n^2} + \frac{1}{\varepsilon_n^2}} = \sqrt{2}\,\frac{\delta_n}{\varepsilon_n} \le \frac{\sqrt{2}}{3}\varepsilon_n.\end{aligned}$$

Then $\|f_n - f_m\| = \|y_n - y_m\| \le \|y_n - \mathbf{j}\| + \|\mathbf{j} - y_m\| \le \varepsilon_n/3 + \sqrt{2}\varepsilon_n/3 < \varepsilon_n + \varepsilon_m$, which contradicts the choice of the sequence (ε_k) and completes the proof of Claim 2.10. ∎

Now we can continue the proof of Lemma 2.8. By induction for every $n \in \omega$ we shall choose a positive real number δ_n such that

(1) $\delta_n \le \varepsilon_n^2/3$;
(2) $[x_k + \delta_k y_k, x_n + \delta_n y_n] \cap C \neq \emptyset$ for any $k < n$.

To start the inductive construction put $\delta_0 = \varepsilon_0^2/3$. Assume that for some $n \in \omega$ we have constructed positive real numbers δ_k, $k < n$, satisfying the conditions (1)–(2). By Claim 2.10, for every $k < n$ the intersection $[x_k + \delta_k y_k, x_n] \cap C$ is not empty. Since the set C is open, we can choose a positive $\delta_n \le \varepsilon_n^2/3$ so small that for every $k < n$ the intersection $[x_k + \delta_k y_k, x_n + \delta_n y_n] \cap C$ is still not empty. This completes the inductive construction.

It follows from (2) that the infinite set $A = \{x_n + \delta_n y_n\}_{n\in\omega}$ is hidden behind the convex set $\bar{C}$. ∎

Lemmas 2.6 and 2.8 complete the proof of Lemma 2.4. ∎

3. Proof of Theorem 1.1. The implications (1)⇒(2)⇒(3)⇒(1) of Theorem 1.1 are proved in the following three lemmas.

LEMMA 3.1. *If C is polyhedral in $\overline{\mathrm{lin}}(C)$, then C is polyhedric in* $\mathrm{aff}(C)$.

Proof. If C is polyhedral in $\overline{\mathrm{lin}}(C)$, then $C = \bigcap_{i=1}^n f_i^{-1}((-\infty, a_i])$ for some linear functionals $f_1, \dots, f_n : \overline{\mathrm{lin}}(C) \to \mathbb{R}$ and some real numbers $a_1, \dots, a_n$. For every $i \le n$ consider the convex set $H_i = \mathrm{aff}(C) \cap f_i^{-1}((-\infty, a_i])$ and observe that its complement $\mathrm{aff}(C) \setminus H_i = \mathrm{aff}(C) \cap f_i^{-1}((a_i, +\infty))$ is also convex. Since $C = \bigcap_{i=1}^n H_i$, the set C is polyhedric in $\mathrm{aff}(C)$. ∎

LEMMA 3.2. *If a convex subset C of a linear space X is polyhedric in* $\mathrm{aff}(C)$, *then C hides no infinite subset $A \subset X \setminus C$.*

Proof. Assume for contradiction that some infinite subset $A \subset X \setminus C$ can be hidden behind C. First we show that $A \setminus \mathrm{aff}(C)$ contains at most two distinct points. Assume for contradiction that there are three pairwise distinct points $a_1, a_2, a_3 \in A \setminus \mathrm{aff}(C)$. Let $P = \{t_1 a_1 + t_2 a_2 + t_3 a_3 : t_1 + t_2 + t_3 = 1\}$ be the affine subspace of X spanned by a_1, a_2, a_3. The subspace P has dimension 1 or 2. The intersection $P \cap \mathrm{aff}(C)$ is an affine subspace of P that intersects the open segments $]a_1, a_2[$, $]a_1, a_3[$ and $]a_2, a_3[$ and hence coincides with P, which is not possible as $a_1, a_2, a_3 \in P \setminus \mathrm{aff}(C)$. So, $|A \setminus \mathrm{aff}(C)| \le 2$ and we lose no generality assuming that $A \subset \mathrm{aff}(C)$.

Being polyhedric in $\mathrm{aff}(C)$, the set C can be written as a finite intersection $C = \bigcap_{i=1}^n H_i$ of convex subsets $H_1, \dots, H_n \subset \mathrm{aff}(C)$ having convex complements $\mathrm{aff}(C) \setminus H_i$, $i \le n$. Since $A \setminus C = \bigcup_{i=1}^n \mathrm{aff}(C) \setminus H_i$, by the Pigeonhole Principle, there is an index $i \in \{1, \dots, n\}$ such that the convex set $\mathrm{aff}(C) \setminus H_i$ contains two distinct points $a, b \in A$ and hence contains the segment $[a,b]$, which is not possible as $[a,b]$ meets $C \subset H_i$. ∎

LEMMA 3.3. *If a closed convex subset C of a complete linear metric space X is not polyhedral in $\overline{\mathrm{lin}}(C)$, then some infinite $A \subset X \setminus C$ can be hidden behind C.*

Proof. Assume that C is not polyhedral in $\overline{\mathrm{lin}}(C)$. It is easy to check that

$$\mathrm{Ker}(C) = \{x \in X : \forall c \in C\ \forall t \in \mathbb{R}\ \ c + tx \in C\}$$

is a closed linear subspace of X and $C = C + \mathrm{Ker}(C)$. Let $Y = X/\mathrm{Ker}(C)$ be the quotient linear metric space and $Q : X \to Y$ be the quotient operator. By [7, 2.3.1], the operator Q is open, and by [7, 1.4.10], Y is a complete linear metric space. Let $D = Q(C)$. The equality $C = C + \mathrm{Ker}(C)$ implies that $C = Q^{-1}(D)$ and $Y \setminus D = Q(X \setminus C)$ is an open set. So, D is a closed convex set in Y. By Lemma 2.1, the set D is not polyhedral in $\overline{\mathrm{lin}}(D)$.

If the linear space $\overline{\mathrm{lin}}(D)$ is finite-dimensional, then it is isomorphic to a finite-dimensional Hilbert space H. Let $T : H \to \overline{\mathrm{lin}}(D)$ be the corresponding isomorphism. Since D is not polyhedral in $\overline{\mathrm{lin}}(D)$, the preimage $E = T^{-1}(D)$ is not polyhedral in H. Being finite-dimensional, the closed convex set E is a convex body in $\mathrm{aff}(E) \subset H$. Then for every $e_0 \in E$ the convex set $E_0 = E - e_0$ is a convex body in the linear subspace $H_0 = \mathrm{aff}(E) - e_0$ of H. Since E is not polyhedral in H, the shift $E_0 = E - e_0$ is not polyhedral in H_0. By Lemma 2.4 the set E_0 hides an infinite subset $A_0 \subset H_0 \setminus E_0$. Then E hides the infinite set $A_0 + e_0$ and the set $T(E) = D$ hides the infinite set $B = T(A_0 + e_0)$. Choose any subset $A \subset X$ such that $Q|A : A \to B$ is bijective. By Lemma 2.2 the infinite set A is hidden behind the convex set $C = Q^{-1}(D)$ and we are done.

Next, assume that $\overline{\mathrm{lin}}(D)$ is infinite-dimensional. Then the convex set D is also infinite-dimensional. By Lemma 2.3, there is a continuous injective affine operator $T : l_2 \to \overline{\mathrm{lin}}(D)$ such that $E = T^{-1}(D)$ is a closed convex body in l_2. Since $\mathrm{Ker}(D) = \{0\}$, we get $\mathrm{Ker}(E) = \{0\}$. This implies that E is not polyhedral in l_2. By Lemma 2.4, the convex set E hides some infinite subset $A_0 \subset l_2 \setminus E$. Then the infinite set $B = T(A_0) \subset Y \setminus D$ is hidden behind the convex set D. Choose any subset $A \subset X$ such that $Q|A : A \to B$ is bijective. By Lemma 2.2 the infinite set A is hidden behind the convex set $C = Q^{-1}(D)$ and we are done. ∎

4. Open problems. It would be interesting to know whether a relative version of Theorem 1.1 is true.

PROBLEM 4.1. *Let $C \subset D$ be two closed convex subsets of a complete linear metric space. Is it true that C hides no infinite subset $A \subset D \setminus C$ if and only if C is polyhedric in $D \cap \mathrm{aff}(C)$?*

In fact, the notions of polyhedric and hidden sets can be defined in a general context of convex structures (see [9]). Let us recall that a *convex structure* on a set X is a family $\mathcal{C}$ of subsets of X such that

- $\emptyset, X \in \mathcal{C}$;
- $\bigcap \mathcal{A} \in \mathcal{C}$ for any subfamily $\mathcal{A} \subset \mathcal{C}$;
- $\bigcup \mathcal{A} \in \mathcal{C}$ for any linearly ordered subfamily $\mathcal{A} \subset \mathcal{C}$.

For a convex structure $(X, \mathcal{C})$ and a subset $A \subset C$ the intersection $\mathrm{conv}(A) = \bigcap\{C \in \mathcal{C} : A \subset C\}$ is called the *convex hull* of A.

We say that a subset $C \subset X$ *hides* a subset $A \subset X$ if $\mathrm{conv}(\{a,b\}) \cap C \neq \emptyset$ for any distinct $a, b \in A$.

A subset C is *polyhedric in a subset* $D \subset X$ if $C = \bigcap_{i=1}^n H_i$ for some subsets $H_1, \ldots, H_n \subset D$ such that $H_i, D \setminus H_i \in \mathcal{C}$ for all $i \leq n$.

PROBLEM 4.2. *Given a convex structure* $(X, \mathcal{C})$ (*possibly with topology*) *characterize* (*closed*) *convex sets* $C \in \mathcal{C}$ *that hide no infinite subset* $A \subset X \setminus C$.

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Taras Banakh
Ivan Franko National University of Lviv
Ukraine
and
Jan Kochanowski University
Kielce, Poland
E-mail: t.o.banakh@gmail.com

Ivan Hetman
Ivan Franko National University of Lviv
Universytetska 1
Lviv, 79000, Ukraine
E-mail: ihromant@gmail.com

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