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A Radon–Nikodym derivative for positive linear functionals

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Abstract. An exact Radon–Nikodym derivative is obtained for a pair (I, J) of positive linear functionals, with J absolutely continuous with respect to I, using a notion of exhaustion of I on elements of a function algebra lattice.

1. Introduction and preliminaries. In [2] an abstract integration theory for extended-real-valued functions was developed, with the integral being a linear form, not necessarily continuous on monotone sequences, defined on a vector lattice. In [3], with local integral metrics, this analogue to Daniell's extension process was generalized. Moreover, in [1], given two positive Daniell integrals J and I, with J absolutely continuous with respect to I, by a constructive procedure, sufficient conditions were found for the existence of an exact Radon–Nikodym derivative of J with respect to I.

The aim of this paper is to contribute to the differentiation theory (in the sense of Radon–Nikodym derivatives) in the context of abstract integration, without any use of measure-theoretic methods.

It is known that the classical Radon–Nikodym theorem fails to be true in the finitely additive case unless some further assumptions hold. Necessary and sufficient conditions for the existence of exact Radon–Nikodym derivatives were obtained by Maynard [9], in the scalar case, illustrating the role of exhaustions and boundedness of the average range of the represented measure λ with respect to the integrating measure μ over a set. In [8] Hagood generalize Maynard's result to the case of a Banach-valued finitely additive measure using the Dunford–Schwartz integration theory ([5, Chap. III]).

In this paper, the results of abstract integration theory, along with techniques used for the finitely additive case, are employed to give an "exact" Radon–Nikodym derivative. The main instruments are certain basic ideas and natural results in abstract integration, which are not more complicated

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than the traditional ones, but more powerful, with applications to additive set functions.

For terminology and results not explained in this section, we refer the reader to [2]–[4].

One starts with a nonempty set X, a vector lattice B of real-valued functions on X (with pointwise operations and order) and a positive linear functional I on B. The triple (X, B, I) is called a *Loomis system.*

As usual, for any $f \in \mathbb{R}^X$, its *Riemann upper integral* is defined by

$$I^-(f) := \inf\{I(g) : g \in B,\ f \le g\},$$

with $\inf \emptyset := +\infty$.

We adopt the convention $\infty - \infty := 0$ for $\overline{\mathbb{N}}$. For every $A \subset \overline{\mathbb{R}}^X$, $+A$ denotes the set of all positive elements in A.

By $\overline{B}(B, I)$, or simply $\overline{B}$, we denote the Riesz space of all real-valued functions belonging to the closure of B in $\overline{\mathbb{R}}^X$ with respect to the integral metric (seminorm) $I^-(|\cdot|)$; equivalently,

$$\overline{B} = \{f \in \overline{\mathbb{R}}^X : I^+(f) = I^-(f) \in \mathbb{R}\}$$

where $I^+(f) := -I^-(-f)$.

A function $f \in \overline{\mathbb{R}}^X$ is said to be *I-integrable* if there exists an I-Cauchy sequence $(h_n) \subset B$ which converges to f (in symbols, $h_n \to f\ (I^-)$); i.e.,

$$I(|h_m - h_n|) \to 0 \quad \text{as } m, n \to \infty,$$

and for each fixed $h \in +B$,

$$I^-(|h_n - f| \wedge h) \to 0 \quad \text{as } n \to \infty.$$

We denote by $R_1(B, I)$, or R_1, the class of all extended-real-valued I-integrable functions. The sequence (h_n) is called an *I-approximate* (or *I-defining*) *sequence* of f; and $I(f) := \lim_n I(h_n)$ for $f \in R_1(B, I)$.

For any $I^- : \overline{\mathbb{R}}^X \to \mathbb{R} \cup \{+\infty\}$ and $f \in \overline{\mathbb{R}}^X$, we define the *localization*

$$I_\ell^-(f) := \sup\{I^-(f \wedge h) : h \in +B\}.$$

Then $R_1(B, I)$ is the closure of B in $\overline{\mathbb{R}}^X$ with respect to the integral seminorm $I_\ell^-(|\cdot|)$. The set $R_1(B, I)$ is I_ℓ^-*-closed* in the sense that for any I_ℓ^--Cauchy sequence $(f_n) \subset R_1(B, I)$ such that $f_n \to f\ (I_\ell^-)$ for some $f \in \overline{\mathbb{R}}^X$, we have $I_\ell^-(|f - f_n|) \to 0$ as $n \to \infty$, and $f \in R_1(B, I)$.

If $f \in R_1(B, I)$ then $f \in \overline{B}(B, I)$ if, and only if, there exists $h \in +B$ such that $|f| \le h$ (see [3, Cor. XI] or [4]).

If Ω is a semiring of sets in X and $\mu : \Omega \to [0, \infty]$ is finitely additive, then $B = B_\Omega :=$ real-valued step functions over Ω and $I = I_\mu := \int \cdot\, d\mu$ are admissible. We call (X, B_Ω, I_μ) the *induced Loomis system.* In general, $R_1(B_\Omega, I_\mu)$ contains strictly the set $L(X, \Omega, \mu)$ of Dunford–Schwartz [5]. If

Ω is an algebra, $\mu(X) < +\infty$ and $f \in \overline{\mathbb{R}}^X$ is bounded, then $f \in L(X, \Omega, \mu)$ if, and only if, $f \in \overline{B}(B_\Omega, I_\mu)$ (see [7, pp. 70, 199]).

We now briefly discuss the I-integrability for the product of two I-integrable functions. Example 4.1 below shows that the pointwise product of I-integrable functions with respect to a Loomis system need not be I-integrable. We have the following sufficient conditions.

LEMMA 1. *Let (X, B, I) be a Loomis system. If $k \in \overline{\mathbb{R}}^X$ is a bounded function such that $hk \in R_1(B, I)$ for all $h \in B$, then $fk \in R_1(B, I)$ for all $f \in R_1(B, I)$.*

Proof. Let $(h_n) \subset B$ be an I-approximate sequence of f. It is easily verified that $(h_n k) \subset R_1(B, I)$ is an I_ℓ^--approximating sequence of kf with

$$I_\ell^-(|h_n k - kf|) \le M I_\ell^-(|h_n - f|) \to 0$$

as $n \to \infty$, where M is an upper bound of $|k|$. In view of the closedness property of R_1, one has $fk \in R_1(B, I)$. ■

In particular, if $f \in R_1(B, I)$ (resp. $\overline{B}$), $h \in B$ is a bounded function and $BB \subset R_1$ (for instance, if B is an algebra lattice), then $fh \in R_1(B, I)$ (resp. $fh \in \overline{B}$). Now, the following especially useful corollary can be formulated.

COROLLARY 2. *Let $f, g \in R_1(B, I)$ be two bounded functions and suppose that $BB \subset R_1$. Then $fg \in R_1(B, I)$.*

Although boundedness is a sufficient condition, Example 4.2 below will show that it is not necessary. Nevertheless, a bit more can be said concerning the I-integrability of the product of two functions.

We say that a Loomis system (X, B, I) has the *c_{00}-property* if B is stonian (i.e., $h \wedge 1 \in B$ for all $h \in +B$) and $I(h - h \wedge n) \to 0$ as $n \to \infty$ for all $h \in +B$. A Loomis system (X, B, I) is said to be an *algebra Loomis system* if B is additionally an algebra lattice, i.e., $BB \subset B$.

LEMMA 3. *Let (X, B, I) be an algebra Loomis system with the c_{00}-property and let $f, g \in R_1(B, I)$. If either of the two functions is bounded, then $fg \in R_1(B, I)$.*

Note that the induced algebra Loomis system (X, B_Ω, I_μ) always has the c_{00}-property. Note that when I is a Daniell integral, i.e., $I(h_n) \to 0$ if $0 \le h_{n+1} \le h_n \in B$ and $h_n \to 0$ pointwise on X, no boundedness conditions are needed: fg will be integrable when f and g are.

We consider a Loomis system (X, B, I) and another positive linear functional $J : B \to \mathbb{R}$.

DEFINITION 4. We say that J is *absolutely continuous* with respect to I on B, and we write $J \ll I$, if for every $\varepsilon > 0$ and $h \in +B$ there exists $\delta > 0$

(depending on ε and h) such that $I(u) < \delta$ implies $J(u) < \varepsilon$ for all $u \in +B$ with $u \leq h$.

Given a finitely additive measure space (X, Ω, μ) and the induced Loomis system (X, B_Ω, I_μ), if $\lambda : \Omega \to [0, \infty[$ is another finitely additive measure on the semiring Ω such that $\lambda \ll \mu$ (i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that $A \in \Omega$ and $\mu(A) < \delta$ implies $\lambda(A) < \varepsilon$), it is easy to check that $\lambda \ll \mu$ implies $I_\lambda \ll I_\mu$. If, additionally, Ω is an algebra, the converse is true.

We record some consequences of the above definition.

Suppose that $J \ll I$. Then the following assertions hold:

(i) Let $(h_n) \subset +B$ be such that there exists $h \in +B$ with $h_n \leq h$ for all $n \in \mathbb{N}$. Then $I(h_n) \to 0$ implies $J(h_n) \to 0$.

(ii) Let $f, (f_n) \subset \overline{\mathbb{R}}^X$, then $f_n \to f$ (I^-) implies $f_n \to f$ (J^-).
In particular, if $f \in \overline{\mathbb{R}}^X$ is an I-null function (i.e., $f \in R_1(B, I)$ and $I(|f|) = 0$), then f is J-null.

(iii) $\overline{B}_I \subset \overline{B}_J$ and $I \ll J$ on $\overline{B}_I$.

To prove (iii), observe that $f \in \overline{B}$ if, and only if, for every $\varepsilon > 0$ there exist $h, k \in B$ such that $h \leq f \leq k$ and $I(k - h) < \varepsilon$.

Note that the following two conditions are equivalent:

(i) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $h \in +B$ and $I(h) < \delta$ implies $J(h) < \varepsilon$.

(ii) There exists $M > 0$ such that $J(h) \leq M I(h)$ for all $h \in +B$.

Example 4. shows that absolute continuity, $J \ll I$, is strictly weaker than the above condition (ii).

We recall that in the finitely additive measure case, a notion of great importance in the construction of the Radon–Nikodym derivatives is that of exhaustion, where the underlying set is partitioned into subsets having a certain prescribed property. Also, since the measures involved need not have Hahn decompositions, it is essential to consider the behaviour of the "average range" and the "ε-approximate range". Our results are proved by translating the finitely additive case into the abstract linear functional one. Example 4.4 exhibits a functional J that is absolutely I-continuous, $J \ll I$, but not *strongly I-representable*, i.e., there does not exist $g \in B$ such that $J(f) = I(fg)$ for all $f \in B$.

2. Exhaustions. We denote by $\mathcal{L}$ either the vector lattice B or $\overline{B}$ (associated to (X, B, I)), and I is again an extended positive linear functional defined on B or $\overline{B}$.

DEFINITION 5. Let (X, B, I) be a Loomis system and let $f \in +R_1(B, I)$ with $I(f) > 0$. A countable collection of positive functions in $\mathcal{L}$, $\Phi = (h_i)_{i \in T}$, is said to be an *$\mathcal{L}$-I-exhaustion* on f if the following conditions are satisfied:

(i) $I(h_i) > 0$ for all $i \in T$ and $\sum_{i\in P} h_i \le f$ for all finite $P \subset T$.
(ii) $\sup_P\{\sum_{i\in P} I(h_i)\} = I(f)$, where the supremum is taken over all finite subsets P of T.

In case $\sum_{i\in T} h_i = f$, we say that Φ is a *complete* $\mathcal{L}$-I-*exhaustion* of f.

For simplicity, we shall only consider exhaustions on functions f with $I(f) > 0$.

It is easy to see that if $(h_i)_{i\in T}$ is a (complete) $\mathcal{L}$-I-exhaustion on f and for each $i \in T$ the countable collection $(h_{i,j})_{j\in T'}$ is a (complete) $\mathcal{L}$-I-exhaustion on h_i, then $(h_{i,j})_{(i,j)\in T\times T'}$ is a (complete) $\mathcal{L}$-I-exhaustion on f.

LEMMA 6. *Let* $f \in +\mathcal{L}$ *and* $\Phi = (h_i)_{i\in T}$ *an* $\mathcal{L}$-I-*exhaustion on* f. *If* $J \ll I$, *then the collection* $\{h \in \Phi : J(h) > 0\}$ *is an* $\mathcal{L}$-J-*exhaustion on* f.

Proof. It suffices to treat the case $T = \mathbb{N}$, since if T is finite the result is clear.

First, we prove that $I(f) > 0$ is equivalent to $\{h \in \Phi : J(h) > 0\} \neq \emptyset$. If $I(f) > 0$, let $0 < \varepsilon < J(f)$, and for each $n \in \mathbb{N}$, set $g_n := f - \sum_{i=1}^{n} h_i$. We have $g_n \in +\mathcal{L}$, $g_n \le f$ and $I(g_n) \to I(f)$ as $n \to \infty$; hence, there exists $k \in \mathbb{N}$ such that $I(g_{n_k}) < \delta$. In view of Definition 1.1 of absolute continuity, one has

$$J(g_k) = J(f) - \sum_{i=1}^{k} J(h_i) < \varepsilon < J(f),$$

so that there exists $j \in \mathbb{N}$ such that $J(h_j) > 0$. The converse is evident, since $0 \le J(h_i) \le J(f)$ for all $i \in \mathbb{N}$.

Now, if Φ is an $\mathcal{L}$-I-exhaustion on f, then according to Definition 5(ii), there exists $k \in \mathbb{N}$ with $I(f - \sum_{i=1}^{n} h_i) < \delta$ for all $n \ge k$; and when $J \ll I$, one has $J(f - \sum_{i=1}^{n} h_i) < \varepsilon$ for all $n \ge k$, and the result is established. ∎

LEMMA 7. *Let* (X, B, I) *be an algebra Loomis system. Suppose that* f *is a bounded function in* $+R_1(B, I)$ *and* $\Phi = (h_i)_{i\in T}$ *is an* $\mathcal{L}$-I-*exhaustion on* f. *Then, for each bounded function* $g \in +\mathcal{L}$, *the collection*

$$\{gh : I(gh) > 0,\, h \in \Phi\}$$

is an $\mathcal{L}$-I-*exhaustion on* gf.

Proof. As a consequence of Lemma 1, $gh \in +\overline{B}$ for all $h \in \Phi$ and $gf \in +R_1(B, I)$. Indeed, $\sum_{k=1}^{n} gh_k \le gf$ for all natural n, and since

$$0 \le I\Big(gf - \sum_{k=1}^{n} gh_k\Big) = I\Big[g\Big(f - \sum_{k=1}^{n} h_k\Big)\Big] \le MI\Big(f - \sum_{k=1}^{n} h_k\Big) \to 0$$

as $n \to \infty$, where M is an upper bound of g, we obtain the result. ∎

Observe that $I(gf) = \sum_{i=1}^{\infty} I(gh_i)$, which says that the exhaustions afford a kind of partial continuity.

Example 4.5 below shows that condition (ii) in Definition 5 is necessary.

Given any Loomis system (X, B, I), a property P is said to *exhaust* I on $f \in +R_1(B, I)$ if there exists an $\mathcal{L}$-I-exhaustion Φ on f, each element of which has P. A property P is called *$\mathcal{L}$-I-invariant* if whenever $f, g \in +\mathcal{L}$, $I(f) > 0$, $I(g) > 0$ and $I(|f - g|) = 0$, then either f and g both have P or neither does.

Lemma 8. *Let $f \in +\overline{B}$ and let P be a property such that*:

(i) *P exhausts I on f.*
(ii) *P is $\overline{B}$-I-invariant.*

Then there exists a complete $\overline{B}$-I-exhaustion on f, each element of which has P.

Proof. Let $(h_i)_{i\in\mathbb{N}}$ be a $\overline{B}$-I-exhaustion on f, each element of which has P. For each $n \in \mathbb{N}$,

$$0 \le \sum_{i=1}^{n} h_i \le \sum_{i=1}^{\infty} h_i \le f,$$

with $\sum_{i=1}^{n} h_i \in +\overline{B}$ and

$$I\Big(f - \sum_{i=i}^{n} h_i\Big) \to 0$$

as $n \to \infty$. As a consequence,

$$\sum_{i\in\mathbb{N}} h_i \in +\overline{B} \quad \text{and} \quad I\Big(\sum_{i=1}^{\infty} h_i\Big) = I(f).$$

We now consider the sequence $(g_i)_{i\in\mathbb{N}}$ defined by

$$g_0 := f - \sum_{i=1}^{\infty} h_i, \qquad g_1 := h_1 + g_0, \qquad g_i := h_i, \quad i \ge 2.$$

Since $I(|g_1 - h_1|) = I(g_0) = 0$, we have a $\overline{B}$-exhaustion on f, which is $\overline{B}$-I-invariant. Moreover, it is complete, since

$$\sum_{i\in\mathbb{N}} g_i = g_0 + \sum_{i\in\mathbb{N}} h_i = f. \ \blacksquare$$

For any $f \in +R_1(B, I)$ and $\mathcal{L}$ either the vector lattice B or $\overline{B}$, define

$$\mathcal{L}_f := \{h \in +\mathcal{L} : h \le f \text{ and } I(h) \ne 0\}.$$

The standard average range for finitely additive measures has the following parallel definition for positive linear functionals.

DEFINITION 9. For each $f \in +R_1(B,I)$ the *average range* of J with respect to I over f (relative to $\mathcal{L}$) is

$$A^{\mathcal{L}}(f) := \left\{ \frac{J(h)}{I(h)} : h \in \mathcal{L}_f \right\},$$

and given $\varepsilon > 0$, the *ε-approximate average range* of J with respect to I over f is

$$A^{\mathcal{L}}_{\varepsilon}(f) := \{x \in \mathbb{R} : |J(h) - xI(h)| \le \varepsilon I(h),\ h \in \mathcal{L}_f\}.$$

Note that the property $P := A^{\mathcal{L}}(f) \neq \emptyset$ is hereditary in $\mathcal{L}$, i.e., if $f, g \in +\mathcal{L}$, $f \le g$ and g has P, then f has P.

We need some notation: given $\varepsilon > 0$, we say that $h \in +\mathcal{L}$ *has the property* P_ε if $A^{\mathcal{L}}_{\varepsilon}(f) \neq \emptyset$.

The following lemma summarizes the properties that we will need. They are all easy to verify.

LEMMA 10. *Given an arbitrary Loomis system* (X, B, I), $f \in +R_1(B,I)$ *and* $\varepsilon > 0$, *the following assertions hold*:

- (i) $\delta(A^{\mathcal{L}}_{\varepsilon}(f)) \le 2\varepsilon$, *where* δ *denotes diameter.*
- (ii) *If* $g \in +R_1(B,I)$ *and* $f \le g$, *then* $A^{\mathcal{L}}(f) \subset A^{\mathcal{L}}(g)$, *and so* $A^{\mathcal{L}}_{\varepsilon}(f) \subset A^{\mathcal{L}}_{\varepsilon}(g)$.
- (iii) *If* $0 < \varepsilon < \gamma$, *then* $A^{\mathcal{L}}_{\varepsilon}(f) \subset A^{\mathcal{L}}_{\gamma}(f)$.
- (iv) *If* $h \in +\overline{\mathbb{R}}^X$ *is bounded and* $fh \in +R_1(B,I)$, *then* $A^{\mathcal{L}}_{\varepsilon}(f) \subset A^{\mathcal{L}}_{\varepsilon}(fh)$.
- (v) $A^{\mathcal{L}}_{\varepsilon}(f) \neq \emptyset$ *if*, *and only if*, $A^{\mathcal{L}}(f)$ *is bounded and* $\delta(A^{\mathcal{L}}(f)) \le 2\varepsilon$.
- (vi) *The property* $A^{\mathcal{L}}_{\varepsilon}(f) \neq \emptyset$ *exhausts* I *on each element in* $+\mathcal{L}$ *if*, *and only if*, *the property* $\delta(A^{\mathcal{L}}(f)) < \varepsilon$ *exhausts* I *on each element in* $+\mathcal{L}$.

LEMMA 11. *Given an arbitrary Loomis system* (X, B, I), *let* $J : B \to \mathbb{R}$ *be a positive linear functional such that* $J \ll I$. *If* $f, g \in +\mathcal{L}$ *are such that* $I(|f - g|) = 0$, *then* $A^{\mathcal{L}}(f) = A^{\mathcal{L}}(g)$. *In particular*, *the property* $A^{\mathcal{L}}_{\varepsilon}(f) \neq \emptyset$ *is* $\mathcal{L}$-I-*invariant.*

Proof. (a) Suppose $f \le g$. Then $A^{\mathcal{L}}(f) \subset A^{\mathcal{L}}(g)$. To prove the reverse inclusion, let $\alpha \in A^{\mathcal{L}}(g)$, then there exists $h \in \mathcal{L}_g$ such that $\alpha = J(h)/I(h)$.

Assume that $h \wedge f \in \mathcal{L}_g$. Clearly, $I(h) + I(f) = I(h \vee f) + I(h \wedge f)$ and $0 \le f \vee h - f \le g - f$. By hypothesis, $I(h) = I(h \wedge f)$, and also $J(h) = J(h \wedge f)$, since J is absolutely continuous with respect to I on B or $\overline{B}_I$. Consequently, $\alpha = J(h)/I(h) = J(h \wedge f)/I(h \wedge f) \in A^{\mathcal{L}}(f)$.

(b) For general $f, g \in +\mathcal{L}$, we have $f \wedge g \in +\mathcal{L}$. By applying (a) for $(f, f \wedge g)$ and $(g, f \wedge g)$ we obtain $A^{\mathcal{L}}(f) = A^{\mathcal{L}}(f \wedge g) = A^{\mathcal{L}}(g)$, and this completes the proof. ∎

Proposition 12 (Exhaustion Principle). *Let* (X, B, I) *be an algebra Loomis system such that* $1 \in \overline{B}$ *and let* P *be a hereditary property. Then the following two statements are equivalent*:

(i) P *exhausts* I *on* 1.
(ii) *For each* $\delta > 0$, *there exists* $g \in +B$ *with* $g \leq 1$ *and* $\alpha \in]0, 1[$ *such that*:

(a) $I(1 - g) < \delta$,
(b) *for all* $h \in +B$ *with* $h \leq g$, *there exists* $k \in +B$ *with* $k \leq h$ *such that* $I(k) > \alpha I(h)$ *and* k *has* P.

Proof. The argument is similar to the one in the proof of [8, Prop. 3.2]. We only need to establish the corresponding versions of some results.

(a) Let $(h_i)_{i\in\mathbb{N}}$ be an $\mathcal{L}$-I-exhaustion on 1. For $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{i=n_0+1}^{\infty} I(h_i) < \delta.$$

Put $g := \sum_{i=1}^{n_0} h_i$ and assume, without loss of generality, that $g \leq 1$. Then $I(1 - g) < \delta$.

(b) For all $h \in +B$ with $h \leq g$, we have

$$I(h) = \sum_{i=1}^{n_0} I(h_i h).$$

Observe that, by Lemma 6, $(h_i h)_i$ is an $\mathcal{L}$-I-exhaustion on h, hence $\sum_{i=1}^{n_0} h_i h \leq h$ and $\sum_{i=1}^{\infty} I(h_i h) = I(h)$. Consequently, there exists $j \in \mathbb{N}$ such that $2n_0 I(h_j h) > I(h)$. Now, with $k := h_j h$, (b) holds.

Note that $k \leq g \leq 1$ and k has P since the property is hereditary. ∎

One can specify property P as $P_\varepsilon := A_\varepsilon^{\mathcal{L}}(f) \neq \emptyset$ for $f \in +\mathcal{L}$. As above, we define

$$\overline{B}_1 := \{h \in +\overline{B} : h \leq 1 \text{ and } I(h) \neq 0\}.$$

3. The Radon–Nikodym derivative

Theorem 13. *Let* (X, B, I) *be an algebra Loomis system such that* $1 \in \overline{B}$. *Let* $J : B \to \mathbb{R}$ *be a positive linear functional. Assume that*:

(i) $J \ll I$.
(ii) $A^{\overline{B}}(1) := \{J(h)/I(h) : h \in \overline{B}_1\}$ *is bounded.*
(iii) *For each* $\varepsilon > 0$, *the property* $P_\varepsilon := A_\varepsilon^{\overline{B}}(h) \neq \emptyset$ *exhausts* I.

Then there exists a bounded function $g \in \overline{B}$ *such that* $J(f) = I(fg)$ *for all bounded* $f \in \overline{B}$.

Proof. We assume $I(1) > 0$. Hypotheses (i) and iii), together with Lemmas 7 and 10, imply that there exists a complete $\overline{B}$-I-exhaustion $(g_i^1)_{i\in\mathbb{N}}$ on 1, each element of which has P_ε, i.e.,

$$1 = \sum_{i=1}^{\infty} g_i^1 \quad \text{and} \quad A_\varepsilon^{\overline{B}}(g_i^1) \neq \emptyset \quad \text{for every } i \in \mathbb{N}.$$

We may now decompose each $g_i^1 \in +\overline{B}$ ($i \in \mathbb{N}$) in an exhausting way and, by induction, we construct a sequence of exhaustions:

For every $i \in \mathbb{N}$, $\varepsilon := 1/2^2$, there exists a complete $\overline{B}$-I-exhaustion $(g_{i,j}^2)_{j\in\mathbb{N}}$ on g_i^1.

Given $g_\alpha^k \in +\overline{B}$ ($k \in \mathbb{N}$, $\alpha \in \mathbb{N}^k$), let $(g_{\alpha,j}^{k+1})_{j\in\mathbb{N}}$ be a complete $\overline{B}$-I-exhaustion on g_α^k; here $\varepsilon := 1/2^{k+1}$.

Thus, for every $k \in \mathbb{N}$, $(g_\alpha^k)_{\alpha\in\mathbb{N}^k}$ is a complete $\overline{B}$-I-exhaustion on 1, and for $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^k$, $s \in \mathbb{N}$, $(g_{\alpha,\beta}^{k+s})_{\beta\in\mathbb{N}^s}$ is a complete $\overline{B}$-I-exhaustion on g_α^k. Moreover, for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^k$,

$$A_{1/2^k}^{\overline{B}}(g_\alpha^k) \neq \emptyset.$$

Now, for each $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^k$, let $r_\alpha^k \in A_{1/2^k}^{\overline{B}}(g_\alpha^k)$. From hypothesis (ii), we have

$$|r_\alpha^k| \leq \left| r_\alpha^k - \frac{J(g_\alpha^k)}{I(g_\alpha^k)} \right| + \frac{J(g_\alpha^k)}{I(g_\alpha^k)} \leq \frac{1}{2^k} + M < 1 + M, \tag{1}$$

where $M := \sup A_{1/2^k}^{\overline{B}}(g_\alpha^k)$. Let $g_k : X \to \mathbb{N}$ be defined by

$$g_k := \sum_{\alpha\in\mathbb{N}^k} r_\alpha^k g_\alpha^k.$$

The sequence (g_k) is uniformly Cauchy:

$$|g_k - g_{k+s}| \leq \sum_{\alpha\in\mathbb{N}^k} \left| r_\alpha^k g_\alpha^k - \sum_{\beta\in\mathbb{N}^s} r_{\alpha,\beta}^{k+s} g_{\alpha,\beta}^{k+s} \right| \leq \sum_{\alpha\in\mathbb{N}^k} \sum_{\beta\in\mathbb{N}^s} |r_\alpha^k - r_{\alpha,\beta}^{k+s}| g_{\alpha,\beta}^{k+s}. \tag{2}$$

But $r_\alpha^k, r_{\alpha,\beta}^{k+s} \in A_{1/2^k}^{\overline{B}}(g_{\alpha+\beta}^{k+s})$ and $|r_\alpha^k - r_{\alpha,\beta}^{k+s}| \leq 1/2^{k-1}$; hence, by Lemma 10(i), it follows that $|g_k - g_{k+s}| \leq 1/2^{k-1}$ on X.

With (1) and (2) we conclude that $g := \lim_k g_k$ is bounded.

On the other hand, as $k \to \infty$, since

$$I(|g_k - g_{k+s}|) \leq \frac{1}{2^{k-1}} I(1) \to 0,$$

and, for all $h \in +B$,

$$I^-(|g_k - g| \wedge h) \leq I(|g_k - g|) \leq \frac{1}{2^{k-1}} I(1) \to 0,$$

we deduce that (g_k) is an I-Cauchy sequence such that $g_k \to g$ (I^-); hence, $g \in R_1(B, I)$ and $I(g) = \lim_k I(g_k)$. Moreover, $g \in \overline{B}$, because g is bounded.

Now, for every bounded f, by Corollary 2, fg and fg_k are in $\overline{B}$, and

$$|I(fg_k) - I(fg)| \le I(f|g_k - g|) \le M' \frac{1}{2^{k-1}} I(1) \to 0,$$

where M' is an upper bound for f. Hence,

$$I(fg) = \lim_{k\to\infty} I(fg_k).$$

Now, Lemmas 6 and 7 ensure that

$$J(f) = \sum_{\alpha\in\mathbb{N}^k} J(fg_\alpha^k) \quad (k \in \mathbb{N});$$

and using Lemma 10(iv), we compute

$$|J(f) - I(f_\alpha^k)| \le \sum_{\alpha\in\mathbb{N}^k} |J(fg_\alpha^k) - r_\alpha^k I(fg_\alpha^k)| \le \frac{1}{2^k} \sum_{\alpha\in\mathbb{N}^k} I(fg_k) \xrightarrow[k\to\infty]{} 0.$$

Therefore,

$$J(f) = \lim_{k\to\infty} I(fg_k) = I(fg).$$

For an arbitrary bounded function $f \in \overline{B}$, we consider $f = f^+ - f^-$. ∎

The following lemma illustrates connections between average ranges (see [9, Lemma 3.7]).

Lemma 14. *Let (X, B, I) be a Loomis system and let $J : B \to \mathbb{N}$ be a positive linear functional. Suppose that for each $\varepsilon > 0$ there exists $\alpha > 0$ such that for each $h \in +B$, there exists $k \le h$ with $k \in +\overline{B}$ and $I(k) > \alpha I(h) > 0$. Consider the following two statements*:

(i) $A^B(k) \ne \emptyset$.
(ii) $\delta(A^B(k)) < \varepsilon$.

Then (i)⇒(ii); *and if additionally $J \ll I$, then* (ii)⇒(i).

Its proof follows quite easily from the above definitions.

Corollary 15. *Let (X, B, I) be an algebra Loomis system with unit $(1 \in B)$ and let $J : B \to \mathbb{N}$ be a positive linear functional with these two properties*:

(a) $J \ll I$.
(b) *For all $\varepsilon > 0$ and $\delta > 0$, there exists $g \in +B$ with $g \le 1$ and $\alpha \in [0, 1]$ such that*:
- $I(1 - g) < \delta$.
- $A^B(g)$ *is bounded.*
- *For all $h \in +B$ with $h \le g$, there exists $k \in +B$ with $k \le h$ such that $I(k) > \alpha I(h)$ and $\delta(A^B(k)) < \varepsilon$.*

Then there exists a bounded function $g \in \overline{B}$ such that $J(f) = I(fg)$ for all bounded $f \in \overline{B}$.

Proof. For an arbitrary bounded function f in $\overline{B}$, with (h_n) an $\mathcal{L}$-I-exhaustion on f, by Lemma 6,

$$J(f) = \sup_{P \subset T} \Big\{ \sum_{n \in P} J(h_n) \Big\}.$$

Now, for every $\varepsilon > 0$, by (b) and the Exhaustion Principle, property $\delta(A^B(k)) < \varepsilon$ exhausts I on each element of $+B$. By Lemma 10(vi), the same is true for $A^B_\varepsilon(k) \neq \emptyset$. Hence, by Lemma 14 and Theorem 13, there exists $g \in \overline{B}$ such that

$$J(h_n) = I(h_n g), \qquad \forall n \in \mathbb{N}.$$

But Lemma 7 says that $(h_n g)$ is an $\mathcal{L}$-I-exhaustion on fg, and we conclude that

$$J(f) = \sup_{P \subset T} \Big\{ \sum_{n \in P} J(h_n) \Big\} = \sup_{P \subset T} \Big\{ \sum_{n \in P} I(h_n g) \Big\} = I(fg), \qquad \forall f \in \overline{B}. \ ∎$$

PROPOSITION 16. *Let (X, B, I) be an algebra Loomis system with unit and let $J : B \to \mathbb{N}$ be a positive linear functional. Then*:

(i) *For $g \in \overline{B}$ and $\delta > 0$, there exists $k \in +\overline{B}$ such that $I(1 - k) < \delta$.*
(ii) *If there exists a bounded function $g \in \overline{B}$ such that $J(f) = I(gf)$ for all bounded functions $f \in \overline{B}$, then $J \ll I$ and $A^{\overline{B}}(1)$ is bounded.*

Proof. For $g \in \overline{B}$ and $\delta > 0$, there exist $h_1, h_2 \in +\overline{B}$ such that $h_1 \leq g \leq h_2$ and $I(h_2 - h_1) < \delta$. With $k := 1 - (h_2 - h_1)$, (i) holds. But (ii) is immediate by Definition 9. ∎

Proposition 16 provides necessary conditions for J to have a Radon–Nikodym derivative with respect to I.

4. Examples

4.1. Let $X := \mathbb{N}$ and $B := \{f \in \mathbb{N}^{\mathbb{N}} : I(f) := \lim f(n)/n \in \mathbb{R} \text{ exists}\}$. Hence, B is a vector lattice such that $1 \in B$. Here,

$$B = \overline{B} = \{f \in R_1 : |f(n)| < \infty, \forall n \in \mathbb{N}\}.$$

If

$$f(n) := \begin{cases} 0, & n \text{ is odd}, \\ 1, & n \text{ is even}, \end{cases} \qquad \text{and } g(n) := n, \ \forall n \in \mathbb{N},$$

then $fg \notin B$.

4.2. Let $X :=]-1,+1[$, $B :=$ the class of all step functions on X, and $I : B \to \mathbb{N}$ given in the usual canonical form:

$$I\Big(\sum_{k=1}^{n} a_k \chi_{A_k}\Big) := \sum_{k=1}^{n} a_k \lambda(A_k), \quad \forall \sum_{k=1}^{n} a_k \chi_{A_k} =: h \in B,$$

where λ is the Lebesgue measure on X. We now consider the following functions:

$$f(x) := \begin{cases} 0, & -1 < x \le 0, \\ \sum_{n=1}^{\infty} n\chi_{[1/(n+1)^3, 1/n^3[}, & 0 < x < 1, \end{cases}$$

$$g(x) := \begin{cases} \sum_{n=1}^{\infty} n\chi_{[1/(n+1)^3-1, 1/n^3-1[}, & -1 < x < 0, \\ 0, & 0 \le x < 1. \end{cases}$$

Both are in $R_1(B,I)$ (it is not difficult to show that $f \wedge h$ and $g \wedge h$ are in $R_1(B,I)$ for all h in $+B$, and $I^+(|f|)$ and $I^+(|g|)$ are finite); but neither is bounded, and their product is zero everywhere; hence fg is I-integrable.

4.3. Let $X := [0,1]$, $B := \mathcal{C}(X)$ and $I(f) := \int_0^1 f\,d\lambda$, for all real continuous functions f on X, and λ the Lebesgue measure on X. We consider the λ-integrable function $h : X \to \mathbb{N}$ given by

$$h(x) := \begin{cases} 0, & x = 0, \\ 1/\sqrt{x}, & x \in]0,1]. \end{cases}$$

If we now define

$$J : B \to \mathbb{N}, \quad J(f) := I(hf), \quad \forall f \in B,$$

then $J \ll I$. Namely, J is a Radon measure with density h on the compact space X, having the Radon measure I as basis (see [10]). If we assume that there exists $M > 0$ such that $J \le MI$ on B, we derive a contradiction by constructing $(u_n) \subset +B$ such that $\|u_n\| := I(|u_n|) \to 0$ and $J(u_n) \ge \ln 2$ for all $n \in \mathbb{N}$. For every $n \ge 2$, u_n will be affine on $[0, 1/2n]$ and $[1/n, 1/(n-1)]$, and will coincide with h on $[1/2n, 1/n]$.

4.4. Let X and B be as in Example 4.3. Let us enumerate $\mathbb{Q} \cap X$ as $\{x_0 = 0, x_1, x_2, \ldots\}$ with the x_n all distinct, and let $(a_n)_{n\ge0} \subset +\mathbb{N}$ be such that $a_0 = 1$ and the series $\sum_{n\ge0} a_n$ converges. If we define the positive linear functional $I : B \to \mathbb{N}$ by

$$I(f) := \sum_{n=0}^{\infty} a_n f(x_n) = f(0) + \sum_{n=1}^{\infty} a_n f(x_n),$$

and the constant functional (hence, positive and linear)

$$J(f) := f(0), \quad \forall f \in B,$$

it is clear that $J \ll I$.

Assume that there exists $g \in B$ such that

$$J(f) = I(gf), \qquad \forall f \in B.$$

Then

$$J(f) = f(0) = I(fg) = f(0)g(0) + \sum_{n=1}^{\infty} a_n f(x_n) g(x_n).$$

Hence, for $f(x) := x$ on X, we have

$$J(f) = f(0) = 0 = \sum_{n=1}^{\infty} a_n x_n g(x_n).$$

But all a_n's are positive, which means that $g(x_n) = 0$ for all $n \in \mathbb{N}$. By continuity, $g \equiv 0$, and so $J = 0$, which is a contradiction. Observe that the functionals I and J are Daniell integrals (use Dini's theorem).

4.5. Let B be the Banach lattice c consisting of all convergent sequences of real numbers, and define the following positive linear functional:

$$I(f) := \sum_{n=1}^{\infty} f(n) 2^{-n} + f(\infty).$$

Let $h_n := \chi_{\{n\}}$. Then (h_n) satisfies (i) in Definition 5 (for $f = 1 \in B$), yet

$$I(1) = 2 \neq 1 = \sum_{n=1}^{\infty} I(h_n).$$

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