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Minimal ball-coverings in Banach spaces and their application

by

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Abstract. By a ball-covering $\mathcal{B}$ of a Banach space X, we mean a collection of open balls off the origin in X and whose union contains the unit sphere of X; a ball-covering $\mathcal{B}$ is called minimal if its cardinality $\mathcal{B}^{\#}$ is smallest among all ball-coverings of X. This article, through establishing a characterization for existence of a ball-covering in Banach spaces, shows that for every $n \in \mathbb{N}$ with $k \leq n$ there exists an n-dimensional space admitting a minimal ball-covering of $n + k$ balls. As an application, we give a new characterization of superreflexive spaces in terms of ball-coverings. Finally, we show that every infinite-dimensional Banach space admits an equivalent norm such that there is an infinite-dimensional quotient space possessing a countable ball-covering.

1. Introduction. The study of geometric and topological properties of unit balls of normed spaces plays a central rule in the geometry of Banach spaces. Almost all properties of Banach spaces, such as convexity, smoothness, reflexivity, the Radon–Nikodým property, etc., can be viewed as properties of the unit ball. We should also mention here several topics concerning the behavior of families of balls, for example, the Mazur intersection property (see, for instance, [14], [16], [17]), the sphere packing problem (see, for instance, [7] and [13]), and the measure of non-compactness with respect to topological degree (see, for instance, [1], [2], [10]), which have also attracted attention of many mathematicians.

Starting from a different viewpoint, this article is devoted to studying the behavior of families $\mathcal{B}$ of open balls off the origin in a Banach space X whose union contains the unit sphere of X. We call such a family $\mathcal{B}$ a *ball-covering* of X. This notion was first introduced in [3]. For a ball-covering $\mathcal{B} \equiv \{B(x_i, r_i)\}_{i\in I}$ of X, we denote by $\mathcal{B}^{\#}$ its cardinality and by $r(\mathcal{B})$ the least upper bound of the radius set $\{r_i\}_{i\in I}$, and we call it the *radius* of $\mathcal{B}$. We say that a ball-covering is *minimal* if its cardinality is the smallest of

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all cardinalities of ball-coverings. We call a given ball-covering $\mathcal{B}$ *α-off the origin* if $\inf\{\|x\| : x \in \bigcup \mathcal{B}\} \geq \alpha$.

Cheng [3] showed the following results. Let $\mathcal{B}_{\min} = \mathcal{B}_{\min}(X)$ be any minimal ball-covering of X. If $\dim X = n$, then $n + 1 \leq \mathcal{B}^{\#}_{\min} \leq 2n$; if, in addition, X is smooth, then $\mathcal{B}^{\#}_{\min} = n + 1$. For any Banach space X, by the separation theorem we can easily show that $\mathcal{B}^{\#}_{\min} = \mathbb{N}^{\#}$ implies that X^* is w^*-separable. Cheng, Cheng and Liu [4] proved that the converse is not true by putting different norms on l^∞. Recently, Cheng and Shi [5] further showed that every Banach space X with a w^*-separable dual admits a $1+\varepsilon$-equivalent norm such that X has a countable ball-covering with respect to the new norm.

This paper brings the following results.

THEOREM 1. *Suppose that X is an n-dimensional Banach space. Then $\mathcal{B}^{\#}_{\min}(X) = n + k$ for some $k \in \mathbb{N}$ if there exist k nontrivial subspaces X_j of X for $j = 1, \ldots, k$ such that*

(i) $X = \sum_{j=1}^{k} \oplus X_j$ *and* $\|x\| = \max_{1\leq j\leq k} \|x_j\|$ *for all* $x = \sum_{j=1}^{k} x_j$ *with* $x_j \in X_j$;
(ii) $\mathcal{B}^{\#}_{\min}(X_j) = \dim(X_j) + 1$ *for* $j = 1, \ldots, k$; *and in particular,*
(iii) $\mathcal{B}^{\#}_{\min}(X) = 2n$ *if and only if X is isometric to* $(\mathbb{R}^n, \|\cdot\|_\infty)$.

THEOREM 2. *Suppose that X is a Banach space. Then it is supperreflexive if and only if there exists an equivalent norm on X such that* (*with respect to the new norm*) *there are positive-valued functions $f, g : \mathbb{N} \to \mathbb{R}^+$ such that for every $n \in \mathbb{N}$ and every n-dimensional subspace Y, there is a minimal ball-covering $\mathcal{B}$ of Y satisfying*

(i) $\mathcal{B}^{\#} = n + 1$;
(ii) $r(\mathcal{B}) \leq f(n)$,
(iii) $\mathcal{B}$ *is $g(n)$-off the origin.*

THEOREM 3. *Suppose that X is an infinite-dimensional Banach space. Then there exists an equivalent norm on X and a closed subspace Y such that with respect to the new norm, $\mathcal{B}^{\#}_{\min}(X/Y) = \mathbb{N}^{\#}$.*

2. A characterization for existence of ball-coverings. In this paper, the letter X will always stand for a Banach space, and X^* for its dual. We denote by $B(x, r)$ (resp. $\overline{B}(x, r)$) the open (resp. closed) ball centered at x with radius r. B_X stands for the closed unit ball of X, and S_X for the sphere of B_X. $\mathcal{B}(X)$ (resp. $\mathcal{B}_{\min}(X)$) always represents a ball-covering (resp. minimal ball-covering) of X; we also write simply $\mathcal{B}$ (resp. $\mathcal{B}_{\min}$) for $\mathcal{B}(X)$ (resp. $\mathcal{B}_{\min}(X)$) if it does not lead to confusion. For any set A, $A^{\#}$ denotes the cardinality of A.

We recall some definitions which will be used in the following.

DEFINITION 2.1. Suppose that X is a Banach space.

(i) The *subdifferential mapping* $\partial\|\cdot\|$ of the norm: $X \to 2^{B_{X^*}}$ is defined by

$\partial\|x\| = \{x^* \in X^* : \langle x^*, x\rangle = \|x\| \text{ and } \langle x^*, z\rangle \leq \|z\| \text{ for all } z \in X\}.$

(ii) The norm $\|\cdot\|$ is said to be *Gateaux differentiable* at x if $\partial\|x\| \equiv \{x^*\}$ is a singleton; in this situation, x^* is called the *Gateaux derivative* of the norm at x, and this is equivalent to

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t} = \langle x^*, y\rangle \quad \text{for all } y \in X.$$

DEFINITION 2.2. Suppose that C is a nonempty closed convex set of a Banach space X.

(i) $x \in C$ is called an *exposed point* of C if there exists $x^* \in X^*$ such that

$$\langle x^*, x\rangle > \langle x^*, y\rangle \quad \text{for all } y \in C \text{ with } y \neq x.$$

(ii) If $C \subset X^*$, then $x \in C$ is called a w^*-*exposed point* of C if there exists $x^* \in X$ such that

$$\langle x^*, x\rangle > \langle x^*, y\rangle \quad \text{for all } y \in C \text{ with } y \neq x.$$

Clearly, a w^*-exposed point is an exposed point, and the two notions coincide if X is reflexive, in particular, if $\dim X < \infty$. We denote by $\exp C$ (resp. w^*-$\exp C$) the exposed (resp. w^*-exposed) point set of C.

DEFINITION 2.3. A Banach space X is called a *Gateaux differentiability* space (GDS) if every equivalent norm is densely Gateaux differentiable in X.

PROPOSITION 2.4 ([16]). *Suppose that X is a Banach space. Then the norm $\|\cdot\|$ is Gateaux differentiable at $x \in X$ with the Gateaux derivative $d\|x\| \equiv x^* \in X^*$ if and only if x^* is a w^*-exposed point of B_{X^*} and it is exposed by x.*

THEOREM 2.5 ([16]). *Suppose that X is a Banach space. Then X is a GDS if and only if every nonempty w^*-compact convex set in X^* is the w^*-closed convex hull of its w^*-exposed points.*

Now, we present the following result.

THEOREM 2.6. *Suppose that X is a Banach space, I is an index set with $I^{\#} = m$, and $\{x_i\}_{i\in I} \subset S_X$. Then $\mathcal{B} \equiv \{B(y_i, r_i)\}_{i\in I}$ forms a ball-covering of X for some $y_i \in \mathbb{R}^+ x_i$ with $\|y_i\| \geq r_i$ for all $i \in I$ if and only if for every selection ϕ of the subdifferential mapping $\partial\|\cdot\|$, $\{\phi(x_i)\}_{i\in I}$ positively separates points of X, that is, $\sup_{i\in I}\langle\phi(x_i), x\rangle > 0$ for every $x \neq 0$ in X.*

Proof. Sufficiency. We want to prove that there exist $\{y_i\}_{i\in I} \subset \bigcup_{i\in I} \mathbb{R}^+ x_i$ and $\{r_i\}_{i\in I} \subset \mathbb{R}^+$ with $\|y_i\| \geq r_i$ such that $S_X \subset \bigcup_{i\in I} B(y_i, r_i)$.

Let $B_{ij} = B(jx_i, j - j^{-1})$ for all $i \in I$ and $j \in \mathbb{N}$.

First, we claim that $S_X \subset G \equiv \bigcup_{i\in I,\, j\in\mathbb{N}} B_{ij}$. Suppose, to the contrary, that there exists $y \in S_X \setminus G$. Then

$$j - j^{-1} \leq \|jx_i - y\| \quad \text{for all } j \in \mathbb{N} \text{ and } i \in I.$$

For each fixed $i \in I$, let $t = j^{-1}$. We obtain

$$-t \leq \frac{\|x_i - ty\| - \|x_i\|}{t},$$

and this implies

$$d^+\|x_i\|(-y) \equiv \lim_{t\searrow 0} \frac{\|x_i - ty\| - \|x_i\|}{t} \geq 0.$$

Note that

$$d^+\|x_i\|(-y) = -d^-\|x_i\|(y) \equiv -\lim_{t\nearrow 0} \frac{\|x_i + ty\| - \|x_i\|}{t}$$

and

$$d^-\|x_i\|(y) = \min\{\langle x^*, y\rangle : x^* \in \partial\|x_i\|\}.$$

We know that there exists a selection ϕ of $\partial\|\cdot\|$ such that

$$\langle \phi(x_i), y\rangle \leq 0 \quad \text{for all } i \in I;$$

this contradicts the hypothesis of sufficiency.

If $m \geq \mathbb{N}^\#$, the proof of sufficiency is finished, since $\{B_{ij} : j \in \mathbb{N},\, i \in I\}$ is the desired ball-covering.

If $m \in \mathbb{N}$, then again by the hypothesis of sufficiency, for each selection ϕ of $\partial\|\cdot\|$, the set $\{\phi(x_i)\}_{i\in I}$ of m vectors in S_{X^*} positively separates points of X. Therefore, $\dim X < \infty$. Since for every fixed $i \in I$, $B_{ij} \subset B_{i,j+1}$ for all $j \in \mathbb{N}$, compactness of S_X implies that there exists $k \in \mathbb{N}$ such that

$$S_X \subset \bigcup\{B_{ik} : i = 1, \ldots, m\}.$$

Now we complete the proof of sufficiency by letting

$$y_i = kx_i \quad \text{and} \quad r_i = k - k^{-1}, \qquad i = 1, \ldots, m.$$

Necessity. Suppose that $\{B_i\}_{i\in I}$ (with $B_i = B(y_i, r_i)$ and $\|x_i\| \geq r_i > 0$) is a ball-covering of X. Let ϕ be a selection of $\partial\|\cdot\|$ such that there exists $y \neq 0$ satisfying

$$\langle \phi(x_i), y\rangle \leq 0 \quad \text{for all } i \in I.$$

Let $z = y/\|y\|$. Then there exists $j \in I$ such that $z \in B_j$. Thus

$$\|x_j\| \geq r_j > \|x_j - z\| \geq \langle \phi(x_j), x_j - z\rangle \geq \|x_j\|.$$

This is a contradiction. ∎

PROPOSITION 2.7. *Suppose that X is a separable Banach space, and I is an index set with $I^{\#} = m$. If there exists a ball-covering of X consisting of m balls, then there is a ball-covering $\mathcal{B} = \{B(x_i, r_i) : i \in I\}$ of X such that $\{x_i\}_{i\in I}$ are Gateaux differentiability points of the norm.*

Proof. Our proof is divided into two cases: (i) $m = \infty$, and (ii) $m < \infty$.

(i) Since X is a separable, it is a GDS. The ball B_{X^*} is w^*-sequentially compact [9], and it is the w^*-(sequentially) closed convex hull of its w^*-exposed points $w^*\text{-}\exp B_{X^*}$. Let $\{B(y_i, s_i) : i \in I\}$ be a ball-covering of X. Then, by Theorem 2.6, for every selection ϕ of the subdifferential mapping $\partial\|\cdot\|$, $\{\phi(y_i)\}_{i\in I}$ positively separates points of X. For each fixed $i \in I$, there exists a sequence $\{y^*_{i,j} : j \in \mathbb{N}\}$ in $\mathrm{co}(w^*\text{-}\exp B_{X^*})$ such that $y^*_{i,j} \xrightarrow{w^*} \phi(y_i)$ $(j \to \infty)$. For each pair (i, j) in $I \times \mathbb{N}$, there are p $(\equiv p(i, j) \in \mathbb{N})$ w^*-exposed points $A_{i,j} \equiv \{y^*_{i,j,k}\}_{k=1}^p$ and p nonnegative numbers $\{\lambda_k\}_{k=1}^p$ with $\sum_{k=1}^p \lambda_k = 1$ such that $y^*_{i,j} = \sum_{k=1}^p \lambda_k y^*_{i,j,k}$. Let $A = \bigcup\{A_{i,j} : i \in I,\ j \in \mathbb{N}\}$. Clearly, $A^{\#} = I^{\#} = m$, since $I^{\#} \geq \mathbb{N}^{\#}$. It is also not difficult to check that $\sup\{\langle x^*, x\rangle : x^* \in A\} \geq \sup\{\langle \phi(y_i), x\rangle : i \in I\}$ for every $x \in X$. Thus A positively separates points of X.

For each $y^*_{i,j,k}$ in A, choose any Gateaux differentiability point $y_{i,j,k} \in S_X$ with Gateaux derivative $d\|y_{i,j,k}\| = y^*_{i,j,k}$, and let $E = \{y_{i,j,k} : i \in I,\ j \in \mathbb{N},\ 1 \leq k \leq p(i, j)\}$. Theorem 2.6 asserts that there exists a ball-covering $\mathcal{B}$ with $\mathcal{B}^{\#} = E^{\#} = m$ such that the center of each ball in $\mathcal{B}$ is a Gateaux differentiability point of the norm.

(ii) Assume that $\{B(y_i, s_i)\}_{i\in I}$ (with $I^{\#} = m < \infty$) is a ball-covering of X.

Since S_X is compact, $S_X \subset \bigcup_{i\in I} B(y_i, s_i)$ implies that there is $\varepsilon > 0$ such that $\bigcup_{i\in I} B(y_i, s_i) \supset S_X + \varepsilon B_X$. For each fixed $i \in I$, we can find a Gateaux differentiability point $x_i \in B(y_i, \varepsilon)$ with $\|x_i\| \geq \|y_i\|$ $(\geq s_i)$. Therefore $S_X \subset \bigcup_{i\in I} B(x_i, s_i)$, and so $\mathcal{B} \equiv \{B(x_i, s_i)\}_{i\in I}$ is again a ball-covering of X. ■

Recall that a nonempty bounded set $A \subset X^*$ is said to be a *norming set* of X if there exists $\alpha > 0$ such that $p(x) \equiv \sigma_A(x) \equiv \sup_{x^*\in A}\langle x^*, x\rangle \geq \alpha\|x\|$ for all $x \in X$. Now, we have

COROLLARY 2.8. *If $\dim X < \infty$, then $\mathcal{B}^{\#}_{\min} \leq m$ if and only if there exists a norming set of X consisting of m w^*-exposed points of B_{X^*}.*

Proof. This is a direct consequence of Theorems 2.6 and 2.7. ■

COROLLARY 2.9. *If X is isometric to l^n_∞, then $\mathcal{B}^{\#}_{\min} = 2n$.*

Proof. Let $X = l^n_\infty$. By Corollary 2.8, it suffices to note that the set of all exposed points $\exp B_{X^*}$ is just $\{\pm e_i\}_{i=1}^n$, and every norming set of X

consisting of exposed points of B_{X^*} is also $\{\pm e_i\}_{i=1}^n$, where e_i $(i = 1, \dots, n)$ denote the standard unit vectors in $\mathbb{R}^n$. ∎

3. Structure of n-dimensional spaces with $\mathcal{B}^{\#}_{\min} = n + k$. This section presents examples of those n-dimensional spaces X satisfying $\mathcal{B}^{\#}_{\min}(X) = n + k$ for every $1 \le k \le n$.

We first show the following lemma.

LEMMA 3.1. *Suppose that X is a Gateaux differentiability space and X_i $(i = 1, 2)$ are two closed subspaces which are again Gateaux differentiability spaces such that $X = X_1 \oplus X_2$. If for every $x^* \in w^*\text{-}\exp B_{X^*}$, either $x^*|_{X_1} = 0$ or $x^*|_{X_2} = 0$, then*

(i) $w^*\text{-}\exp B_{X^*} = w^*\text{-}\exp B_{X_1^*} \cup w^*\text{-}\exp B_{X_2^*}$;

(ii) $\|x\| = \max\{\|x_1\|, \|x_2\|\}$ *for all* $x = x_1 + x_2$ *with* $x_1 \in X_1$ *and* $x_2 \in X_2$;

(iii) $\mathcal{B}^{\#}_{\min}(X) = \mathcal{B}^{\#}_{\min}(X_1) + \mathcal{B}^{\#}_{\min}(X_2)$.

Proof. (i) It is easy to see that this is true by definition of w^*-exposed points.

(ii) Note that $w^*\text{-}\exp B_{X^*}$ is always an exact norming set of X if X is a Gateaux differentiability space. By (i), for every $x = x_1 + x_2$ with $x_i \in X_i$ $(i = 1, 2)$,

$$\begin{aligned}\|x\| &= \sup\{\langle x^*, x\rangle : x^* \in w^*\text{-}\exp B_{X^*}\}\\ &= \max\{\langle x_i^*, x\rangle : x_i^* \in w^*\text{-}\exp B_{X_i^*},\ i = 1, 2\}\\ &= \max\{\langle x_i^*, x_i\rangle : x_i^* \in w^*\text{-}\exp B_{X_i^*},\ i = 1, 2\}\\ &= \max\{\|x_i\| : i = 1, 2\}.\end{aligned}$$

(iii) Assume $\mathcal{B}^{\#}_{\min}(X) = m$. Then, by Theorem 2.7, there exists a subset A of w^*-exposed points of B_{X^*} with $A^{\#} = m$ such that A positively separates points of X. Thus, A positively separates points of both X_1 and X_2. Let $A_i = \{x^* \in A : x^*|_{X_i \neq 0}\}$, $i = 1, 2$. By hypothesis, $A_1 \cap A_2 = \emptyset$ and $A = A_1 \cup A_2$. Therefore, A_i positively separates points of X_i and $A_i \subset w^*\text{-}\exp B_{X_i^*}$ for $i = 1, 2$. This says that $\mathcal{B}^{\#}_{\min}(X_i) \le A_i^{\#}$ for $i = 1, 2$, and

$$\mathcal{B}^{\#}_{\min}(X) = m = A_1^{\#} + A_2^{\#} \ge \mathcal{B}^{\#}_{\min}(X_1) + \mathcal{B}^{\#}_{\min}(X_2).$$

Conversely, suppose that $\mathcal{B}^{\#}_{\min}(X_i) = m_i$ for $i = 1, 2$, and that $A_i \subset w^*\text{-}\exp B_{X_i^*}$ with $A_i^{\#} = m_i$ are such that A_i positively separates points of X_i for $i = 1, 2$. Then $A_1 \cup A_2 (\equiv A \subset w^*\text{-}\exp B_{X^*})$ positively separates points of X, which implies that

$$\mathcal{B}^{\#}_{\min}(X) \le A^{\#} = A_1^{\#} + A_2^{\#} = \mathcal{B}^{\#}_{\min}(X_1) + \mathcal{B}^{\#}_{\min}(X_2). \ ∎$$

THEOREM 3.2. *Let $n \in \mathbb{N}$. Then for any $n+1 \le k \le 2n$, there is a space of dimension n admitting a minimal ball-covering of k balls.*

Proof. Let $X = \mathbb{R}^n, X_1 = \sum_{i=1}^m \oplus \mathbb{R}e_i$ and $X_2 = \sum_{i=m+1}^n \oplus \mathbb{R}e_i$, where $m = k-(n+1)$. Put $\|\cdot\|$ on X by

$$\|x\| = \max\{\|x_1\|_\infty, \|x_2\|_2\} \quad \text{for } x = x_1 + x_2 \in X_1 \oplus X_2$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Then

$$\exp B_{X^*} = \exp B_{X_1^*} \cup \exp B_{X_2^*} = \{\pm e_i\}_{i=1}^m \cup \{z \in X_2 : \|z\|_2 = 1\}.$$

By Lemma 3.1,

$$\mathcal{B}^{\#}_{\min}(X) = \mathcal{B}^{\#}_{\min}(X_1) + \mathcal{B}^{\#}_{\min}(X_2) = 2m + [(n-m)+1] = k. \ \blacksquare$$

By Lemma 3.1, more generally, we have

THEOREM 3.3. *Suppose that X is an n-dimensional space, and let $1 \le m \le n$. Then $\mathcal{B}_{\min}(X) = n+m$ if there exist m positive integers n_j and m subspaces X_j of X $(j = 1, \ldots, m)$ such that*

(i) $\sum_{j=1}^m n_j = n$;
(ii) $\sum_{j=1}^m \oplus X_j = X$;
(iii) $\mathcal{B}^{\#}_{\min}(X_j) = n_j + 1$ *for* $j = 1, \ldots, m$;
(iv) $\|x\| = \max_{1 \le j \le m} \|x_j\|$, *for* $X = \sum_{i=1}^n x_j$ *with* $x_j \in X_j$.

4. A characterization of superreflexive spaces. First, we recall some definitions.

DEFINITION 4.1 ([11]). A Banach space X is called *uniformly nonsquare* if l_∞^2 cannot be represented in X, that is, there exists $\varepsilon > 0$ such that for every two-dimensional subspace X_2 of X, if $T : X_2 \to l_\infty^2$ is a linear isomorphism, then $\|T\|\,\|T^{-1}\| \ge 1+\varepsilon$.

DEFINITION 4.2. Suppose that X is a Banach space and $\mathcal{B} \equiv \{B(x_i, r_i)\}$ is a ball-covering.

(i) The number $r(\mathcal{B}) \equiv \sup_{i \ge 1} r_i$ is called the *radius* of $\mathcal{B}$.
(ii) We say that $\mathcal{B}$ is *α-off the origin* if $\inf_i\{\|x_i\| - r_i\} \ge \alpha$.

THEOREM 4.3. *Suppose that X is a Banach space. If there exist two constants $\beta, \alpha > 0$ such that for every two-dimensional subspace Y of X, there exists a ball-covering $\mathcal{B}$ of Y with $\mathcal{B}^{\#} = 3$ which is α-off the origin and $r(\mathcal{B}) \le \beta$, then X is uniformly nonsquare.*

Proof. Let $0 < \delta < \alpha/\beta$ and assume that there are a two-dimensional subspace $Y \subset X$ and an isomorphism $T : Y \to l_\infty^2$ with $\|T\| = 1$ and $\|T^{-1}\| < 1+\delta$. If $V = T^{-1}(B_{l_\infty^2})$ then

$$B_Y \subset V \subset (1+\delta)B_Y. \tag{1}$$

Clearly, V as a unit ball generates on Y the l_∞^2-norm. By the assumption of the theorem,

$$S_Y \subset \bigcup_{i=1}^{3}(x_i + r_i B_Y), \qquad \|x_i\| - r_i \geq \alpha,\ r_i \leq \beta,\ i = 1,2,3.$$

It follows from the left inclusion in (1) that

$$S_Y \subset \bigcup_{i=1}^{3}(x_i + r_i V). \tag{2}$$

Next we write

$$\|x_i\| - (1+\delta)r_i = \|x_i\| - r_i - \delta_i \geq \alpha - \delta\beta > 0,$$

which together with the right inclusion in (1) gives $0 \notin x_i + r_i V$, $i = 1,2,3$, contradicting (2) (for any three l_∞-balls on the plane which do not contain the origin there is a ray starting from the origin that does not meet these balls). ∎

DEFINITION 4.4. Suppose that X is a Banach space.

(i) The *modulus of smoothness* of X, $\varrho_X : \mathbb{R}^+ \to \mathbb{R}^+$, is defined by

$$\varrho_X(\tau) = \sup\left\{\left\|\frac{x+y}{2}\right\| + \left\|\frac{x-y}{2}\right\| - 1 : \|x\| \leq 1,\ \|y\| \leq \tau\right\}.$$

(ii) The space X is called *uniformly smooth* if

$$\varrho_X(\tau)/\tau \to 0 \quad \text{as } \tau \to 0^+.$$

Clearly, a finite-dimensional space X is uniformly smooth if its norm is everywhere Gateaux differentiable off the origin, i.e., X is (Gateaux) smooth.

LEMMA 4.5. *Suppose that X is an n-dimensional smooth Banach space. Then there exist $n+1$ exposed points $\{x_i^*\}_{i=0}^n$ of B_{X^*} such that*

$$\max_{0\leq i\leq n}\langle x_i^*, x\rangle \geq \frac{1}{3n}\|x\| \quad \textit{for every } x \in X.$$

Proof. Note that, in this case, w^*-$\exp B_{X^*} = \exp B_{X^*} = S_{X^*}$. By the Auerbach Theorem (see, for instance, [12, p. 16]), there exist $\{x_i\}_{i=1}^n \subset S_X$ and $\{x_i^*\}_{i=1}^n \subset S_{X^*}$ such that

$$\langle x_i^*, x_j\rangle = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad i,j = 1,\ldots,n.$$

Let $x_0^* = -\|\sum_{i=1}^n x_i^*\|^{-1}\sum_{i=1}^n x_i^*$. Then $\{x_i^*\}_{i=0}^n$ are $n+1$ exposed points of B_{X^*}. We want to show

$$\max_{0\leq i\leq n}\langle x_i^*, x\rangle \geq \frac{1}{3n}\|x\| \quad \text{for all } x \in X.$$

Without loss of generality, we assume that $n \geq 2$ and $\|x\| = 1$. Let $x = \sum_{i=1}^{n} \alpha_i x_i$. If $\max_{1 \leq i \leq n} \langle x_i^*, x \rangle = \max_{1 \leq i \leq n} \alpha_i \geq 1/3n$, then we are done. Otherwise, let $I_x^+ = \{i : \alpha_i > 0\}$. Then

$$\begin{aligned}\langle x_0^*, x \rangle &= -\Big\|\sum_{i=1}^{n} x_i^*\Big\|^{-1} \sum_{i-1}^{n} \alpha_i = \Big\|\sum_{i=1}^{n} x_i^*\Big\|^{-1} \Big[\sum_{i=1}^{n} |\alpha_i| - 2 \sum_{i \in I_x^+} \alpha_i\Big] \\ &\geq \Big\|\sum_{i=1}^{n} x_i^*\Big\|^{-1} \Big[1 - \frac{2}{3}\Big] \geq \frac{1}{3n}. \ \blacksquare\end{aligned}$$

THEOREM 4.6. *Suppose that X is a uniformly smooth space. Then there exist functions $\alpha, \beta : \mathbb{N} \to \mathbb{R}^+$ such that for every $n \in \mathbb{N}$ and every n-dimensional subspace Y of X, there exists a ball-covering of Y satisfying*

(i) $\mathcal{B}^{\#} = n + 1$,
(ii) $r(\mathcal{B}) \leq \beta(n)$,
(iii) $\mathcal{B}$ *is $\alpha(n)$-off the origin.*

Proof. Let ϱ_X be the modulus of smoothness of X. Since $\varrho_X(\tau)/\tau \to 0$ as $\tau \to 0^+$, for each fixed $n \in \mathbb{N}$ we can choose $j \in \mathbb{N}$ such that

$$\varrho_X\Big(\frac{2}{j-1}\Big) < \frac{1}{j-1}\Big(\frac{1}{3n} - \frac{1}{j}\Big).$$

Let $\alpha(n) = 1/j$ and $\beta(n) = j$. We claim that the functions $\alpha, \beta : \mathbb{N} \to \mathbb{R}^+$ have the desired properties.

Let Y be an n-dimensional subspace of X, and let $\{y_j\}_{j=0}^{n} \subset S_Y$ and $\{y_i^*\}_{i=1}^{n} \subset S_{Y^*}$ satisfy

(a) $\langle y_i^*, y_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$;
(b) $\max_{0 \leq i \leq n} \langle y_i^*, y \rangle \geq (1/3n)\|y\|$ for all $y \in Y$.

Then each $B_i \equiv B(jy_i, j - j^{-1})$ is $1/j$-off the origin of Y and its radius satisfies $j - j^{-1} < j = \beta(n)$. It remains to show that $\mathcal{B} \equiv \{B_i\}_{i=0}^{n}$ is a ball-covering of Y.

For each $y \in S_Y$, choose y_i^* such that $\langle y_i^*, y \rangle \equiv \gamma \geq 1/3n$. Then there exists $h \in Y$ with $\langle y_i^*, h \rangle = 0$ and $\|h\| \leq 2$ such that $y = \gamma y_i + h$. We assert that $y \in B_i$. Otherwise,

$$j - j^{-1} \leq \|jy_i - y\| = \|(j - \gamma)y_i - h\|.$$

Let $\tau = 1/(j - \gamma)$. Then we know

$$\frac{\|y_i - \tau h\| - \|y_i\|}{\tau} - \gamma \geq -j^{-1}.$$

Note $\|y_i + th\| \geq \langle y_i^*, y_i + th \rangle = \|y_i\| = 1$ for all $t \in \mathbb{R}$.

Since $\varrho_X(\tau)/\tau$ is nondecreasing in $\tau \in \mathbb{R}^+$,

$$\frac{\varrho_X\left(\frac{2}{j-1}\right)}{1/(j-1)} \geq \frac{\varrho_X(2\tau)}{\tau} \geq 2\,\frac{\|y_i + \tau h\| + \|y_i - \tau h\| - 2}{\tau\|h\|}$$
$$\geq 2\,\frac{\|y_1 - \tau h\| - 1}{\tau\|h\|} \geq \frac{\|y_i - \tau h\| - 1}{\tau} \geq \gamma - j^{-1}.$$

Therefore

$$\varrho_X\left(\frac{2}{j-1}\right) \geq (j-1)^{-1}(\gamma - j^{-1}) \geq (j-1)^{-1}\left(\frac{1}{3n} - \frac{1}{j}\right).$$

This is a contradiction. ■

THEOREM 4.7. *Suppose that X is a Banach space. Then it is superreflexive if and only if there exist an equivalent norm $|\cdot|$ on X and two positive-valued functions $\alpha, \beta : \mathbb{N} \to \mathbb{R}^+$ such that, with respect to $|\cdot|$, for every $n \in \mathbb{N}$ and every n-dimensional subspace Y of X, there is a ball-covering $\mathcal{B}$ of Y satisfying*

(i) $\mathcal{B}^{\#} = n + 1$;
(ii) $r(\mathcal{B}) \leq \beta(n)$;
(iii) *$\mathcal{B}$ is $\alpha(n)$-off the origin.*

Proof. Sufficiency is an immediate consequence of Theorem 4.3, since a uniformly nonsquare space is necessarily superreflexive [11]. Necessity is contained in Theorem 4.6, since every superreflexive space is uniformly smoothable [8]. ■

5. Ball-covering property of quotient spaces. In this section, we show that every infinite dimensional Banach space can be renormed so that it has an infinite-dimensional quotient space admitting the ball-covering property.

DEFINITION 5.1. We say a Banach space X has the *ball-covering property* if S_X can be covered by a sequence of balls off the origin.

DEFINITION 5.2. Suppose that X is a Banach space.

(i) A pair of sequences $\{x_n\}_{n=1}^m$ in X and $\{x_n^*\}_{n=1}^m$ ($m \in \mathbb{N} \cup \{\mathbb{N}^{\#}\}$) is called a *biorthogonal system* if $\langle x_j^*, x_i\rangle = \delta_{ij}$ for all $i, j \in \mathbb{N}$ with $i, j \leq m$.
(ii) For a biorthogonal system $\{(x_i, x_i^*)\}_{i=1}^m \subset X \times X^*$, the constant $k = \sup_i \|x_i^*\|\,\|x_i\|$ ($\leq \infty$) is said to be the *system constant.*
(iii) A biorthogonal system $\{(x_i, x_i^*)\}_{i=1}^m$ is called *normalized* if $\|x_i\| = 1$ for all $i \in \mathbb{N}$ with $i \leq m$.

LEMMA 5.3. *Suppose that X is a Banach space, $\varepsilon > 0$, and $\{(x_i, x_i^*)\}_{i=1}^m$ is a normalized biorthogonal system with the system constant $K \le 1+\varepsilon$. Then there exists an equivalent norm $|\cdot|$ on X such that*

(i) $(1+\varepsilon)^{-1}\|x\| \le |x| \le (1+\varepsilon)\|x\|$ *for all $x \in X$;*
(ii) $|x_i| = 1 = |x_j^*|$ *for all $i, j \in \mathbb{N}$ with $i, j \le m$;*
(iii) $|\cdot|$ *is Fréchet differentiable at $\{\pm x_i\}_{i=1}^m$ with the Fréchet derivatives $|\pm x_i|' = \pm x_i^*$;*
(iv) $|y| = (1+\varepsilon)^{-1}\|y\|$ *for all $y \in [\{x_j^*\}_{j=1}^m]^\top \equiv Y$, where $[\{x_j^*\}_{j=1}^m]^\top = \{x \in X : \langle x_j^*, x\rangle = 0$ for $j = 1, \ldots, m\}$.*

Proof. Let C^* be the w^*-closed convex hull of $\{\pm x_j^*\}_{j=1}^m$, and let $D^* = (1+\varepsilon)^{-1}B_{X^*}$ (where B_{X^*} denotes the closed unit ball of X^* with respect to the original dual norm $\|\cdot\|$). Put $B^* = \overline{\mathrm{co}}^{w^*}(C^* \cup D^*)$ and define $|\cdot| : X \to \mathbb{R}$ by

$$|x| = \sup_{x^* \in B^*} \langle x^*, x\rangle \quad \text{for all } x \in X.$$

It is not difficult to see that both (i) and (ii) hold.

To show (iii), by Proposition 2.4, it suffices to prove that for each $j \in \mathbb{N}$ with $j \le m$, x_j^* is a w^*-strongly exposed point of B^* and w^*-strongly exposed by x_j. Note $\overline{\mathrm{co}}^{w^*}(C^* \cup D^*) = \mathrm{co}(C^* \cup D^*)$. Let $z_n^* = \lambda_n c_n^* + (1-\lambda_n)d_n^* \in B^*$ be such that $\langle z_n^*, x_j\rangle \to \langle x_j^*, x_j\rangle = 1$. Then

$$\begin{aligned}\langle z_n^*, x_j\rangle &= \lambda_n\langle c_n^*, x_j\rangle + (1-\lambda_n)\langle d_n^*, x_j\rangle \le \lambda_n\langle c_n^*, x_j\rangle + (1-\lambda_n)(1+\varepsilon)^{-1}\\ &\le \lambda_n + (1-\lambda_n)(1+\varepsilon)^{-1}.\end{aligned}$$

Therefore $\lambda_n \to 1$ and $\langle c_n^*, x_j\rangle \to 1$.

Next, put $C_k^* = \overline{\mathrm{co}}^{w^*}\{\pm x_i^*\}_{i\ne k}$. Then

$$C^* = \mathrm{co}(C_j^* \cup \{\pm x_j^*\}) \quad \text{and} \quad C_j^* \subseteq \{x_j\}^\perp.$$

For each $n \in \mathbb{N}$, let

$$c_n^* = \alpha_{n,1}c_{n,j}^* + \alpha_{n,2}x_j^* + \alpha_{n,3}(-x_j^*)$$

where $\alpha_{n,i} \ge 0$ $(i = 1, 2, 3)$ with $\sum_{i=1}^3 \alpha_{n,i} = 1$ and $c_{n,k}^* \in C_j^*$. So we have

$$\langle c_n^*, x_j\rangle = \alpha_{n,2} - \alpha_{n,3} \to 1,$$

which implies that $\alpha_{n,1} \to 0$, $\alpha_{n,2} \to 1$ and $\alpha_{n,3} \to 0$ as $n \to \infty$. Thus, we have proved $z_n^* \to x_j^*$ and so x_j^* is a w^*-strongly exposed point, and it is w^*-strongly exposed by x_j.

(iv) is clear since for all $y \in Y = [\{x_j^*\}_{j=1}^m]^\top$,

$$|y| = \max\{\sup_{x^* \in C^*}\langle x^*, y\rangle, \sup_{x^* \in D^*}\langle x^*, y\rangle\} = \sup_{x^* \in D^*}\langle x^*, y\rangle = (1+\varepsilon)^{-1}\|y\|. \ \blacksquare$$

Theorem 5.4. *Suppose that X is an infinite-dimensional Banach space. Then for every $\varepsilon > 0$ there exist an equivalent norm $|\cdot|$ on X and a closed subspace Y of X with $\dim X/Y = \infty$ such that*

(i) $(1+\varepsilon)^{-1}\|x\| \le |x| \le (1+\varepsilon)\|x\|$ *for all* $x \in X$;
(ii) X/Y *has the ball-covering property with respect to* $|\cdot|$.

Proof. Without loss of generality we assume that X is nonseparable; otherwise, X itself has the ball-covering property. Fix any separable infinite-dimensional closed subspace $X_0 \subset X$, and for every $0 < \varepsilon < 1$, applying a theorem of Pełczyński [14] to X_0, we find that there exists a normalized biorthogonal system $\{(x_i, x_i^*)\}_{i=1}^\infty$ in $X \times X^*$ such that

(a) $\sup_{j\in\mathbb{N}} \|x_j^*\| \le 1 + \varepsilon$;
(b) $\operatorname{span}\{x_i\}_{i=1}^\infty$ is dense in X_0;
(c) $\{x_j^*\}_{j=1}^\infty$ separates points of X_0.

By Lemma 5.3, there exists an equivalent norm $|\cdot|$ on X such that

(d) $(1+\varepsilon)^{-1}\|x\| \le |x| \le (1+\varepsilon)\|x\|$ for all $x \in X$;
(e) $|x_i| = |x_j^*| = 1$ for all $i, j \in \mathbb{N}$;
(f) $|\cdot|$ is Fréchet differentiable at $\{\pm x_i\}$ with $|\pm x_i|' = \pm x_i^*$ for all $i \in \mathbb{N}$;
(g) $|y| = (1+\varepsilon)^{-1}\|y\|$ for all $y \in Y \equiv [\{x_i^*\}_{i=1}^\infty]^\top$.

Clearly, $\dim X/Y = \infty$. We claim that the quotient space X/Y has the ball-covering property with respect to $|\cdot|$.

For every $x \in X$, we write $\overline{x} = x + Y \in X/Y$. It is easy to check that $|\overline{x}_i| = |x_i^*| = 1$ for all $i \in \mathbb{N}$, and all $\{\pm x_j^*\}_{j=1}^\infty$ are w^*-strongly exposed points of the unit ball $B_{(X/Y)^*} = B_{Y^\perp}$ of $(X/Y)^* = Y^\perp$, and they are w^*-strongly exposed by $\{\pm\overline{x}_j\}_{j=1}^\infty$, respectively. This implies that $\{\pm\overline{x}_j\}_{j=1}^\infty$ are Fréchet differentiability points of the quotient norm $|\cdot|$ and with $|\pm\overline{x}_j|' = \pm x_j^*$ for all $j \in \mathbb{N}$. Since $\{\pm x_j^*\}_{j=1}^\infty$ positively separates points of X/Y and since every selection ϕ of the subdifferential mapping of the quotient norm $|\cdot|$ satisfies $\phi(\pm x_i) = \pm x_i^*$ for all $i \in \mathbb{N}$, we complete the proof by Theorem 2.6.

Proposition 5.5. *A Banach space X admits an infinite-dimensional separable quotient space if and only if it can be renormed so that, with respect to the new norm, it admits an infinite-dimensional quotient space whose unit sphere has a countable ball-covering $\mathcal{B}$ with $r(\mathcal{B}) < 1$.*

Proof. This can be directly obtained from Theorem 3.1 of [3], where we prove that every Banach space admitting a countable ball-covering with radii at most $r < 1$ is separable. ∎

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