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Weighted variable L^p integral inequalities for the maximal operator on non-increasing functions

by

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Abstract. Let B_p be the Ariño–Muckenhoupt weight class which controls the weighted L^p-norm inequalities for the Hardy operator on non-increasing functions. We replace the constant p by a function $p(x)$ and examine the associated $L^{p(x)}$-norm inequalities of the Hardy operator.

1. Introduction. The weights $w : \mathbb{R}_+ \to \mathbb{R}_+$ for which the Hardy operator

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

on non-negative non-increasing functions f (we write simply $f\downarrow$) is bounded:

$$\int_0^\infty Hf(x)^p w(x)\,dx \le c_* \int_0^\infty f(x)^p w(x)\,dx, \qquad 1 \le p < \infty, \tag{1}$$

have been characterized by Ariño and Muckenhoupt [1] by the condition

$$w \in B_p : \quad \int_r^\infty \left(\frac{r}{x}\right)^p w(x)\,dx \le c\int_0^r w(x)\,dx. \tag{2}$$

A different proof of (1)$\Leftrightarrow$(2) was given by me in [7] where it is also apparent that in the implication (2)$\Rightarrow$(1) the constant c_* can be taken to be $(c+1)^p$. For (1)$\Rightarrow$(2) one uses the test function $f = \chi_{[0,r]}$ and (2) follows with $c = c_*$. We also note that for $f\downarrow$, $Hf(x)$ equals $Mf(x)$, the Hardy–Littlewood maximal function.

In the past few years a great deal of attention has been paid to the problem of the boundedness of M on variable L^p-spaces. If $p : \mathbb{R}^n \to [1,\infty)$ and $w : \mathbb{R}^n \to \mathbb{R}_+$, let $L^{p(x)}(w)$ be the collection of all functions $f : \mathbb{R}^n \to \mathbb{R}$

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such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} w(x)\, dx < \infty,$$

equipped with the Luxemburg norm

$$\|f\|_{p(x),w} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} w(x)\, dx \le 1\right\}.$$

This makes $L^{p(x)}(w)$ into a Banach space; for the properties of these spaces see [5]. Cruz-Uribe, Fiorenza, and myself have shown in [3] that for $w \equiv 1$,

$$\|Mf\|_{p(x)} \le c\|f\|_{p(x)} \tag{3}$$

provided $1 < p_* \le p(x) < \infty$, and

$$|p(x) - p(y)| \le \begin{cases} \dfrac{c}{\log \frac{1}{|x-y|}}, & |x-y| \le 1/2, \\ \dfrac{c}{\log(e + |x|)}, & |y| \ge |x|, \end{cases}$$

and that the condition on $p(x)$ is nearly sharp (see [3] for further details and additional references).

However, a characterization of the weights $w : \mathbb{R}^n \to \mathbb{R}_+$ so that

$$\|Mf\|_{p(x),w} \le c\|f\|_{p(x),w} \tag{B}$$

is not known. Some necessary and some sufficient conditions are contained in a forthcoming paper [4]. We are therefore led to the "easier" problem of characterizing (B) for $f\downarrow$ since from (2) the natural condition appears to be

$$w \in B_{p(x)} : \quad \int_r^\infty \left(\frac{r}{x}\right)^{p(x)} w(x)\, dx \le c\int_0^r w(x)\, dx. \tag{C}$$

The primary purpose of this paper is to establish for certain $p : \mathbb{R}_+ \to [1, \infty)$ a connection between (B) and (C), and the related integral inequality

$$\int_0^\infty Mf(x)^{p(x)} w(x)\, dx \le c \int_0^\infty f(x)^{p(x)} w(x)\, dx, \qquad f\downarrow. \tag{A}$$

Remark. If the hypothesis $f\downarrow$ is omitted in (A) and $0 < p(x) < p_+ < \infty$, then $p(x)$ is constant. This surprising result is due to A. K. Lerner [6] for $w \equiv 1$. The same proof, with only minor changes, works for positive $w(x)$. A related result is contained in [2] where a variable exponent $B_{p(x)}$ is introduced. It is the same as (C) except for an additional parameter $s > 0$:

$$\int_r^\infty \left(\frac{r}{sx}\right)^{p(x)} w(x)\, dx \le c\int_0^r \frac{w(x)}{s^{p(x)}}\, dx.$$

The main result is that this condition is equivalent to (A) and to $p(x) = p_0$, a constant, if the oscillation of $p(x)$ at $x = 0$ is zero, and then $w \in B_{p_0}$.

It turns out that there is a relationship between (A), (B), and (C) under some natural restrictions which are illustrated by the following examples.

(1) Let $p(x) = 4\chi_{[0,1]}(x) + 2\chi_{[1,\infty)}(x)$. Then $w(x) \equiv 1$ is in $B_{p(x)}$. Let $f_\alpha = \alpha\chi_{[0,1]}$. Then

$$\int_0^\infty f_\alpha(x)^{p(x)}\,dx = \alpha^4 \quad \text{and} \quad \int_0^\infty Hf_\alpha(x)^{p(x)}\,dx = \alpha^4 + \alpha^2,$$

and (A) cannot hold as $\alpha \to 0$. This explains the restriction that $p(x)$ be non-decreasing (written $p\uparrow$).

(2) Let now $p(x) = 2\chi_{[0,1]}(x) + 4\chi_{[1,\infty)}(x)$. Again $w(x) \equiv 1$ is in $B_{p(x)}$. If $f_N = N\chi_{[0,1]}$, then

$$\int_0^\infty f(x)^{p(x)}\,dx = N^2 \quad \text{and} \quad \int_0^\infty Hf_N(x)^{p(x)}\,dx = N^2 + N^4/3,$$

and (A) cannot hold as $N \to \infty$. This shows that in addition to $f\downarrow$ we must assume that $0 \le f(x) \le 1$.

2. The inequality (A). Let w be a weight: $w \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and non-negative, and let $p : \mathbb{R}_+ \to [1,\infty)$.

LEMMA 1. *$w \in B_{p(x)}$ if and only if there exists $0 < c < \infty$ such that for every $r\downarrow$,*

$$\int_0^\infty \chi^{r(x)}(x)\left(\frac{r(x)}{x}\right)^{p(x)} w(x)\,dx \le c\int_0^\infty \chi_{r(x)}(x)w(x)\,dx,$$

where for $a > 0$, $\chi_a(x) = \chi_{[0,a]}(x)$ and $\chi^a(x) = \chi_{[a,\infty)}(x)$.

Proof. We only have to show that $w \in B_{p(x)}$ implies the condition with $r\downarrow$, since the reverse follows by taking $r(x) = r$.

Since $y = r(x)$ is non-increasing and $y = x$ is increasing there is a unique point i_r such that

$$(r(x) - x)(i_r - x) > 0, \qquad x \ne i_r.$$

In fact, $i_r = \sup\{x : r(x) > x\} = \inf\{x : r(x) < x\}$.

The right side is

$$R = \int_{\{x\,:\,x<r(x)\}} w(x)\,dx = \int_0^{i_r} w(x)\,dx,$$

and the left side is

$$L = \int\limits_{\{x\,:\,r(x)<x\}} \left(\frac{r(x)}{x}\right)^{p(x)} w(x)\,dx \le \int\limits_{i_r}^{\infty} \left(\frac{i_r}{x}\right)^{p(x)} w(x)\,dx,$$

since for $x \ge t > i_r$ we have $r(x) \le r(t) \le t$ and thus $r(x) \le i_r$. ∎

Let $\mathcal{D}$ be the collection of all $f\downarrow$ with $f(0+) \le 1$, and let

$$Hf(x) = \frac{1}{x}\int\limits_0^x f(t)\,dt = Mf(x)$$

be the Hardy operator for $f \in \mathcal{D}$. Then H maps $\mathcal{D}$ into $\mathcal{D}$.

THEOREM 2. *Let $p : \mathbb{R}_+ \to [1,\infty)$ and $p\uparrow$. Then there exists a constant $0 < c < \infty$ such that*

$$\int\limits_0^\infty Hf(x)^{p(x)} w(x)\,dx \le c\int\limits_0^\infty f(x)Hf(x)^{p(x)-1} w(x)\,dx$$

for every $f \in \mathcal{D}$ if and only if $w \in B_{p(x)}$.

Proof. The choice $f = \chi_r$ gives one implication, and for the reverse direction we only need to prove the integral inequality for functions in $\mathcal{D}$ supported in $[0,K]$, continuous and strictly decreasing on $[0,K]$, with a constant c depending only upon the $B_{p(x)}$-constant of w. An arbitrary $f \in \mathcal{D}$ can be approximated by such functions so that the integral inequality is obtained as a limit.

Let $r : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $t = r(x,y)$, be decreasing in x for each y and continuous and strictly decreasing in y for each x. For a fixed x we denote by $r^{-1}(x,t)$ the inverse of $t = r(x,y)$, i.e. $t = r(x, r^{-1}(x,t))$. Then $r^{-1}(x,t)$ is decreasing in x for each t and continuous and strictly decreasing in t for each x. Later we will choose

$$r^{-1}(x,t) = f(t)Hf(t)^{p(x)-1}.$$

From Lemma 1 for each $r(x,y)$ as above we have

$$\int\limits_0^\infty \chi^{r(x,y)}(x)\left(\frac{r(x,y)}{x}\right)^{p(x)} w(x)\,dx \le c\int\limits_0^\infty \chi_{r(x,y)}(x)w(x)\,dx.$$

We integrate this in y and get

$$\int\limits_0^\infty\int\limits_0^\infty \chi^{r(x,y)}(x)\left(\frac{r(x,y)}{x}\right)^{p(x)} w(x)\,dx\,dy \le c\int\limits_0^\infty\int\limits_0^\infty \chi_{r(x,y)}(x)w(x)\,dx\,dy.$$

We interchange the order of integration and then the left side equals

$$L = \int\limits_0^\infty \int\limits_{\{y\,:\,r(x,y)\le x\}} r(x,y)^{p(x)}\,dy\,\frac{w(x)}{x^{p(x)}}\,dx.$$

If $r^{-1}(x,x) = i_r(x)$, then $\{y : r(x,y) \le x\} = [i_r(x), \infty)$. Thus

$$L = \int_0^\infty \int_{i_r(x)}^\infty r(x,y)^{p(x)}\,dy\,\frac{w(x)}{x^{p(x)}}\,dx.$$

The inner integral is

$$\int_{i_r(x)}^\infty r(x,y)^{p(x)}\,dy = \int_0^{x^{p(x)}} r^{-1}(x, t^{1/p(x)})\,dt - x^{p(x)} i_r(x).$$

The substitution $t = u^{p(x)}$ gives

$$\int_{i_r(x)}^\infty r(x,y)^{p(x)}\,dy = \int_0^x r^{-1}(x,u)p(x)u^{p(x)-1}\,du - x^{p(x)} i_r(x).$$

Now we choose $r^{-1}(x,u) = f(u)Hf(u)^{p(x)-1}$. Then

$$\begin{aligned}\int_{i_r(x)}^\infty r(x,y)^{p(x)}\,dy &= \int_0^x f(u)\Big(\int_0^u f(\tau)\,d\tau\Big)^{p(x)-1} p(x)\,du - x^{p(x)} i_r(x)\\ &= \Big(\int_0^x f(t)\,dt\Big)^{p(x)} - x^{p(x)} i_r(x).\end{aligned}$$

Hence

$$L = \int_0^\infty Hf(x)^{p(x)} w(x)\,dx - \int_0^\infty i_r(x)w(x)\,dx.$$

The right side is

$$R = c\int_0^\infty \int_{\{y\,:\,r(x,y)\ge x\}} w(x)\,dy\,dx = c\int_0^\infty i_r(x)w(x)\,dx.$$

We combine the above estimates and get

$$\int_0^\infty Hf(x)^{p(x)} w(x)\,dx \le (c+1)\int_0^\infty i_r(x)w(x)\,dx.$$

The proof is completed now by noting that

$$i_r(x) = r^{-1}(x,x) = f(x)Hf(x)^{p(x)-1}.$$

Note moreover that, if c_1 equals the $B_{p(x)}$-constant of w, then the constant c of the integral inequality is at most $c_1 + 1$. ∎

THEOREM 3. *Let $p : \mathbb{R}_+ \to [1,\infty)$, $p\uparrow$, and $1 \le p(x) \le p^* < \infty$. Then there is a constant $0 < c_* < \infty$ such that*

$$\int_0^\infty Hf(x)^{p(x)} w(x)\,dx \le c_* \int_0^\infty f(x)^{p(x)} w(x)\,dx, \qquad f \in \mathcal{D},$$

if and only if $w \in B_{p(x)}$.

Proof. The choice $f = \chi_r$ proves the necessity. For the sufficiency we first note that $w_N = w\chi_N$ is in $B_{p(x)}$ with the same constant and hence, by Theorem 2,

$$\int_0^\infty Hf(x)^{p(x)} w_N(x)\,dx \le c_0 \int_0^\infty f(x)^{p(x)} Hf(x)^{p(x)-1} w_N(x)\,dx, \qquad f \in \mathcal{D},$$

where $c_0 > 1$ does not depend on N. (Below, we need the integrals to be finite and that is the reason for the restriction to w_N). We now fix $\lambda_0 > c_0 > 1$. Then $f/\lambda_0 \in \mathcal{D}$ if $f \in \mathcal{D}$. Replace f by f/λ_0 in the above inequality and use Young's inequality to obtain

$$\begin{aligned}\int_0^\infty \left(\frac{Hf(x)}{\lambda_0}\right)^{p(x)} w_N(x)\,dx &\le \frac{c_0}{\lambda_0}\int_0^\infty f(x) H(f/\lambda_0)(x)^{p(x)-1} w_N(x)\,dx \\ &\le \frac{c_0}{\lambda_0}\int_0^\infty \left(\frac{f(x)^{p(x)}}{p(x)} + \frac{H(f/\lambda_0)(x)^{p(x)}}{q(x)}\right) w_N(x)\,dx \\ &\le \frac{c_0}{\lambda_0}\int_0^\infty f(x)^{p(x)} w_N(x)\,dx + \frac{c_0}{\lambda_0}\int_0^\infty \left(\frac{Hf(x)}{\lambda_0}\right)^{p(x)} w_N(x)\,dx,\end{aligned}$$

where $p(x)^{-1} + q(x)^{-1} = 1$. From this we get

$$(1 - c_0/\lambda_0)\int_0^\infty \left(\frac{Hf(x)}{\lambda_0}\right)^{p(x)} w_N(x)\,dx \le \frac{c_0}{\lambda_0}\int_0^\infty f(x)^{p(x)} w_N(x)\,dx,$$

and the left side is

$$\ge \frac{\lambda_0 - c_0}{\lambda_0^{p^*+1}}\int_0^\infty Hf(x)^{p(x)} w_N(x)\,dx.$$

Thus

$$\int_0^\infty Hf(x)^{p(x)} w_N(x)\,dx \le c_* \int_0^\infty f(x)^{p(x)} w_N(x)\,dx,$$

where $c_* = \lambda_0^{p_*} c_0/(\lambda_0 - c_0)$. Let $N \to \infty$ to complete the proof. ∎

REMARK. The constant c_* can be chosen to depend only on the $B_{p(x)}$-constant c of w: in fact, if $\lambda_0 = 2c_0$, then $c_* = (2c_0)^{p^*} = (2(c+1))^{p^*}$.

3. The inequality (B)

THEOREM 4. *Let $p : \mathbb{R}_+ \to [1,\infty)$ and $w : \mathbb{R}_+ \to \mathbb{R}_+$. Assume there exists a constant $1 \le c_* < \infty$ such that*

$$\text{(A)} \qquad \int_0^\infty Hf(x)^{p(x)} w(x)\,dx \le c_* \int_0^\infty f(x)^{p(x)} w(x)\,dx, \qquad f \in \mathcal{D}.$$

Then

(B) $$\|Hf\|_{p(x),w} \le c_*\|f\|_{p(x),w}$$

if either

(i) $f \in \mathcal{D}$ *and* $\|f\|_{p(x),w} \ge 1/c_*$, *or*
(ii) f *is non-increasing on* $\mathbb{R}_+$ *and* $f(x)/\|f\|_{p(x),w} \in \mathcal{D}$.

Proof. (i) Since $c_* \ge 1$ we have

$$\begin{aligned}
\|Hf\|_{p(x),w} &= \inf\Big\{\lambda > 0 : \int_0^\infty \Big(\frac{Hf(x)}{\lambda}\Big)^{p(x)} w(x)\,dx \le 1\Big\} \\
&\le \inf\Big\{\lambda \ge 1 : \int_0^\infty \Big(\frac{Hf(x)}{\lambda}\Big)^{p(x)} w(x)\,dx \le 1\Big\} \\
&\le \inf\Big\{\lambda \ge 1 : c_*\int_0^\infty \Big(\frac{f(x)}{\lambda}\Big)^{p(x)} w(x)\,dx \le 1\Big\} \\
&\le \inf\Big\{\lambda \ge 1 : \int_0^\infty \Big(\frac{f(x)}{\lambda/c_*}\Big)^{p(x)} w(x)\,dx \le 1\Big\} \\
&= \inf\Big\{c_*\sigma \ge 1 : \int_0^\infty \Big(\frac{f(x)}{\sigma}\Big)^{p(x)} w(x)\,dx \le 1\Big\} \\
&= c_* \inf\Big\{\sigma \ge 1/c_* : \int_0^\infty \Big(\frac{f(x)}{\sigma}\Big)^{p(x)} w(x)\,dx \le 1\Big\} \le c_*\|f\|_{p(x),w}.
\end{aligned}$$

(ii) Let $g(x) = f(x)/\|f\|_{p(x),w}$. By hypothesis $g \in \mathcal{D}$ and $\|g\|_{p(x),w} = 1$. Hence

$$\int_0^\infty Hg(x)^{p(x)} w(x)\,dx \le c_* \int_0^\infty g(x)^{p(x)} w(x)\,dx \le c_*.$$

This implies $\|Hf\|_{p(x),w} \le c_*\|f\|_{p(x),w}$. ∎

Remark. By Theorem 3 the hypothesis of Theorem 4 is satisfied if $1 \le p(x) \le p^* < \infty$, $p\uparrow$ and $w \in B_{p(x)}$. The constant c_* depends only on the $B_{p(x)}$-constant of w.

Example. We will now show that (i) of Theorem 4 does not imply the norm inequality (B) with a constant depending on the $B_{p(x)}$-constant of w only if the $L^{p(x)}(w)$-norm of f is not bounded away from zero. Let $0 < a < 1$ and let $p_a(x) = 2\chi_a(x) + 4\chi^a(x)$. It is easily checked that $w(x) \equiv 1$ is in $B_{p_a(x)}$ with constant independent of a. Let $f = \chi_a$. Then

$$\|f\|_{p_a(x),w} = \inf\Big\{\lambda > 0 : \int_0^a \Big(\frac{1}{\lambda}\Big)^2 dx \le 1\Big\} = a^{1/2},$$

and

$$\|Hf\|_{p_a(x),w} \geq \inf\left\{\lambda > 0 : \int_a^\infty \left(\frac{a}{\lambda x}\right)^4 dx \leq 1\right\} = \left(\frac{a}{3}\right)^{1/4}.$$

Hence the norm inequality of Theorem 4 cannot hold with a constant independent of a.

4. The equivalence (A)$\Leftrightarrow$(B)$\Leftrightarrow$(C). We need the following lemma.

Lemma 5. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\|f\|_{p(x),w} > 0$, where $1 \leq p(x) \leq p^* < \infty$, and let $0 < a < \infty$. Then there exists $0 < \sigma < \infty$ such that $\|f\|_{p(x),\sigma w} = a$.*

Proof. For $\sigma \geq 1$,

$$\int_0^\infty \left(\frac{f(x)}{\lambda}\right)^{p(x)} \sigma w(x)\,dx \geq \int_0^\infty \left(\frac{f(x)}{\lambda/\sigma^{1/p^*}}\right)^{p(x)} w(x)\,dx,$$

which implies that $\|f\|_{p(x),\sigma w} \geq \sigma^{1/p^*}\|f\|_{p(x),w}$. Hence the set $S_a = \{\sigma > 0 : \|f\|_{p(x),\sigma w} \geq a\}$ is not empty. Let $\sigma_0 = \inf\{\sigma : \sigma \in S_a\}$. Then a straightforward argument shows that $\|f\|_{p(x),\sigma_0 w} = a$. ∎

Since the conditions (A) and (C) remain unchanged when $w(x)$ is replaced by $\sigma w(x)$, $0 < \sigma < \infty$, the condition (B) has to be modified to reflect this.

Theorem 6. *The following statements are equivalent for $1 \leq p(x) \leq p^* < \infty$, $p\uparrow$, and $w : \mathbb{R}_+ \to \mathbb{R}_+$.*

- *There exists $1 \leq c_* < \infty$ such that*

$$\text{(A)} \qquad \int_0^\infty Hf(x)^{p(x)} w(x)\,dx \leq c_* \int_0^\infty f(x)^{p(x)} w(x)\,dx, \qquad f \in \mathcal{D}.$$

- *For each $0 < \gamma \leq 1$ there is $1 \leq c_\gamma < \infty$ such that*

$$\text{(B)} \qquad \|Hf\|_{p(x),\sigma w} \leq c_\gamma \|f\|_{p(x),\sigma w}$$

for every $f \in \mathcal{D}$ and every $0 < \sigma < \infty$ for which $\|f\|_{p(x),\sigma w} \geq \gamma$.

- *We have*

$$\text{(C)} \qquad w \in B_{p(x)}.$$

Proof. (A)$\Rightarrow$(B). Let $0 < \gamma \leq 1$ and let $c_\gamma = \max(c_*, 1/\gamma)$. Then (A) holds with c_* replaced by c_γ and $w(x)$ replaced by $\sigma w(x)$. Theorem 4 gives (B).

(B)$\Rightarrow$(C). We have to show that

$$\int_r^\infty \left(\frac{r}{x}\right)^{p(x)} w(x)\,dx \le c\int_0^r w(x)\,dx.$$

Let $f = \chi_r$. Then $f \in \mathcal{D}$. Fix $0 < \gamma < 1$ and then by Lemma 5 we can choose $0 < \sigma < \infty$ such that

$$\gamma \le \|f\|_{p(x),\sigma w} \equiv \lambda_0 \le 1.$$

Then

$$\int_0^r \frac{\sigma w(x)}{\lambda_0^{p(x)}}\,dx = 1,$$

which implies, since $\lambda_0 \le 1$, that

$$\int_0^r \sigma w(x)\,dx \ge \lambda_0^{p^*}.$$

Let $c = \max(c_\gamma, 1/\gamma)$. Since $\|Hf\|_{p(x),\sigma w} \le c\lambda_0$, we have

$$\int_0^\infty \left(\frac{Hf(x)}{c\lambda_0}\right)^{p(x)} \sigma w(x)\,dx \le 1.$$

Because $c\lambda_0 \ge 1$, the left side is

$$\ge \frac{1}{(c\lambda_0)^{p^*}} \int_r^\infty \left(\frac{r}{x}\right)^{p(x)} \sigma w(x)\,dx,$$

and consequently

$$\frac{1}{(c\lambda_0)^{p^*}} \int_r^\infty \left(\frac{r}{x}\right)^{p(x)} \sigma w(x)\,dx \le 1 \le \frac{1}{\lambda_0^{p^*}} \int_0^r \sigma w(x)\,dx.$$

Hence $w \in B_{p(x)}$ with constant c^{p^*}.

(C)$\Rightarrow$(A). This is contained in Theorem 3. ∎

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