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Invertibility in tensor products of Q-algebras

by

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Abstract. We consider, using various tensor norms, the completed tensor product of two unital lmc algebras one of which is commutative. Our main result shows that when the tensor product of two Q-algebras is an lmc algebra, then it is a Q-algebra if and only if pointwise invertibility implies invertibility (as in the Gelfand theory). This is always the case for Fréchet algebras.

In 1973 L. Waelbroeck [26] introduced the concept of Banach-valued spectrum for commutative unital Banach algebras using the projective tensor product. This concept has been extended by various authors in the last twenty five years and more recently in a systematic fashion by S. Dineen, R. E. Harte and C. Taylor [9]–[11]. These authors considered arbitrary tensor norms and arbitrary unital Banach algebras. In this article we extend the scope of this investigation to wider collections of algebras. Initially we considered arbitrary locally multiplicatively convex (lmc) algebras, but as our investigations proceeded we found that the main results centered around Fréchet and Q-algebras. For these overlapping collections we prove $\{\text{pointwise invertibility}\} \Rightarrow \{\text{invertibility}\}$ results using different methods. In the Fréchet algebra case we use techniques due to R. Arens [3], and for the Q-algebra case we develop and apply a vector-valued spectral theory following [9]–[11]. By combining the two cases we obtain necessary and sufficient conditions under which certain tensor products of Q-algebras are Q-algebras.

In §1 we define uniform tensor norms and topologies, lmc algebras and Q-algebras and give a number of examples. In §2 we define a number of vector-valued spectra and discuss their basic properties. We specialize to algebra-valued spectra in §3 and prove invertibility results. In §4 we use the Gelfand transform to obtain further criteria for invertibility.

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We refer to A. Defant and K. Floret [7] and H. Jarchow [18] for tensor products and to A. Mallios [20] and A. E. Michael [22] for the theory of topological algebras. The authors wish to thank Domingo García and Manuel Maestre for fruitful conversations and Maria Fragoulopoulou for helpful comments and for drawing our attention to relevant literature.

1. Uniform tensor norms and Q-algebras

DEFINITION 1.1. A method τ of assigning a norm $\| \ \|_\tau$ on $E\otimes F$ to every pair of normed linear spaces is called a *uniform tensor norm* if the following hold:

(1) for normed linear spaces E, F, $\|x\otimes y\|_\tau = \|x\|\cdot\|y\|$ for $x\in E$ and $y\in F$,

(2) for normed linear spaces E_i, F_i, $i=1,2$, and $T_i\in\mathcal{L}(E_i;F_i)$ (continuous linear mappings from E_i to F_i), $i=1,2$, we have $T_1\otimes T_2\in\mathcal{L}(E_1\otimes_\tau E_2;F_1\otimes_\tau F_2)$ and

$$\|T_1\otimes T_2\|\le\|T_1\|\cdot\|T_2\|.$$

A uniform tensor norm satisfies the *projective limit condition* if for any set of projective limits

$$\varprojlim_i E_i = \varprojlim_j F_j,\qquad \varprojlim_\alpha G_\alpha = \varprojlim_\beta H_\beta$$

of normed linear spaces we have

$$\varprojlim_{i,\alpha}(E_i\otimes_\tau G_\alpha) = \varprojlim_{j,\beta}(F_j\otimes_\tau H_\beta).$$

When the projective limit condition is satisfied we let

$$(\varprojlim_i E_i)\otimes_\tau(\varprojlim_\alpha G_\alpha) = \varprojlim_{i,\alpha}(E_i\otimes_\tau G_\alpha).$$

For locally convex spaces we let $E\ \widehat{\otimes}_\tau\ F$ denote the completion of $E\otimes_\tau F$. The projective (π) and injective (ε) norms are uniform tensor norms which satisfy the projective limit condition. Further examples are given by the Lapresté tensor norms [7, §12.5 and §12.7].

If τ is a uniform tensor norm which satisfies the projective limit condition then

(1) for any pair of locally convex spaces the canonical bilinear mapping $E\times F\to E\otimes_\tau F$ is separately continuous,

(2) if E_i, F_i, $i=1,2$, are locally convex spaces and $M_i\subseteq\mathcal{L}(E_i;F_i)$, $i=1,2$, are equicontinuous sets, then $M_1\otimes M_2\subseteq\mathcal{L}(E_1\otimes_\tau E_2;F_1\otimes_\tau F_2)$ is an equicontinuous set.

All algebras considered are over the field of complex numbers. An algebra $\mathcal{A}$ which is also a locally convex space is a *locally multiplicatively convex* (*lmc*) algebra if its topology is generated by a family $(p_\alpha)_{\alpha\in\Gamma}$ of continuous seminorms such that

$$p_\alpha(x \cdot y) \le p_\alpha(x) \cdot p_\alpha(y) \tag{1}$$

for all $\alpha \in \Gamma$ and all $x, y \in \mathcal{A}$.

If $\mathcal{A}$ is a complete lmc algebra then there exists a collection $(\mathcal{A}_\alpha)_\alpha$ of Banach algebras and continuous homomorphisms $\pi_\alpha : \mathcal{A} \to \mathcal{A}_\alpha$ such that $\pi_\alpha(\mathcal{A})$ is dense in $\mathcal{A}_\alpha$ and $\mathcal{A} = \varprojlim_\alpha(\mathcal{A}_\alpha, \pi_\alpha)$ (a projective limit representation such that $\pi_\alpha(\mathcal{A})$ is dense in $\mathcal{A}_\alpha$ for all α is called *reduced*).

An algebra with identity $1_{\mathcal{A}}$ is called a *unital* algebra. We shall also use $1_{\mathcal{A}}$ to denote the identity operator on $\mathcal{A}$, i.e. $1_{\mathcal{A}}(x) = x$ for all $x \in \mathcal{A}$. A complete unital lmc algebra is a *Q-algebra* if the set $\mathcal{A}_{\mathrm{inv}}$ of invertible elements in $\mathcal{A}$ is open (a more general definition of Q-algebra is given in [20] but we have used the above for convenience). This is the case if and only if the identity has a neighbourhood consisting of invertible elements. The spectrum of $\mathcal{A}$, $\mathfrak{M}(\mathcal{A})$ (the set of all continuous non-zero $\mathbb{C}$-valued homomorphisms on $\mathcal{A}$), is a weak*-compact subset of $\mathcal{A}'$ when $\mathcal{A}$ is a Q-algebra ([27, Proposition 10]). All unital Banach algebras are Q-algebras. An important classical example of a Q-algebra which is not a Banach algebra is the space $\mathcal{C}^\infty[0,1]$ endowed with the topology of uniform convergence of functions and all their derivatives over $[0,1]$.

We now connect the concepts of tensor product and lmc algebra. If $\mathcal{A}$ and $\mathcal{B}$ are algebras then universal properties of the tensor product show that the product

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := a_1a_2 \otimes b_1b_2$$

for $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ extends to define a product on the algebra $\mathcal{A} \otimes \mathcal{B}$ ([9, 20]).

Definition 1.2. If $\mathcal{A}$ and $\mathcal{B}$ are complete lmc algebras and τ is a uniform tensor norm which satisfies the projective limit condition, then we say that $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ is *representable* if there exist reduced projective Banach algebra representations $\mathcal{A} = \varprojlim_\alpha(\mathcal{A}_\alpha, \pi_\alpha)$ and $\mathcal{B} = \varprojlim_\beta(\mathcal{B}_\beta, \varrho_\beta)$ such that $\mathcal{A}_\alpha \mathbin{\widehat{\otimes}_\tau} \mathcal{B}_\beta$ is a Banach algebra for all α and β.

If $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ is representable then, by the projective limit condition, $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B} = \varprojlim_{\alpha,\beta}(\mathcal{A}_\alpha \mathbin{\widehat{\otimes}_\tau} \mathcal{B}_\beta, \pi_\alpha \otimes \varrho_\beta)$ is a complete lmc algebra.

Results in [9] show that $\mathcal{A} \mathbin{\widehat{\otimes}_\pi} \mathcal{B}$ is representable for any pair of lmc algebras $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{A} \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{B}$ is representable if there exists a reduced projective limit representation $\mathcal{A} = \varprojlim_\alpha(\mathcal{A}_\alpha, \pi_\alpha)$ where each $\mathcal{A}_\alpha$ is a uniform Banach algebra and $\mathcal{B}$ is an lmc algebra. Further examples can be obtained using Lapresté's tensor norms [7, §12.5 and §12.7].

We call a complete lmc algebra whose topology is generated by a countable set of seminorms a *Fréchet algebra*; a more general definition is given in [20]. Thus every Fréchet algebra has a countable reduced projective representation by Banach algebras and continuous homomorphisms. An element a in a unital algebra is *left invertible* if there exists $b \in \mathcal{A}$ such that $b \cdot a = 1_{\mathcal{A}}$. The following result for unital lmc algebras is due to M. Fragoulopoulou ([13, Lemma 6.8] and [14, Proposition 4. 13]). For the sake of completeness we include a proof for Fréchet algebras, a case that we require later.

Proposition 1.3. *If $\mathcal{A}$ and $\mathcal{B}$ are unital Fréchet algebras with $\mathcal{A}$ commutative, τ is a uniform tensor norm that satisfies the projective limit condition and $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ is representable then the following are equivalent for* $\mathsf{a} \in \mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$:

(1) a *is left invertible in* $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$.
(2) $(h \otimes I_{\mathcal{B}})(\mathsf{a})$ *is left invertible for all* $h \in \mathfrak{M}(\mathcal{A})$.

Proof. Since algebra homomorphisms map left invertibles to left invertibles we clearly have (1)⇒(2).

Now suppose that (2) is satisfied. By our hypothesis we have $\mathcal{A} = \varprojlim_n(\mathcal{A}_n, \pi_n)$, $\mathcal{B} = \varprojlim_n(\mathcal{B}_n, \varrho_n)$ and $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B} = \varprojlim_n(\mathcal{A}_n \mathbin{\widehat{\otimes}_\tau} \mathcal{B}_n, \pi_n \otimes \varrho_n)$ where each $\mathcal{A}_n \mathbin{\widehat{\otimes}_\tau} \mathcal{B}_n$ is a Banach algebra and $\pi_n \otimes \varrho_n(\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B})$ is dense in $\mathcal{A}_n \mathbin{\widehat{\otimes}_\tau} \mathcal{B}_n$.

If $h_n \in \mathfrak{M}(\mathcal{A}_n)$ then $h_n \circ \pi_n \in \mathfrak{M}(\mathcal{A})$ and, by (2), $((h_n \circ \pi_n) \otimes I_{\mathcal{B}})(\mathsf{a})$ is left invertible in $\mathcal{B}$. Hence $\varrho_n((h_n \circ \pi_n) \otimes I_{\mathcal{B}}(\mathsf{a}))$ is left invertible in $\mathcal{B}_n$. By first considering elements of $\mathcal{A} \otimes \mathcal{B}$ and then using continuity we see that

$$\varrho_n((h_n \circ \pi_n) \otimes I_{\mathcal{B}}(\mathsf{a})) = (h_n \otimes I_{\mathcal{B}})(\pi_n \otimes \varrho_n(\mathsf{a})).$$

By [9, Proposition 20], $\pi_n \otimes \varrho_n(\mathsf{a})$ is left invertible in $\mathcal{A}_n \mathbin{\widehat{\otimes}_\tau} \mathcal{B}_n$, and by [3, Theorem 4.2], a is left invertible in $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$. ■

Example 1.4. Let X be a completely regular hemicompact $k_{\mathbb{R}}$-space (see [16, Chapter 3], [20, Theorem 1.2, p. 223]). Then $\mathcal{C}(X)$, the complex-valued functions on X endowed with the compact open topology, is a unital Fréchet algebra and $\mathfrak{M}(\mathcal{C}(X))$ can be identified with point evaluations at points of X ([16, 4.1.7]). If $\mathcal{B}$ is unital Fréchet algebra then $\mathcal{C}(X, \mathcal{B}) \cong \mathcal{C}(X) \mathbin{\widehat{\otimes}_\varepsilon} \mathcal{B}$ ([18, 16.6.3]). If δ_x is point evaluation at $x \in X$ then, by the above identification $\delta_x \otimes I_{\mathcal{B}}(f) = f(x)$ for all $f \in \mathcal{C}(X, \mathcal{B})$. By Proposition 1.3, $f \in \mathcal{C}(X, \mathcal{B})$ is left invertible if and only if $f(x)$ is left invertible in $\mathcal{B}$ for all $x \in X$. This result is a special case of [8, Proposition 1] due to S. Dierolf and K. Aye Aye.

We remark that the left inverse of an element of an algebra is not necessarily unique and thus this result says that we may make a continuous selection of left inverses. The same result is true for right inverses and combining these two cases we obtain the result for inverses. However, the result for inverses is trivial since $f^{-1}(x) := (f(x))^{-1}$ defines a continuous inverse.

EXAMPLE 1.5. Let U denote a connected pseudo-convex Riemann domain over $E \cong F'_{\mathrm{c}}$, where F'_{c} is endowed with the compact open topology and F is a Fréchet space with the approximation property. Results by J. Mujica [23] and M. Schottenloher [24] show that $(H(U), \tau_0)$ is a Fréchet space and $\mathfrak{M}(H(U), \tau_0) = U$ (via point evaluations). Note that if F is separable then U is a hemicompact topological space but, in general, U will not be hemicompact. We have

$$(H(U, \mathcal{B}), \tau_0) \cong (H(U), \tau_0) \mathbin{\widehat{\otimes}_{\varepsilon}} \mathcal{B}$$

for any unital Banach algebra $\mathcal{B}$ ([12]). Using the method of the previous example and Proposition 1.3 we see that if $f \in H(U, \mathcal{B})$ is left invertible at each point, i.e. if $f(z) \in \mathcal{B}$ is left invertible for all $z \in U$, then there exists $g \in H(U, \mathcal{B})$ such that $g(z)f(z) = 1_{\mathcal{B}}$ (identity on $\mathcal{B}$) for all $z \in U$. This result is due to G. Allan [1], [2] when E is finite-dimensional. An example in [6] shows that the pseudo-convexity condition is necessary. Connectedness can be removed by considering each connected component separately.

2. Vector-valued spectra. We require three different left spectra in this article. For the reader's convenience we collect here the notation that we shall subsequently use. If $\mathcal{A}$ is a unital lmc algebra, E is a locally convex space and τ is a uniform tensor norm we let:

- $\sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$ denote the left joint Harte spectrum of a collection $\mathsf{a} := (a_i)_{i \in I} \subseteq \mathcal{A}$,
- $\sigma^{\mathrm{left}}(\mathsf{a})$ denote the vector-valued left spectrum of $\mathsf{a} \in \mathcal{A} \mathbin{\widehat{\otimes}_{\tau}} E$, and
- $\sigma_{\mathcal{A}}^{\mathrm{left}}(\mathsf{a})$ denote the (usual) left spectrum of $\mathsf{a} \in \mathcal{A}$.

In this section we develop a theory of vector-valued spectra similar to that developed for Banach algebras in [9]–[11]. We use this theory to extend Proposition 1.3 to Q-algebras in §3.

DEFINITION 2.1. For a family $\mathsf{a} = (a_i)_{i \in I} \subseteq \mathcal{A}$, a unital lmc algebra, the *left joint Harte spectrum* $\sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$ of a is the set of all $(\lambda_i)_{i \in I} \in \mathbb{C}^I$ such that

$$1_{\mathcal{A}} \notin \Big\{ \sum_{\substack{i \in F \subseteq I \\ F \text{ finite}}} b_i(a_i - \lambda_i 1_{\mathcal{A}}) : b_i \in \mathcal{A} \Big\}.$$

Note that the set of all finite sums above is the left ideal generated by $(a_i)_{i \in I}$. When the family consists of just one element, the joint left spectrum is the scalar left spectrum. If $\lambda := (\lambda_i)_{i \in I} \in \mathbb{C}^I$ then $\lambda \in \sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$ if and only if $(\lambda_j)_{j \in J} \in \sigma_{\mathrm{H}}^{\mathrm{left}}((a_j)_{j \in J})$ for each finite subset J of I. If the indexing set I is a locally convex space E, then we can interpret $\mathsf{a} := (a_x)_{x \in E}$ as a mapping $\mathsf{a} : E \to \mathcal{A}$ by letting $\mathsf{a}(x) = a_x$, and $\lambda = (\lambda_x)_{x \in E}$ as a mapping $\lambda : E \to \mathbb{C}$ by letting $\lambda(x) = \lambda_x$. Under certain conditions, properties of a as a mapping are inherited by elements of the spectrum.

Lemma 2.2. *If $\mathcal{A}$ is a Q-algebra, $\mathsf{a} \in \mathcal{L}(E;\mathcal{A})$ and $\lambda \in \sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$, then $\lambda \in E'$. If $E = \mathcal{B}$ is an lmc algebra and a is a non-zero algebra homomorphism, then so also is $\lambda \in \sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$.*

Proof. The algebraic properties follow from the proof of [9, Lemma 3]. It remains to show that λ is continuous. Let $(x_\alpha)_\alpha \subseteq E$ and suppose $x_\alpha \to x$ as $\alpha \to \infty$. If $\lambda \in \sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$ and $\lambda(x_\alpha) \not\to \lambda(x)$ as $\alpha \to \infty$ then we can find $\delta > 0$ such that (on taking a subnet if necessary) $|\lambda(x_\alpha) - \lambda(x)| \geq \delta$ for all α. We have

$$(\mathsf{a}(x_\alpha) - \lambda(x_\alpha)1_{\mathcal{A}}) - (\mathsf{a}(x) - \lambda(x)1_{\mathcal{A}}) = (\lambda(x) - \lambda(x_\alpha))\left(\frac{\mathsf{a}(x_\alpha) - \mathsf{a}(x)}{\lambda(x) - \lambda(x_\alpha)} + 1_{\mathcal{A}}\right)$$

for all α. Since $\mathcal{A}$ is a Q-algebra there exists a neighbourhood U of 0 such that $1_{\mathcal{A}} + U \subseteq \mathcal{A}_{\mathrm{inv}}$. For α sufficiently large

$$\frac{a_{x_\alpha} - a_x}{\lambda(x) - \lambda(x_\alpha)} \in U$$

and hence $(\mathsf{a}(x_\alpha) - \lambda(x_\alpha)1_{\mathcal{A}}) - (\mathsf{a}(x) - \lambda(x)1_{\mathcal{A}})$ is invertible. This contradicts the fact that $\lambda \in \sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$. Hence λ is continuous.

If a is an algebra homomorphism then so also is λ by [9, Lemma 3]. If $\mathsf{a} \neq 0$ then $\mathsf{a}(1_{\mathcal{B}}) = 1_{\mathcal{A}}$ and

$$1_{\mathcal{A}} \in \Big\{ \sum_{\substack{i \in F \subseteq \mathcal{B} \\ F \text{ finite}}} b_i \mathsf{a}(i) : b_i \in \mathcal{A} \Big\}$$

and $\lambda \equiv 0$ does not belong to $\sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$. This contradicts the fact that $\lambda \in \sigma_{\mathrm{H}}^{\mathrm{left}}(\mathsf{a})$ and completes the proof. ∎

Definition 2.3. Let $\mathcal{A}$ be a unital lmc algebra, E a locally convex space and let τ be a uniform tensor norm on $\mathcal{A} \otimes E$. If $\mathsf{a} \in \mathcal{A} \widehat{\otimes}_\tau E$ we define the *left vector spectrum* $\sigma^{\mathrm{left}}(\mathsf{a})$ of a to be $\sigma_{\mathrm{H}}^{\mathrm{left}}(\{[1_{\mathcal{A}} \otimes x'](\mathsf{a})\}_{x' \in E'})$.

By Lemma 2.2,

$$\sigma^{\mathrm{left}}(\mathsf{a}) = \Big\{ x'' \in E'' : 1_{\mathcal{A}} \neq \sum_{\substack{i \in F \\ F \text{ finite}}} b_i([1_{\mathcal{A}} \otimes x_i'](\mathsf{a}) - x''(x_i')1_{\mathcal{A}}),\ b_i \in \mathcal{A},\ x_i' \in E' \Big\}.$$

We now prove results which allow us to rewrite $\sigma^{\mathrm{left}}(\mathsf{a})$ in a more convenient fashion. Let J_E denote the canonical mapping from a locally convex space into its bidual. The following result extends [9, Proposition 6] from unital Banach algebras to Q-algebras.

Proposition 2.4. *Let $\mathcal{A}$ be a Q-algebra, E a complete locally convex space and τ a uniform tensor topology. If $\mathsf{a} \in \mathcal{A} \widehat{\otimes}_\tau E$ and $x'' \in \sigma^{\mathrm{left}}(\mathsf{a})$ then $x'' = J_E(x)$ for some $x \in E$.*

Proof. If $\mathsf{a} \in \mathcal{A} \otimes E$ then the mapping

$$x' \in (E', \sigma(E', E)) \mapsto [1_{\mathcal{A}} \otimes x'](\mathsf{a}) \in \mathcal{A}$$

is easily seen to be continuous.

Let M denote a closed equicontinuous subset of E' and suppose $(x'_\alpha)_\alpha \subseteq M$ converges in the $\sigma(E', E)$ topology to $x' \in M$ as $\alpha \to \infty$. Let p denote a continuous seminorm on $\mathcal{A}$ and let $\varepsilon > 0$ be arbitrary. Since $\{1_{\mathcal{A}}\} \otimes M \subseteq \mathcal{L}(\mathcal{A} \mathbin{\widehat{\otimes}_\tau} E; \mathcal{A})$ is equicontinuous there exists a continuous seminorm q on $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} E$ such that

$$\sup_{x' \in M} p([1_{\mathcal{A}} \otimes x'](\mathsf{b})) \le q(\mathsf{b}) \tag{2}$$

for all $\mathsf{b} \in \mathcal{A} \mathbin{\widehat{\otimes}_\tau} E$. Now choose $\mathsf{b} \in \mathcal{A} \otimes E$ such that $q(\mathsf{a} - \mathsf{b}) < \varepsilon$. Since b is a finite tensor there exists α_0 such that

$$p([1_{\mathcal{A}} \otimes x'_\alpha](\mathsf{b}) - [1_{\mathcal{A}} \otimes x'](\mathsf{b})) < \varepsilon \tag{3}$$

for all $\alpha \ge \alpha_0$. By (2) and (3),

$$p([1_{\mathcal{A}} \otimes x'_\alpha](\mathsf{a}) - [1_{\mathcal{A}} \otimes x'](\mathsf{a})) \le p([1_{\mathcal{A}} \otimes x'_\alpha](\mathsf{b}) - [1_{\mathcal{A}} \otimes x'](\mathsf{b})) + 2q(\mathsf{a} - \mathsf{b}) \le 3\varepsilon$$

for all $\alpha \ge \alpha_0$. Hence the mapping $x' \in E' \mapsto [1_{\mathcal{A}} \otimes x'](\mathsf{a})$ is $\sigma(E', E)$-continuous on equicontinuous subsets of E'.

Let $x'' \in \sigma^{\mathrm{left}}(\mathsf{a})$. Suppose $(x'_\alpha)_\alpha$ is an equicontinuous net in E' which converges to $x' \in E'$ in the $\sigma(E', E)$ topology as $\alpha \to \infty$. If $x''(x'_\alpha)$ does not converge to $x''(x')$, then, by taking a subnet if necessary, we can suppose there exists $\delta > 0$ such that $|x''(x'_\alpha) - x''(x')| > \delta$ for all α. Since $\mathcal{A}$ is a Q-algebra we can choose a convex balanced neighbourhood U of 0 such that $1_{\mathcal{A}} + U \subseteq \mathcal{A}_{\mathrm{inv}}$. Since $x' \mapsto [1_{\mathcal{A}} \otimes x'](\mathsf{a})$ is $\sigma(E', E)$-continuous on equicontinuous sets, there exists α_0 such that

$$-[1_{\mathcal{A}} \otimes (x'_\alpha - x')](\mathsf{a}) \in \delta U$$

for all $\alpha \ge \alpha_0$. Hence

$$[1_{\mathcal{A}} \otimes (x'_\alpha - x')](\mathsf{a}) - x''(x'_\alpha - x')1_{\mathcal{A}} = -x''(x'_\alpha - x')\left(1_{\mathcal{A}} - \frac{[1_{\mathcal{A}} \otimes (x'_\alpha - x')](\mathsf{a})}{x''(x'_\alpha - x')}\right) \in \mathcal{A}_{\mathrm{inv}}.$$

This contradicts the fact that $x'' \in \sigma^{\mathrm{left}}(\mathsf{a})$. Hence x'' is $\sigma(E', E)$-continuous on equicontinuous sets of E', and Grothendieck's completeness criterion [17, Chapter 4, Section 11, Corollary 3] implies $x'' \in J_E(x)$ for some $x \in E$. ∎

Because of Proposition 2.4 we identify $\sigma^{\mathrm{left}}(\mathsf{a})$ with a subset of E for any $\mathsf{a} \in \mathcal{A} \mathbin{\widehat{\otimes}_\tau} E$.

By first considering elements of $\mathcal{A} \otimes E$ and then using density and continuity we see that

$$\begin{aligned} b([1_{\mathcal A}\otimes x'](\mathsf a) - x'(x)1_{\mathcal A}) &= b([1_{\mathcal A}\otimes x'](\mathsf a) - [1_{\mathcal A}\otimes x'](1_{\mathcal A}\otimes x)) \\ &= b([1_{\mathcal A}\otimes x'](\mathsf a - 1_{\mathcal A}\otimes x)) \\ &= [b\otimes x'](\mathsf a - 1_{\mathcal A}\otimes x) \end{aligned}$$

for $b \in \mathcal A$, $x' \in E'$, $x \in E$ and $\mathsf a \in \mathcal A \mathbin{\widehat\otimes}_\tau E$. This allows us to rewrite $\sigma^{\mathrm{left}}(\mathsf a)$, $\mathsf a \in \mathcal A \mathbin{\widehat\otimes}_\tau E$ for $\mathcal A$ a Q-algebra, E a complete locally convex space and τ a uniform tensor norm, as follows:

$$\begin{aligned} (4)\quad & \sigma^{\mathrm{left}}(\mathsf a) \\ &= \Big\{x\in E : 1_{\mathcal A}\notin \Big\{ \sum_{\substack{i\in F\\ F\text{ finite}}} a_i([1_{\mathcal A}\otimes x_i'](\mathsf a - 1_{\mathcal A}\otimes x)) : a_i\in\mathcal A,\ x_i\in E'\Big\}\Big\} \\ &= \{x\in E : 1_{\mathcal A}\neq \mathsf b\cdot(\mathsf a - 1_{\mathcal A}\otimes x) \text{ for any } \mathsf b\in\mathcal A\otimes E'\}. \end{aligned}$$

In (4), $\mathcal A\otimes E'$ acts on $\mathcal A\otimes E$ by a linear extension of the action $(a\otimes x')\cdot(b\otimes x) := x'(x)ab$. This final description is similar in form to the classical one.

In the commutative case we recover the definition of Waelbroeck [26]. This extends the Banach algebra result in [9, Proposition 7].

PROPOSITION 2.5. *Let $\mathcal A$ be a commutative Q-algebra, E a complete locally convex space and τ a uniform tensor norm. If $\mathsf a \in \mathcal A \mathbin{\widehat\otimes}_\tau E$, then*

$$\sigma^{\mathrm{left}}(\mathsf a) = \{[h\otimes I_E](\mathsf a) : h\in\mathfrak M(\mathcal A)\}.$$

Proof. Suppose $x\notin\sigma^{\mathrm{left}}(\mathsf a)$. Then there exists $\mathsf b\in\mathcal A\otimes E'$ such that $\mathsf b\cdot(\mathsf a - 1_{\mathcal A}\otimes x) = 1_{\mathcal A}$. If $h\in\mathfrak M(\mathcal A)$ then by first considering elements in $\mathcal A\otimes E$ and a density argument we see that

$$h(\mathsf b\cdot\mathsf a - 1_{\mathcal A}\otimes x) = 1 = (h\otimes I_{E'}(\mathsf b))(h\otimes I_E(\mathsf a) - x).$$

Hence $x\neq h\otimes I_E(\mathsf a)$ for any $h\in\mathfrak M(\mathcal A)$ and $h\otimes I_E(\mathsf a)\in\sigma^{\mathrm{left}}(\mathsf a)$.

Conversely, if $x\in\sigma^{\mathrm{left}}(\mathsf a)$ then, as $\mathcal A$ is a Q-algebra and so all its maximal ideals are closed ([22, p. 80]), there is $h\in\mathfrak M(\mathcal A)$ such that $h(\mathsf b\cdot(\mathsf a - 1_{\mathcal A}\otimes x)) = 0$ for all $\mathsf b\in\mathcal A\otimes E'$. In particular taking $\mathsf b = 1_{\mathcal A}\otimes x'$ we obtain

$$0 = h(1_{\mathcal A}\otimes x'\cdot\mathsf a - 1_{\mathcal A}\otimes x) = x'(h\otimes I_E(\mathsf a) - x).$$

Since x' was arbitrary the Hahn–Banach theorem implies $[h\otimes I_E](\mathsf a) = x$ and this completes the proof. ■

The following result generalizes [5, Proposition 2.3] to the non-commutative setting.

PROPOSITION 2.6. *Let $\mathcal A = \varprojlim_i(\mathcal A_i,\pi_i)$ be a reduced projective limit representation of the Q-algebra $\mathcal A$ by complete unital lmc algebras $\mathcal A_i$ where each π_i is an algebra homomorphism, let E be a locally convex space and suppose τ is a uniform tensor norm which satisfies the projective limit condition. If*

$\mathsf{a} \in \mathcal{A} \mathbin{\widehat{\otimes}_\tau} E$ *and* $\mathsf{a}_i = \pi_i \otimes I_E(\mathsf{a})$ *for all* $i \in I$ *then*

$$\sigma^{\text{left}}(\mathsf{a}) = \bigcup_{i\in I} \sigma^{\text{left}}(\mathsf{a}_i).$$

Proof. If $x \notin \sigma^{\text{left}}(\mathsf{a})$ then there is $\mathsf{b} \in \mathcal{A} \otimes E'$ such that $\mathsf{b} \cdot (\mathsf{a} - 1_{\mathcal{A}} \otimes x) = 1_{\mathcal{A}}$. By first considering elements of $\mathcal{A} \otimes E$ and then using density and continuity and the fact that the projective limit is reduced we obtain

$$\begin{aligned} 1_{\mathcal{A}_i} = \pi_i(1_{\mathcal{A}}) &= \pi_i(\mathsf{b} \cdot (\mathsf{a} - 1_{\mathcal{A}} \otimes x)) \\ &= (\pi_i \otimes I_{E'})(\mathsf{b}) \cdot (\pi_i \otimes I_E(\mathsf{a}) - \pi_i(1_{\mathcal{A}}) \otimes x) \\ &= (\pi_i \otimes I_{E'})(\mathsf{b}) \cdot (\mathsf{a}_i - 1_{\mathcal{A}_i} \otimes x). \end{aligned}$$

Since $(\pi_i \otimes I_{E'})(\mathsf{b}) \in \mathcal{A}_i \otimes E'$ this implies $x \notin \sigma^{\text{left}}(\mathsf{a}_i)$ and

$$\bigcup_{i\in I} \sigma^{\text{left}}(\mathsf{a}_i) \subseteq \sigma^{\text{left}}(\mathsf{a}).$$

Now suppose $x \in \sigma^{\text{left}}(\mathsf{a})$. Let $\mathcal{I}$ denote the left ideal in $\mathcal{A}$ generated by $\{(1_{\mathcal{A}} \otimes x') \cdot (\mathsf{a} - 1_{\mathcal{A}} \otimes x)\}_{x'\in E'}$. Since $\mathcal{A}$ is a Q-algebra and $x \in \sigma^{\text{left}}(\mathsf{a})$ we have $1_{\mathcal{A}} \notin \overline{\mathcal{I}}$. By the Hahn–Banach theorem we can choose $\psi \in \mathcal{A}'$ such that $\psi(1_{\mathcal{A}}) = 1$ and $\psi(\overline{\mathcal{I}}) = 0$. By the projective limit representation there exist $i \in I$ and $\psi_i \in \mathcal{A}_i'$ such that $\psi = \psi_i \circ \pi_i$. Since π_i is an algebra homomorphism,

$$\psi_i(1_{\mathcal{A}_i}) = \psi_i(\pi_i(1_{\mathcal{A}})) = \psi(1_{\mathcal{A}}) = 1.$$

Let $\mathcal{I}_i$ denote the left ideal in $\mathcal{A}_i$ generated by

$$\{(1_{\mathcal{A}_i} \otimes x') \cdot (\mathsf{a}_i - 1_{\mathcal{A}_i} \otimes x)\}_{x'\in E'}.$$

Since $\pi_i((1_{\mathcal{A}} \otimes x') \cdot (\mathsf{a} - 1_{\mathcal{A}} \otimes x)) = (1_{\mathcal{A}_i} \otimes x') \cdot (\mathsf{a}_i - 1_{\mathcal{A}_i} \otimes x)$ we have $\pi_i(\mathcal{I}) \subseteq \mathcal{I}_i$ and $\overline{\pi_i(\mathcal{I})} \subseteq \overline{\mathcal{I}}_i$. Since the projective limit representation of $\mathcal{A}$ is reduced $\pi_i(\mathcal{A})$ is dense in $\mathcal{A}_i$. Let $b_i \in \mathcal{A}_i$ and suppose $(b_\alpha)_\alpha \subseteq \mathcal{A}$ and $\pi_i(b_\alpha) \to b_i$ as $\alpha \to \infty$. If $x' \in E'$ then

$$\begin{aligned} (b_i \otimes x') \cdot (\mathsf{a}_i - 1_{\mathcal{A}_i} \otimes x) &= \lim_\alpha (\pi_i(b_\alpha) \otimes x') \cdot (\mathsf{a}_i - 1_{\mathcal{A}_i} \otimes x) \\ &= \lim_\alpha \pi_i(b_\alpha \otimes x') \cdot (\mathsf{a}_i - 1_{\mathcal{A}_i} \otimes x) \end{aligned}$$

and $\mathcal{I}_i \subseteq \overline{\pi_i(\mathcal{I})}$. Hence $\overline{\mathcal{I}}_i = \overline{\pi_i(\mathcal{I})}$.

If $w \in \overline{\mathcal{I}}_i$ then there exists $(w_\beta)_\beta \subseteq \mathcal{I}$ such that $w = \lim_\beta \pi_i(w_\beta)$. Hence

$$\psi_i(w) = \psi_i(\lim_\beta \pi_i(w_\beta)) = \lim_\beta \psi_i \circ \pi_i(w_\beta) = \lim_\beta \psi(w_\beta) = 0.$$

On the other hand $\psi_i(1_{\mathcal{A}_i}) = 1$. Hence $\overline{\mathcal{I}}_i$ is a proper closed ideal in $\mathcal{A}_i$. This implies $x \in \sigma^{\text{left}}(\mathsf{a}_i)$ and completes the proof. ∎

Our next example generalizes [3, Theorem 4.2] concerning invertibility in a projective limit of algebras.

EXAMPLE 2.7. If $\mathcal{A} = \varprojlim_i \mathcal{A}_i$ is a Q-algebra then $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathbb{C} \cong \mathcal{A}$. Therefore, $\lambda \notin \sigma^{\mathrm{left}}(a \otimes 1_{\mathbb{C}})$ if and only if there is a finite sum, with $a_i \in \mathcal{A}$ and $\mu_i \in \mathbb{C}$, such that

$$\begin{aligned} 1_{\mathcal{A}} &= \sum_{j=1}^{n} a_j([1_{\mathcal{A}} \otimes \mu_j](a \otimes 1_{\mathbb{C}}) - \lambda \mu_j 1_{\mathcal{A}}) \\ &= \sum_{j=1}^{n} a_j(\mu_j a - \mu_j \lambda 1_{\mathcal{A}}) = \sum_{j=1}^{n} a_j \mu_j (a - \lambda 1_{\mathcal{A}}) \\ &= b(a - \lambda 1_{\mathcal{A}}) \end{aligned}$$

and this is equivalent to $\lambda \notin \sigma_{\mathcal{A}}^{\mathrm{left}}(a)$. Then $\sigma^{\mathrm{left}}(a \otimes 1_{\mathbb{C}}) = \sigma_{\mathcal{A}}^{\mathrm{left}}(a)$. If $\pi_i(a) = a_i$ then Proposition 2.6 implies

$$\sigma_{\mathcal{A}}^{\mathrm{left}}(a) = \bigcup_{i \in I} \sigma_{\mathcal{A}_i}^{\mathrm{left}}(a_i).$$

Since $a \in \mathcal{A}$ is left invertible if and only if $0 \notin \sigma^{\mathrm{left}}(a) = \bigcup_{i \in I} \sigma_{\mathcal{A}_i}^{\mathrm{left}}(a_i)$, this is equivalent to $0 \notin \sigma_{\mathcal{A}_i}^{\mathrm{left}}(a_i)$ for all i. Hence, $a \in \mathcal{A}$ is left invertible if and only if every a_i is left invertible in $\mathcal{A}_i$.

3. Algebra-valued spectra. In this section we use the results in §2 with E an lmc algebra to obtain invertibility results. We consider $\mathsf{a} \in \mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ where τ is a uniform tensor norm and we suppose that $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ itself is an lmc algebra.

To obtain our results we need to extend a number of Banach algebra results in [9] to Q-algebras. The extensions are fairly straightforward, except for the following lemma, which is used to extend [9, Proposition 11]. We include the other results without proof. In the following lemma ∂A denotes the boundary of the set A.

LEMMA 3.1. *Let $\mathcal{A}$ be a Q-algebra and let $z \in \partial \mathcal{A}_{\mathrm{inv}}$. Then there exists a continuous multiplicative seminorm p on $\mathcal{A}$ and a net $(z_\alpha)_\alpha \subseteq \mathcal{A}$ such that $p(z_\alpha) = 1$ for all α and*

$$\lim_{\alpha} p(z_\alpha z) = 0 = \lim_{\alpha} p(z z_\alpha).$$

Proof. Since $z \in \partial \mathcal{A}_{\mathrm{inv}}$, there exists a net $(r_\alpha)_\alpha \subseteq \mathcal{A}_{\mathrm{inv}}$ such that $\lim_\alpha r_\alpha = z$. Since $\mathcal{A}$ is a Q-algebra there exist $\varepsilon > 0$ and a continuous multiplicative seminorm p such that $\{a \in \mathcal{A} : p(a - 1_{\mathcal{A}}) < \varepsilon\} \subseteq \mathcal{A}_{\mathrm{inv}}$. Suppose $(p(r_\alpha^{-1}))_\alpha$ is bounded. Then

$$p(r_\alpha^{-1}(z - r_\alpha)) \le p(r_\alpha^{-1}) p(z - r_\alpha) \to 0.$$

Since $r_\alpha^{-1}(z - r_\alpha) = r_\alpha^{-1} z - 1_{\mathcal{A}}$ we have $r_\alpha^{-1} z \in \mathcal{A}_{\mathrm{inv}}$ for α sufficiently large. Hence $z = r_\alpha(r_\alpha^{-1} z) \in \mathcal{A}_{\mathrm{inv}}$. Since $\mathcal{A}$ is a Q-algebra, $\mathcal{A}_{\mathrm{inv}}$ is open and $\mathcal{A}_{\text{inv}} \cap \partial\mathcal{A}_{\text{inv}} = \emptyset$. This is a contradiction and, by taking a subnet if necessary, we may suppose $\lim_\alpha p(r_\alpha^{-1}) = \infty$.

If $z_\alpha := r_\alpha^{-1}/p(r_\alpha^{-1})$ then $p(z_\alpha) = 1$ for all α. Moreover

$$zz_\alpha = \frac{zr_\alpha^{-1}}{p(r_\alpha^{-1})} = \frac{1_{\mathcal{A}} + zr_\alpha^{-1} - r_\alpha r_\alpha^{-1}}{p(r_\alpha^{-1})} = \frac{1_{\mathcal{A}}}{p(r_\alpha^{-1})} + (z - r_\alpha)z_\alpha$$

for all α. Hence

$$p(zz_\alpha) \le \frac{1}{p(r_\alpha^{-1})} + p(z - r_\alpha)p(z_\alpha) \to 0$$

as $\alpha \to \infty$. In the same way $p(z_\alpha z) \to 0$ as $\alpha \to \infty$. ■

We now state the remaining results we require (the proof of **(1)** requires Lemma 3.1). The references are to the proofs for the Banach algebra cases.

[9, Proposition 11] Let $(a_i)_{i\in I}$ and $(b_j)_{j\in J}$ denote subsets of the Q-algebra $\mathcal{A}$. If $a_i a_k = a_k a_i$ and $a_i b_j = b_j a_i$ for all $i, k \in I$ and $j \in J$ then

$$\textbf{(1)} \qquad \pi_J(\sigma_{\mathrm{H}}^{\text{left}}((a_i)_{i\in I}, (b_j)_{j\in J})) = \sigma_{\mathrm{H}}^{\text{left}}((b_j)_{j\in J})$$

(π_J denotes the usual projection).

[9, Proposition 13, Lemma 14] Let $\mathcal{A}$ be a unital commutative lmc algebra, $\mathcal{B}$ a unital lmc algebra and τ a uniform tensor norm such that $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is an lmc algebra. If $\mathsf{a} \in \mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$, $\mu \in \mathbb{C}$ and $h \in \mathfrak{M}(\mathcal{A})$ then the ideals generated by

$$\textbf{(2)} \qquad \{[(a - h(a)1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}, \mathsf{a} - \mu(1_{\mathcal{A}} \otimes 1_{\mathcal{B}})\}$$

and

$$\{[(a - h(a)1_{\mathcal{A}}) \otimes 1_{\mathcal{B}}]_{a\in\mathcal{A}}, 1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathsf{a}) - \mu 1_{\mathcal{B}})\}$$

coincide.

If $(a_i)_{i\in I} \subseteq \mathcal{A}$, $(b_j)_{j\in J} \subseteq \mathcal{B}$ then

$$\textbf{(3)} \qquad \sigma_{\mathrm{H}}^{\text{left}}((a_i \otimes 1_{\mathcal{B}})_{i\in I}, (1_{\mathcal{A}} \otimes b_j)_{j\in J}) = \sigma_{\mathrm{H}}^{\text{left}}((a_i)_{i\in I}) \times \sigma_{\mathrm{H}}^{\text{left}}((b_j)_{j\in J}).$$

PROPOSITION 3.2. *Let $\mathcal{A}$ be a commutative unital lmc algebra, $\mathcal{B}$ a unital lmc algebra and τ a uniform tensor norm such that $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is a Q-algebra. If $\mathsf{a} \in \mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ then*

$$\sigma_{\mathcal{A}\widehat{\otimes}_\tau\mathcal{B}}^{\text{left}}(\mathsf{a}) = \bigcup_{h\in\mathfrak{M}(\mathcal{A})} \sigma_{\mathcal{B}}^{\text{left}}([h \otimes I_{\mathcal{B}}](\mathsf{a})).$$

Proof. Since $h\otimes I_{\mathcal{B}}$ is a non-zero algebra homomorphism for all $h \in \mathfrak{M}(\mathcal{A})$ we have $[h \otimes I_{\mathcal{B}}](1_{\mathcal{A}} \otimes 1_{\mathcal{B}}) = 1_{\mathcal{B}}$ and

$$\sigma_{\mathcal{B}}^{\text{left}}([h \otimes I_{\mathcal{B}}](\mathsf{a})) \subseteq \sigma_{\mathcal{A}\widehat{\otimes}_\tau\mathcal{B}}^{\text{left}}(\mathsf{a}).$$

Hence

$$\bigcup_{h\in\mathfrak{M}(\mathcal{A})} \sigma_{\mathcal{B}}^{\text{left}}([h \otimes I_{\mathcal{B}}](\mathsf{a})) \subseteq \sigma_{\mathcal{A}\widehat{\otimes}_\tau\mathcal{B}}^{\text{left}}(\mathsf{a}).$$

Now suppose $\mu \in \sigma^{\text{left}}_{\mathcal{A}\widehat{\otimes}_\tau\mathcal{B}}(\mathsf{a})$. Since the mapping $\mathcal{A} \to \mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ given by $a \mapsto a \otimes 1_{\mathcal{B}}$ is a non-zero continuous algebra homomorphism Lemma 2.2 implies that each element of $\sigma^{\text{left}}_{\text{H}}((a \otimes 1_{\mathcal{B}})_{a\in\mathcal{A}})$ belongs to $\mathfrak{M}(\mathcal{A})$. Since $\mathcal{A}$ is commutative, the system $\{(a \otimes 1_{\mathcal{B}})_{a\in\mathcal{A}}, \mathsf{a}\} \subseteq \mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ satisfies the commutativity relations in **(1)**. Hence by **(1)** there exists $h \in \mathfrak{M}(\mathcal{A})$ such that

$$((h(a))_{a\in\mathcal{A}}, \mu) \in \sigma^{\text{left}}_{\text{H}}((a \otimes 1_{\mathcal{B}})_{a\in\mathcal{A}}, \mathsf{a}).$$

By **(2)** and **(3)**,

$$\begin{aligned}((h(a))_{a\in\mathcal{A}}, \mu) &\in \sigma^{\text{left}}_{\text{H}}((a \otimes 1_{\mathcal{B}})_{a\in\mathcal{A}}, 1_{\mathcal{A}} \otimes ([h \otimes I_{\mathcal{B}}](\mathsf{a}))) \\ &= \sigma^{\text{left}}_{\text{H}}((a)_{a\in\mathcal{A}}) \times \sigma^{\text{left}}_{\mathcal{B}}([h \otimes I_{\mathcal{B}}](\mathsf{a})).\end{aligned}$$

Hence $\mu \in \sigma^{\text{left}}_{\mathcal{B}}([h \otimes I_{\mathcal{B}}](\mathsf{a}))$. ∎

Proposition 3.2 generalizes part of [9, Proposition 20] from unital Banach algebras to Q-algebras, and the remaining part is generalized in Proposition 4.5 below. Clearly Proposition 3.2 also holds for right spectra. Combining the results for right and left spectra we obtain an analogous result for the usual spectrum and recover a special case of results due to M. Fragoulopoulou [13, Proposition 6.9] and [14, Proposition 4.13].

Remark 3.3. If $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is a Q-algebra then, since τ is a uniform tensor norm, there exists a continuous multiplicative seminorm p on $\mathcal{A}$ such that $a \otimes 1_{\mathcal{B}} \in (\mathcal{A} \widehat{\otimes}_\tau \mathcal{B})_{\text{inv}}$ for all a such that $p(a - 1_{\mathcal{A}}) < 1$. If $\mathsf{b} \in \mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ satisfies

$$(a \otimes 1_{\mathcal{B}})\mathsf{b} = \mathsf{b}(a \otimes 1_{\mathcal{B}}) = 1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$$

and $\varphi \in \mathcal{B}'$ is chosen so that $\varphi(1_{\mathcal{B}}) = 1$ then, by density and continuity,

$$(1_{\mathcal{A}} \otimes \varphi)((a \otimes 1_{\mathcal{B}})\mathsf{b}) = a(1_{\mathcal{A}} \otimes \varphi)(\mathsf{b}) = (1_{\mathcal{A}} \otimes \varphi)(\mathsf{b})a = 1_{\mathcal{A}}$$

and $\{a \in \mathcal{A} : p(a - 1_{\mathcal{A}}) < 1\} \subseteq \mathcal{A}_{\text{inv}}$. On interchanging the roles of $\mathcal{A}$ and $\mathcal{B}$ we thus see that if $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is a Q-algebra then so also are $\mathcal{A}$ and $\mathcal{B}$. Our next result shows that the converse is true if and only if pointwise invertibility implies invertibility.

Theorem 3.4. *If $\mathcal{A}$ is a commutative Q-algebra, $\mathcal{B}$ is a Q-algebra and τ is a uniform tensor norm such that $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is an lmc algebra then the following are equivalent*:

(1) *$\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is a Q-algebra.*
(2) *If $\mathsf{a} \in \mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ then*

$$\sigma^{\text{left}}_{\mathcal{A}\widehat{\otimes}_\tau\mathcal{B}}(\mathsf{a}) = \bigcup_{h\in\mathfrak{M}(\mathcal{A})} \sigma^{\text{left}}_{\mathcal{B}}([h \otimes I_{\mathcal{B}}](\mathsf{a})).$$

Proof. We have (1)⇒(2) by Proposition 3.2. Suppose (2) holds. Since $\mathcal{A}$ is a Q-algebra, $\mathfrak{M}(\mathcal{A})$ is an equicontinuous subset of $\mathcal{A}'$ [20, p. 187]. Hence there exists a continuous multiplicative seminorm p on $\mathcal{A}$ such that

$|h(a)| \le p(a)$ for all $h \in \mathfrak{M}(\mathcal{A})$ and all $a \in \mathcal{A}$. Since $\mathcal{B}$ is a Q-algebra there exists a continuous multiplicative seminorm q on $\mathcal{B}$ such that $\{1_{\mathcal{B}} + y : q(y) < 1\} \subseteq \mathcal{B}_{\mathrm{inv}}$.

Now suppose $\mathsf{a} \in \mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ and $p \otimes_\tau q(\mathsf{a}) < 1$. If $h \in \mathfrak{M}(\mathcal{A})$ then

$$[h \otimes I_{\mathcal{B}}](1_{\mathcal{A}} \otimes 1_{\mathcal{B}} + \mathsf{a}) = 1_{\mathcal{B}} + [h \otimes I_{\mathcal{B}}](\mathsf{a}).$$

Next,

$$q([h \otimes I_{\mathcal{B}}](\mathsf{a})) = \sup_{\substack{\varphi \in \mathcal{B}' \\ \varphi \in B_q^\circ}} |\varphi([h \otimes I_{\mathcal{B}}](\mathsf{a}))| = \sup_{\substack{\varphi \in \mathcal{B}' \\ \varphi \in B_q^\circ}} |[h \otimes \varphi](\mathsf{a})| \le p \otimes_\tau q(\mathsf{a}) < 1.$$

Hence $1_{\mathcal{B}} + [h \otimes I_{\mathcal{B}}](\mathsf{a}) \in \mathcal{B}_{\mathrm{inv}}$ and, by (2), $\mathsf{a} \in (\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B})_{\mathrm{inv}}$. Hence $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ is a Q-algebra. ∎

The following result for the projective tensor product is due to H. A. Smith [25, Theorem 3] and to A. Mallios [19, Proposition 4.2] and [20, Lemma, p. 412] for more general tensor norms when $\mathcal{A}$ and $\mathcal{B}$ are both commutative. Special cases when $\mathcal{A} = \mathcal{C}(K)$, K compact Hausdorff, and $\mathcal{A} = \mathcal{C}^\infty(X)$, X a finite-dimensional compact manifold, and $\mathcal{B}$ is non-commutative are considered in [19, Lemma 2.1] and [15, Example 3.4].

THEOREM 3.5. *If $\mathcal{A}$ is a commutative Fréchet Q-algebra, $\mathcal{B}$ is a Fréchet Q-algebra, and τ is a uniform tensor norm that satisfies the projective limit condition and such that $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ is a representable lmc algebra, then $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ is a Q-algebra.*

Proof. By Proposition 1.3, condition (2) of Theorem 3.4 is satisfied. Hence $\mathcal{A} \mathbin{\widehat{\otimes}_\tau} \mathcal{B}$ is a Q-algebra. ∎

Examples can now be obtained by looking at our remarks after Definition 1.2. For instance $\mathcal{A} \mathbin{\widehat{\otimes}_\pi} \mathcal{B}$ is a Q-algebra whenever $\mathcal{A}$ is a commutative Fréchet Q-algebra and $\mathcal{B}$ is an arbitrary Fréchet Q-algebra.

4. The Gelfand transform

DEFINITION 4.1. Let $\mathcal{A}$ be an lmc algebra, E a complete locally convex space and τ a uniform tensor norm. For each $\mathsf{a} \in \mathcal{A} \mathbin{\widehat{\otimes}_\tau} E$ we define its *Gelfand transform*:

$$\widehat{\mathsf{a}} : \mathfrak{M}(\mathcal{A}) \to E, \qquad \widehat{\mathsf{a}}(h) = [h \otimes I_E](\mathsf{a}).$$

We endow $\mathfrak{M}(\mathcal{A})$ with the topology induced by $\sigma(\mathcal{A}', \mathcal{A})$.

PROPOSITION 4.2. *Let $\mathcal{A}$ be a Q-algebra, E a complete locally convex space and τ a uniform tensor norm. If $\mathsf{a} \in \mathcal{A} \mathbin{\widehat{\otimes}_\tau} E$, then $\widehat{\mathsf{a}} \in \mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$.*

Proof. Suppose $(h_\alpha)_\alpha \subseteq \mathfrak{M}(\mathcal{A})$ and $h_\alpha \to h$ as $\alpha \to \infty$. Let q be any continuous seminorm on E and $\varepsilon > 0$ be arbitrary. Since $\mathcal{A}$ is a Q-algebra,

$\mathfrak{M}(\mathcal{A}) \otimes \{I_E\}$ is an equicontinuous subset of $\mathcal{L}(\mathcal{A} \widehat{\otimes}_\tau E; E)$. Hence we can find a continuous seminorm p on $\mathcal{A} \widehat{\otimes}_\tau E$ such that

$$q([g \otimes I_E](\mathsf{b})) \le p(\mathsf{b}) \tag{5}$$

for all $\mathsf{b} \in \mathcal{A} \widehat{\otimes}_\tau E$ and all $g \in \mathfrak{M}(\mathcal{A})$. If $\mathsf{a} := \sum_{i=1}^n a_i \otimes x_i \in \mathcal{A} \otimes E$ then $\widehat{\mathsf{a}}(h) = \sum_{i=1}^n h(a_i)x_i$ is clearly continuous. If a is arbitrary we can choose $\mathsf{b} \in \mathcal{A} \otimes E$ such that $p(\mathsf{a} - \mathsf{b}) < \varepsilon$. Now choose α_0 such that

$$q([h_\alpha \otimes I_E](\mathsf{b}) - [h \otimes I_E](\mathsf{b})) < \varepsilon$$

for all $\alpha \ge \alpha_0$. Then

$$\begin{aligned} q([h_\alpha \otimes I_E](\mathsf{a}) - [h \otimes I_E](\mathsf{a})) &\le q([h_\alpha \otimes I_E](\mathsf{b}) - [h \otimes I_E](\mathsf{b})) \\ &\quad + q([(h_\alpha - h) \otimes I_E](\mathsf{b} - \mathsf{a})) \\ &\le \varepsilon + 2p(\mathsf{a} - \mathsf{b}) \le 3\varepsilon. \end{aligned}$$

Hence $\widehat{\mathsf{a}} \in \mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$. ■

DEFINITION 4.3. If $\mathcal{A}$ is a Q-algebra, E is a complete locally convex space and τ is a uniform tensor norm, the *Gelfand mapping* $\widehat{\ } : \mathcal{A} \widehat{\otimes}_\tau E \to \mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$ is defined by $\widehat{\mathsf{a}}(h) = [h \otimes I_E](\mathsf{a})$.

PROPOSITION 4.4. *If $\mathcal{A}$ is a Q-algebra, E is a complete locally convex space and τ is a uniform tensor norm then the Gelfand mapping is a continuous linear mapping from $\mathcal{A} \widehat{\otimes}_\tau E$ into $\mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$. If $\mathcal{B}$ is a complete lmc algebra and $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is an lmc algebra, then the Gelfand mapping is an algebra homomorphism.*

Proof. The Gelfand mapping is easily seen that to be linear. The estimate (5) in the previous proposition shows that it is continuous. When $\mathcal{B}$ is an algebra and $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is an lmc algebra then the Gelfand mapping is easily seen to be a homomorphism from $\mathcal{A} \otimes E$ into $\mathcal{C}(\mathfrak{M}(\mathcal{A}), E)$. Continuity and density can be applied to complete the proof. ■

PROPOSITION 4.5. *If $\mathcal{A}$ is a commutative unital lmc algebra, $\mathcal{B}$ is a unital lmc algebra and τ is a uniform tensor norm such that $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ is a Q-algebra then for $\mathsf{a} \in \mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$ the following are equivalent:*

(1) a *is left invertible in* $\mathcal{A} \widehat{\otimes}_\tau \mathcal{B}$.
(2) $\widehat{\mathsf{a}}(h)$ *is left invertible in* $\mathcal{B}$ *for every* $h \in \mathfrak{M}(\mathcal{A})$.
(3) $\widehat{\mathsf{a}}$ *is left invertible in* $\mathcal{C}(\mathfrak{M}(\mathcal{A}), \mathcal{B})$.

Proof. By Proposition 3.2, (1)⇔(2). By Proposition 4.4, $\mathsf{a} \mapsto \widehat{a}$ is an algebra homomorphism and $\widehat{1_\mathcal{A} \otimes 1_\mathcal{B}}(h) = I_\mathcal{B}$ for all $h \in \mathfrak{M}(\mathcal{A})$. Hence the Gelfand transform maps left invertible elements to left invertible elements and (1)⇒(3). Since we always have (3)⇒(2) this completes the proof. ■

EXAMPLE 4.6. Let E be Fréchet–Schwartz space whose topology is generated by an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of seminorms such that each $\widehat{E}_n$, which is a Banach space, has the approximation property. Take $K \subseteq E$ compact, balanced and polynomially convex and $\mathcal{B}$ a unital Banach algebra. Let $\mathcal{H}(K,\mathcal{B})$ denote the space of $\mathcal{B}$-valued holomorphic germs on K. We have the following representation (see [4]):

$$(\mathcal{H}(K,\mathcal{B}),\tau_\omega) \cong (\mathcal{H}(K),\tau_\omega)\,\widehat{\otimes}_\varepsilon\,\mathcal{B}.$$

Moreover, since K is polynomially convex, $\mathfrak{M}(\mathcal{H}(K)) \cong K$ by means of the identification $h(f) = f(k)$. Both $\mathcal{H}(K)$ and $\mathcal{H}(K,\mathcal{B})$ are Q-algebras, since they are inductive limits of Banach algebras.

As in Example 1.4, $[h_k \otimes I_{\mathcal{B}}](\mathsf{a}) = \mathsf{a}(k)$ for all $\mathsf{a} \in \mathcal{H}(K)\,\widehat{\otimes}_\varepsilon\,\mathcal{B}$ and $k \in K$. By Proposition 3.2, $F \in \mathcal{H}(K,\mathcal{B})$ is left invertible if and only if $F(k)$ is left invertible in $\mathcal{B}$ for all $k \in K$.

Since the Gelfand mapping $\widehat{\ }: \mathcal{H}(K,\mathcal{B}) \to \mathcal{C}(K,\mathcal{B})$ is the inclusion mapping, Proposition 4.5 implies that F is left invertible in $\mathcal{H}(K,\mathcal{B})$ if and only if it is left invertible in $\mathcal{C}(K,\mathcal{B})$.

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