An embedding theorem for Sobolev type functions with gradients in a Lorentz space

by

ALIREZA RANJBAR-MOTLAGH (Tehran)

Abstract. The purpose of this paper is to prove an embedding theorem for Sobolev type functions whose gradients are in a Lorentz space, in the framework of abstract metric-measure spaces. We then apply this theorem to prove absolute continuity and differentiability of such functions.

Introduction. In this article, we extend Morrey’s embedding theorem (see for instance [GT, Thm. 7.17]) to Sobolev type functions whose generalized gradients are in a Lorentz space, when the underlying space is a metric-measure space. In fact, Morrey’s theorem states that for any function \( f \) in the Sobolev space \( W^{1,p}(\mathbb{R}^n) \), the following inequality is satisfied:

\[
\text{ess sup}_{x,y \in B(a,R)} |f(x) - f(y)| \leq CR^{1-n/p} \|\nabla f\|_{L^p},
\]

whenever \( p > n \), for some constant \( C \) which depends on \( n \) and \( p \). We state and prove an inequality, similar to (1), controlling the value of \( \text{ess sup}_{x,y \in B(a,R)} |f(x) - f(y)| \) by the Lorentz norm of the (generalized) gradient of \( f \), for a real-valued function \( f \) whose domain is a metric-measure space (see Theorem 2.1). We use the definition of Sobolev spaces as in [H], in order to define the (generalized) gradient for functions over abstract spaces. Then we study continuity and differentiability of such functions. Our results sharpen those of [MS] for weighted Sobolev spaces and extend the main results of [KKM] to metric-measure spaces. See also [AC], [DS], [S], [Cc], [CL] and [Ca] for other work on this subject.

1. Preliminaries. First, we recall some basic definitions relating to metric-measure spaces, that is, metric spaces \((X,d)\) with a Radon (outer)
measure \( \mu \). We denote by \( B(x, r) \) the open ball of radius \( r > 0 \) with center at \( x \) in a metric space \((X, d)\). For the basic concepts of metric-measure spaces, see for instance [He] and the references therein.

**Definition 1.1.** Let \((X, d, \mu)\) be a metric-measure space and let \( \Omega \) be a (measurable) subset of \( X \). Then \( \Omega \) is also a metric-measure space with the induced metric and measure. We say that \( \Omega \) is Bishop–Gromov regular of dimension \( n > 0 \) if

\[
\frac{\mu(B(x, s) \cap \Omega)}{s^n} \leq C_0 \frac{\mu(B(y, r) \cap \Omega)}{r^n}
\]

for all \( 0 < r < 2s \), \( x, y \in \Omega \) and \( d(x, y) \leq s \), where \( C_0 \geq 1 \) is a constant.

We say that \( \Omega \) is doubling if there is a constant \( C' > 0 \) such that

\[
\mu(B(x, 2r) \cap \Omega) \leq C' \mu(B(x, r) \cap \Omega)
\]

for all \( x \in \Omega \) and \( r > 0 \).

It is easy to see that if \( \Omega \) is doubling, then it is Bishop–Gromov regular of dimension \( n \) for some \( n > 0 \). Also, if \( \Omega \) is doubling and \( \mu \) is not identically zero on \( \Omega \), then \( \mu(B(x, r) \cap \Omega) > 0 \) for all \( x \in \Omega \) and \( r > 0 \).

Now, we recall the definition of Lorentz spaces. For their basic properties, see for instance [SW], [BS] and [KK]; see also [CRS] and the references therein.

**Definition 1.2.** Let \((X, d, \mu)\) be a metric-measure space and let \( \Omega \) be a (measurable) subset of \( X \). Let \( u : X \to \mathbb{R} \) be a measurable function and let \( m \) be a positive number. We say that \( u \) belongs to the Lorentz space \( L_{m,1}(\Omega) \) if

\[
\|u\|_{L_{m,1}(\Omega)} := \int_0^\infty \mu(\{z \in \Omega : |u(z)| > t\})^{1/m} dt < \infty.
\]

Next, we state the following simple fact about connected doubling metric-measure spaces (cf. Remark 2.2).

**Proposition 1.3.** Let \((X, d, \mu)\) be a metric-measure space and assume that the ball \( B(a, R) \) is a connected doubling subset of \( X \). Suppose that \( \mu \) is not identically zero on \( B(a, R) \) and there exists \( R^- \in ]2R/3, R[ \) such that \( B(a, R^-) \neq B(a, R) \). Then

\[
0 < \mu(B(x, r) \cap B(a, R)) < \mu(B(x, r+t) \cap B(a, R)) < \infty
\]

for all \( x \in B(a, R) \), \( r \in ]0, R/4] \) and \( t > 0 \).

**Proof.** Since \( B(a, R) \) is doubling and \( \mu \) is not identically zero on \( B(a, R) \), we know that \( \mu(B(y, s) \cap B(a, R)) > 0 \) for all \( y \in B(a, R) \) and \( s > 0 \). First, assume \( R/4 < d(x, a) < R \). Since the distance function is continuous and
B(a, R) is connected, there exist \( z \in X \) and \( s > 0 \) such that
\[
B(z, s) \subset [B(x, r + t) \setminus B(x, r)] \cap B(a, R)
\]
for any \( r \in [0, R/4] \) and \( 0 < t < d(x, a) - R/4 \). So, the condition (2) holds.

Also, when \( d(x, a) \leq R/4 \), by a similar method (and using the condition \( B(a, R^-) \neq B(a, R) \)), we can obtain (2).

Now, we recall the concept of geodesic spaces. Let \((X, d)\) be a metric space. We say that \( X \) is a geodesic space if for any \( x, y \in X \), there exists a geodesic \( \gamma_{x,y} \) between \( x \) and \( y \), i.e. there is an isometry \( \gamma_{x,y} : [0, d(x, y)] \to X \) such that \( \gamma(0) = x \) and \( \gamma(d(x, y)) = y \).

**Proposition 1.4.** Let \((X, d, \mu)\) be a metric-measure space and let \( B(a, R) \) be a ball in \( X \). Suppose that \( X \) is geodesic and Bishop–Gromov regular of dimension \( n \), i.e.
\[
\frac{\mu(B(x, s))}{s^n} \leq C_0 \frac{\mu(B(y, r))}{r^n}
\]
for all \( 0 < r < 2s \), \( x, y \in X \) and \( d(x, y) \leq s \), where \( C_0 \geq 1 \) is a constant. Then \( B(a, R) \) is Bishop–Gromov regular of dimension \( n \), more precisely,
\[
\frac{\mu(B(x, s) \cap B(a, R))}{s^n} \leq C \frac{\mu(B(y, r) \cap B(a, R))}{r^n}
\]
for all \( 0 < r < 2s \), \( x, y \in B(a, R) \) and \( d(x, y) \leq s \), where \( C \) is a constant which depends on \( C_0 \) and \( n \), but is independent of \( a \) and \( R \).

**Proof.** Pick \( x, y \in B(a, R) \) and \( r, s \in \mathbb{R} \) with \( 0 < r < 2s \) and \( d(x, y) \leq s \). It is clear that if \( B(y, r) \subset B(a, R) \), we immediately obtain the desired result. Also, without loss of generality, we may assume \( r \leq R/2 \). Since \( X \) is geodesic, by considering a geodesic \( \gamma_{a,y} \) between \( a \) and \( y \), we see that there exists \( z \in B(a, R) \) such that
\[
B(z, r/2) \subset B(y, r) \cap B(a, R).
\]
Moreover, since \( X \) is doubling, we know that
\[
\mu(B(y, r)) \leq C \mu(B(z, r/2)),
\]
where \( C \) is a constant which depends on \( C_0 \) and \( n \). This completes the proof of the proposition. \( \blacksquare \)

**2. Embedding theorem.** In this section, we state and prove an embedding theorem for Sobolev type functions with gradients in the Lorentz space \( L^{n,1}(X) \). We use a Lipschitz type characterization of Sobolev spaces (see [H]) to define the (generalized) gradient for Sobolev type functions over metric-measure spaces.

**Theorem 2.1.** Let \((X, d, \mu)\) be a metric-measure space and assume that the ball \( B(a, R) \) is Bishop–Gromov regular of dimension \( n \). Suppose that
\(B(a, R)\) is connected, \(\mu\) is not identically zero on \(B(a, R)\) and there exists \(R^- \in [2R/3, R]\) such that \(B(a, R^-) \neq B(a, R)\). Suppose that \(f : X \to \mathbb{R}\) is a measurable function satisfying
\[
|f(x) - f(y)| \leq d(x, y)[g(x) + g(y)]
\]
for a.e. \(x, y \in X\), where \(g\) is a non-negative function in the Lorentz space \(L^{n,1}(X)\). Then
\[
|f(x) - f(y)| \leq 1000 \left( \frac{C_0 R^n}{\mu(B(a, R))} \right)^{1/n} \|g\|_{L^{n,1}(B(a, R))}
\]
for a.e. \(x, y \in B(a, R)\), where the constant \(C_0\) is as in Definition 1.1 (with \(\Omega = B(a, R)\)).

**Proof.** Set \(\Omega := B(a, R)\). There is a set \(E \subset X\) of measure zero such that
\[
|f(z) - f(w)| \leq d(z, w)[g(z) + g(w)]
\]
for all \(z, w \in X \setminus E\). For \(i \in \mathbb{Z}\), define
\[
F_i := \{z \in \Omega : g(z) \leq 2^i\} \setminus E,
\]
\[
r_i := \min \left\{ 2R, \left( \mu(\Omega \setminus F_i) \frac{C_0 R^n}{\mu(\Omega)} \right)^{1/n} \right\},
\]
\[
s_i := \begin{cases} r_i & \text{if } r_i \leq R/4, \\ 2R & \text{if } r_i > R/4. \end{cases}
\]

First, we show that
\[
\mu(\Omega \setminus F_i) \leq \mu(B(x, r_i) \cap \Omega)
\]
for all \(i \in \mathbb{Z}\) and \(x \in \Omega\). If \(r_i = 2R\) or \(r_i = 0\), then (4) is obvious; otherwise, \(0 < r_i < 2R\), and since \(\Omega\) is Bishop–Gromov regular, we have
\[
\frac{\mu(\Omega)}{R^n} \leq C_0 \frac{\mu(B(x, r_i) \cap \Omega)}{r_i^n}
\]
for all \(i \in \mathbb{Z}\) and \(x \in \Omega\). This implies (4).

Next, we show that for any \(x_{k+1} \in F_{k+1}\), whenever \(F_k \neq \emptyset\) there exists \(x_k \in F_k\) such that \(d(x_{k+1}, x_k) < s_k^+\) for any \(s_k^+\) such that \(\mu(B(x_{k+1}, s_k^+) \cap \Omega) < \mu(B(x_k + s_k^+) \cap \Omega)\); when \(s_k = 2R\) define \(s_k^+ := s_k = 2R\). This is obvious if \(r_k > R/4\), so suppose that \(r_k \leq R/4\). By contradiction, suppose that \(B(x_{k+1}, s_k^+) \cap F_k = \emptyset\). Then \(B(x_{k+1}, s_k^+) \cap \Omega \subset \Omega \setminus F_k\), and therefore \(\mu(B(x_{k+1}, s_k^+) \cap \Omega) \leq \mu(\Omega \setminus F_k)\). Also, by (4), we obtain
\[
\mu(B(x_{k+1}, s_k^+) \cap \Omega) \leq \mu(\Omega \setminus F_k) \leq \mu(B(x_{k+1}, s_k^+) \cap \Omega) < \mu(B(x_{k+1}, s_k^+) \cap \Omega),
\]
a contradiction.

Now, we prove (3). Suppose that \(x, y \in \Omega \setminus E\). There are \(k_x, k_y \in \mathbb{Z}\) such that \(g(x) \leq 2^{k_x}\) and \(g(y) \leq 2^{k_y}\). Define \(l := \inf\{i : F_i \neq \emptyset\} \in \mathbb{Z} \cup \{-\infty\}\).
Then there are two sequences \( \{x_i\} \) and \( \{y_j\} \) such that
\[
\begin{cases}
  x_{k_x} := x, \\
  x_i \in F_i, \\
  d(x_i, x_{i-1}) < s_{i-1}^+ \quad \text{for } k_x \geq l,
\end{cases}
\]
and
\[
\begin{cases}
  y_{k_y} := y, \\
  y_j \in F_j, \\
  d(y_j, y_{j-1}) < s_{j-1}^+ \quad \text{for } k_y \geq k_y > l.
\end{cases}
\]

Also, we have
\[
|f(z) - f(w)| \leq d(z, w)[g(z) + g(w)] \leq d(z, w)[2^i + 2^i] \leq 2^{i+1} d(z, w)
\]
for all \( z, w \in F_i \). Therefore, we obtain
\[
|f(x_i) - f(x_{i-1})| \leq 2^{i+1} s_{i-1}^+, \quad |f(y_j) - f(y_{j-1})| \leq 2^{j+1} s_{j-1}^+,
\]
for all \( k_x \geq i > l \) and \( k_y \geq j > l \). Without loss of generality, we may assume that \( k_x \geq k_y \). Then, for any integer \( k_y > m > l \), we get
\[
|f(x) - f(y)| \leq \sum_{k_x \geq i > m} |f(x_i) - f(x_{i-1})| + |f(x_m) - f(y)|
\]
\[
\leq \sum_{k_x \geq i > m} |f(x_i) - f(x_{i-1})| + |f(x_m) - f(y_m)| + |f(y_m) - f(y)|
\]
\[
\leq \sum_{k_x \geq i > m} |f(x_i) - f(x_{i-1})| + |f(x_m) - f(y_m)|
\]
\[
+ \sum_{k_y \geq j > m} |f(y_j) - f(y_{j-1})|
\]
\[
\leq \sum_{k_x \geq i > m} 2^{i+1} s_{i-1}^+ + 2^{m+2} R + \sum_{k_y \geq j > m} 2^{j+1} s_{j-1}^+
\]
\[
\leq 2^{m+2} R + 2 \sum_{k_x \geq i > m} 2^{i+1} s_{i-1}^+
\]
\[
\leq \left( \frac{C_0 R^n}{\mu(\Omega)} \right)^{1/n} (2^{m+2} \mu(\Omega)^{1/n}) + 2 \sum_{k_x \geq i > m} 2^{i+1} s_{i-1}^+.
\]

By Proposition 1.3, we can choose \( s_{i-1}^+ \) to be any number greater than \( s_{i-1} \), so
\[
(5) \quad |f(x) - f(y)| \leq \left( \frac{C_0 R^n}{\mu(\Omega)} \right)^{1/n} (2^{m+2} \mu(\Omega)^{1/n}) + 4 \sum_{\max\{k_x, k_y\} \geq i > m} 2^{i+1} s_{i-1}^+
\]
for all \( x, y \in \Omega \setminus E \) and \( \min\{k_x, k_y\} > m > l \).
On the other hand, by definition, we have

\[ \sum_{i \in \mathbb{Z}} 2^{i-1} \mu(\Omega \setminus F_i)^{1/n} \leq \|g\|_{L^{n,1}(\Omega)}. \]

Then, if \( l = -\infty \), by letting \( m \to -\infty \) in (5), we obtain

\[ |f(x) - f(y)| \leq 0 + 600 \left( \frac{C_0 R^n}{\mu(\Omega)} \right)^{1/n} \|g\|_{L^{n,1}(\Omega)}, \]

for all \( x, y \in \Omega \setminus E \). Also, if \( l \neq -\infty \), we know that

\[ 2^{l-1} \mu(\Omega)^{1/n} \leq \int_0^{2^{l-1}} \mu(\{ z \in \Omega : g(z) > t \})^{1/n} dt \leq \|g\|_{L^{n,1}(\Omega)}; \]

then, by choosing \( m = l + 1 \) and (5), we obtain

\[ |f(x) - f(y)| \leq 1000 \left( \frac{C_0 R^n}{\mu(\Omega)} \right)^{1/n} \|g\|_{L^{n,1}(\Omega)} \]

for all \( x, y \in \Omega \setminus E \). This completes the proof of the theorem. \( \blacksquare \)

**Remark 2.2.** In Theorem 2.1, we can replace the connectedness assumption and the condition \( B(a, R^-) \neq B(a, R) \) by the following (weaker) assumption:

- There exists a constant \( K > 1 \) such that, for any \( x \in B(a, R) \) and \( r \in [0, R/4] \), we have \( \mu(B(x, r) \cap B(a, R)) < \mu(B(x, Kr) \cap B(a, R)) \).

3. Applications. In this section, we apply Theorem 2.1 to study the continuity and differentiability of Sobolev type functions. First, we recall some definitions (see [M1] and [KKM]).

**Definition 3.1.** Let \( (X, d, \mu) \) be metric-measure space and let \( f \) be a real-valued function on \( X \). We say that \( f \) is essentially \( m \)-absolutely continuous, for some \( m > 0 \), if for any \( \epsilon > 0 \), there is \( \delta > 0 \) such that for any family \( \{B_i\} \) of pairwise disjoint balls in \( X \), we have

\[ \sum_i \text{ess sup}_{x,y \in B_i} |f(x) - f(y)|^m < \epsilon \quad \text{whenever} \quad \sum_i \mu(B_i) < \delta. \]

Also, the essential \( m \)-variation of \( f \) is defined by

\[ \mathcal{V}_m(f) := \sup_{\{B_i\}} \left\{ \sum_i \text{ess sup}_{x,y \in B_i} |f(x) - f(y)|^m \right\}, \]

where the supremum is taken over all pairwise disjoint families \( \{B_i\} \) of balls in \( X \).
**Theorem 3.2.** Let the notations and assumptions be as in Theorem 2.1. Moreover, assume that $X$ is Bishop–Gromov regular of dimension $n$ and geodesic. Then

(i) For a.e. $a \in X$, we have

$$\limsup_{s \to 0} \esssup_{x,y \in B(a,s)} \frac{|f(x) - f(y)|}{s} < \infty.$$  

(ii) There exists a constant $C$ which depends on $C_0$ and $n$ (as in Definition 1.1 with $\Omega = X$) such that

$$V_n(f) \leq C \left( \sup_{z,s} \frac{s^n}{\mu(B(z,s))} \right)^{n} \|g\|_{L^{n,1}(X)}.$$  

(iii) In addition, if $\sup_{z,s} \frac{s^n}{\mu(B(z,s))} < \infty$, then $f$ is essentially $n$-absolutely continuous.

**Proof.** First, we assume $n > 1$. We have (cf. [M2, Thm. 2.1])

$$\|g\|_{L^{n,1}(B(a,R))} \leq \left( \int_0^\infty \Phi'(t)^{1/(1-n)} dt \right)^{(n-1)/n} \left( \int_0^\infty \Phi'(t) \mu(\{ z \in B(a,R) : g(z) > t \}) dt \right)^{1/n},$$

where $\Phi(t) := \left( \int_0^t \mu(\{ z \in B(a,R) : g(z) > s \})^{(1-n)/n} ds \right)$. Then we obtain

$$\|g\|_{L^{n,1}(B(a,R))} \leq L \|g\|_{L^{n,1}(B(a,R))}^{1/n} \|g\|_{L^{n,1}(B(a,R))}^{1/n} < \infty,$$

where $L := \|g\|_{L^{n,1}(X)}^{(n-1)/n}$. Also, it is easy to see that (6) is meaningful and valid when $n = 1$ (with $L = 1$ and $\Phi(t) \equiv t$).

Then, by Proposition 1.4, (3) and (6), we have

$$\esssup_{x,y \in B(a,s)} \frac{|f(x) - f(y)|}{s} \leq \frac{1000}{s} \left( \frac{C s^n}{\mu(B(a,s))} \right)^{1/n} \|g\|_{L^{n,1}(B(a,s))} \leq LC \left( \frac{1}{\mu(B(a,s))} \int_{B(a,s)} \Phi \circ g d\mu \right)^{1/n},$$

where $C$ is a constant which depends on $C_0$ and $n$ (associated with $X$). Now, Lebesgue’s differentiation theorem (for doubling spaces, see for instance [He, Thm. 1.8]) implies (i). On the other hand, assume that $\{B(a_i, s_i)\}$ is any family of pairwise disjoint balls in $X$. Again, by Proposition 1.4, (2.1) and (3.1), we get...
\[
\sum_i \text{ess sup}_{x,y \in B(a_i, s_i)} |f(x) - f(y)|^n \leq L^n C^n \sum_i \frac{s_i^n}{\mu(B(a_i, s_i))} \int_{B(a_i, s_i)} \Phi \circ g \, d\mu \\
\leq L^n C^n \left( \sup_{z, s} \frac{s^n}{\mu(B(z, s))} \right) \sum_i \int_{B(a_i, s_i)} \Phi \circ g \, d\mu \\
= L^n C^n \left( \sup_{z, s} \frac{s^n}{\mu(B(z, s))} \right) \int_{\bigcup_i B(a_i, s_i)} \Phi \circ g \, d\mu.
\]

This completes the proof of theorem. ■

**Remark 3.3.** If we replace “ess sup” by “sup” in Definition 3.1 and Theorem 3.2(iii), then the conclusion of Theorem 3.2(iii) remains valid after changing \( f \) on a set of measure zero.

Now, by the implications of the Poincaré inequality [HK, Thms. 3.2 and 3.4], properties of the Hardy–Littlewood maximal function [He, Thm. 2.2] (see also [KK, Thm. 1.2.1]), Marcinkiewicz’s interpolation theorem for Lorentz spaces [BS, Ch. 4, Thm. 4.13, Rem. 4.15], characterization of the Lorentz space \( L^{n,1}(X) \) by Orlicz spaces [Cc] (see also [CL], [KKM] and [AC]) and properties of \( A_p \)-weights [KK, Thm. 5.3.1], we are able to recover and sharpen the results of [MS] for \( A_p \)-weighted Sobolev functions. Moreover, Theorem 3.2 extends the main results of [KKM] and [AC, Thm. 1.1] (see also [Cc] and [CL]) for functions whose domain is an abstract metric-measure space (see also [R]). Compare [S], [DS], [M1] and [Ca].

**Acknowledgements.** The author would like to thank the Research Council of Sharif University of Technology for support.

**References**


Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11365-9415
Tehran, Iran
E-mail: ranjbarm@sharif.edu

*Received June 6, 2007
Revised version June 18, 2008*