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Non-commutative martingale VMO-spaces

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Abstract. We study Banach space properties of non-commutative martingale VMO-spaces associated with general von Neumann algebras. More precisely, we obtain a version of the classical Kadets–Pełczyński dichotomy theorem for subspaces of non-commutative martingale VMO-spaces. As application we prove that if $\mathcal{M}$ is hyperfinite then the non-commutative martingale VMO-space associated with a filtration of finite-dimensional von Neumannn subalgebras of $\mathcal{M}$ has property (u).

1. Introduction. The space of functions of *bounded mean oscillation* generally referred to as BMO-space has been instrumental in several aspects of analysis. Its martingale version plays an equally important role in probability.

In this paper, we analyze subspaces of BMO-spaces related to non-commutative martingales. Our main motivation comes primarily from a paper by Müller and Schechtman [14] who studied structural properties of closed subspaces of dyadic martingale VMO (*vanishing mean oscillation*) as Banach spaces. More precisely, they provided, among other things, a version of the classical Kadets–Pełczyński dichotomy theorem for closed subspaces of dyadic martingale VMO-spaces. In order to explain the details, we first recall the celebrated Kadets–Pełczyński dichotomy theorem, which states that every closed subspace of $L^p(0,1)$, $2 < p < \infty$, either is isomorphic to a Hilbert space or contains a subspace which is isomorphic to l^p. This dichotomy plays a crucial role in the development of L^p-spaces and the theory of function spaces in general. Non-commutative analogues of the Kadets–Pełczyński dichotomy has been considered by several authors with the most general result obtained by Raynaud and Xu (see [21]) in the context of Haagerup L^p-spaces when $2 \leq p < \infty$. Clearly, the dichotomy does not extend to closed subspaces of $L^\infty(0,1)$ or any $C(K)$-spaces in general. As a substitute, the following result was obtained by Müller and Schechtman:

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Theorem 1.1 ([14]). *Let X be a subspace of the dyadic VMO. Then either X is isomorphic to a Hilbert space or X contains a subspace isomorphic to c_0.*

The result of Raynaud and Xu naturally leads to the question whether Theorem 1.1 generalizes to non-commutative martingale VMO. Our main result (Theorem 3.2) can be viewed as a characterization of subspaces of non-commutative VMO that contain isomorphic copies of c_0, which for the hyperfinite case (Theorem 3.4) becomes an exact non-commutative analogue of Theorem 1.1. As application, we deduce that as Banach space, non-commutative VMO has the so called property (u).

The paper is organized as follows: in Section 2, we recall the construction of Haagerup L^p-spaces associated with general von Neumannn algebras, review the general setup for non-commutative martingales, and introduce non-commutative VMO-spaces. In Section 3, we formulate the appropriate analogue of the Kadets–Pełczyński dichotomy theorem for VMO-spaces. In the last section, we discuss applications of our main result to Banach space structures of both VMO and BMO.

2. Notation and preliminary definitions. We use standard notation in operator algebras and Banach space theory. We refer to [11, 23] for background on von Neumannn algebra theory and to [3, 12] for unexplained terminology from Banach space theory. Throughout, we assume that $\mathcal{M}$ is a σ-finite von Neumannn algebra acting on a Hilbert space H, and φ is a distinguished normal faithful state on $\mathcal{M}$. We denote by $\sigma_t = \sigma_t^{\varphi}$ the one-parameter modular automorphism group on $\mathcal{M}$ associated with φ. The Haagepup L^p-spaces associated with $(\mathcal{M}, \varphi)$ are defined from the cross-product $\mathcal{R} = \mathcal{M} \rtimes_{\sigma_t} \mathbb{R}$. We recall that $\mathcal{R}$ is the von Neumannn subalgebra of $\mathcal{B}(L^2(\mathbb{R}, H))$ generated by the operators $\pi(x)$, $x \in \mathcal{M}$, and $\lambda(s)$, $s \in \mathbb{R}$, where

$$\pi(x)(\xi(t)) = \sigma_{-t}^{\varphi}(x)(\xi(t)) \quad \text{and} \quad \lambda(s)(\xi(t)) = \xi(t-s)$$

for $t \in \mathbb{R}$ and $\xi \in L^2(\mathbb{R}, H)$. If $W(s)$ is the unitary operator on $L^2(\mathbb{R}, H)$ defined by

$$W(s)(\xi(t)) = e^{-ist}\xi(t),$$

then the dual action θ on $\mathcal{R}$ is given by

$$\theta_s(x) = W(s)xW(s)^*, \qquad x \in \mathcal{R}.$$

The von Neumannn algebra $\mathcal{M}$ can be identified with the subalgebra of $\mathcal{R}$

$$\pi(\mathcal{M}) = \{x \in \mathcal{R} : \theta_s(x) = x \text{ for all } s \in \mathbb{R}\}.$$

Moreover, it is known that $\mathcal{R}$ is a semifinite von Neumannn algebra and admits a canonical normal faithful semifinite trace τ satisfying

$$\tau \circ \theta_s = e^{-s}\tau, \qquad s \in \mathbb{R},\, x \in \mathcal{R}.$$

For $1 \leq p \leq \infty$, the *Haagerup L^p-space* associated with $\mathcal{M}$, denoted by $L^p(\mathcal{M})$, is defined as the space of τ-measurable operators x affiliated with $\mathcal{R}$ such that for all $s \in \mathbb{R}$,

$$\theta_s(x) = e^{-s/p}x.$$

This is clearly a closed self-adjoint linear subspace of the space of τ-measurable operators affiliated with $\mathcal{R}$, and $L^\infty(\mathcal{M})$ coincides with $\mathcal{M}$. Moreover, there is a canonical isomorphism between $L^1(\mathcal{M})$ and the predual $\mathcal{M}_*$ of $\mathcal{M}$ which we now describe:

Every normal semifinite faithful weight $\psi \in (\mathcal{M}_*)_+$ is given by a density $h_\psi \in L^1(\mathcal{M})_+$ satisfying

$$\tau(h_\psi x) = \int_{\mathbb{R}} \psi(\theta_s(x))\, ds$$

for all $x \in \mathcal{R}_+$. Using polar decomposition of an arbitrary element $\psi \in \mathcal{M}_*$, this correspondence between $(\mathcal{M}_*)_+$ and $L^1(\mathcal{M})_+$ extends to a bijection between $\mathcal{M}_*$ and $L^1(\mathcal{M})$. Indeed, if $\psi \in \mathcal{M}_*$, then $\psi = u|\psi|$, where $u \in \mathcal{M}$ and $|\psi|$ is the modulus of ψ. The corresponding $h_\psi \in L^1(\mathcal{M})$ then admits the polar decomposition

$$h_\psi = u|h_\psi| = uh_{|\psi|}.$$

The norm on $L^1(\mathcal{M})$ is defined by setting

$$\|h_\psi\|_1 = |\psi|(\mathbf{1}) = \|\psi\|_{\mathcal{M}_*}, \qquad \psi \in \mathcal{M}_*.$$

With this norm, $L^1(\mathcal{M})$ and $\mathcal{M}_*$ are isometric. Furthermore, one can define a distinguished positive linear functional tr on $L^1(\mathcal{M})$ called *trace* by

$$\operatorname{tr}(h_\psi) = \psi(\mathbf{1}), \qquad \psi \in \mathcal{M}_*.$$

Given $1 \leq p < \infty$ and $x \in L^p(\mathcal{M})$, we have $|x|^p \in L^1(\mathcal{M})$. Define

$$\|x\|_p := \|\, |x|^p\|_1^{1/p} = (\operatorname{tr}(|x|^p))^{1/p}.$$

Equipped with $\|\cdot\|_p$, the space $L^p(\mathcal{M})$ is a Banach space.

Throughout, D denotes the Radon–Nikodym derivative (with respect to τ) of the dual weight $\widetilde{\varphi}$ of the distinguished state φ. The state φ can be recovered from tr by the identity

$$\varphi(x) = \operatorname{tr}(Dx), \qquad x \in \mathcal{M}.$$

If the von Neumannn algebra $\mathcal{M}$ is semifinite equipped with a faithful normal semifinite trace then the Haagerup L^p-spaces reduce to the usual noncommutative L^p-spaces constructed from the theory of non-commutative integration as described in [16]. The reader is referred to [7, 24] for full details of Haagerup's theory.

Let us now recall the general setup for non-commutative martingales. The reader is referred to [4, 2, 5] for the classical (commutative) martingale theory. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing sequence of von Neumannn subalgebras of $\mathcal{M}$ such that the union of the $\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$ and $\sigma_t(\mathcal{M}_n) \subset \mathcal{M}_n$ for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$. We further assume that if $(p_n)_{n\geq 1}$ is the increasing sequence of projections in $\mathcal{M}$ consisting of the units of $\mathcal{M}_n$'s then $\sigma_t(p_n) = p_n$ for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$.

For each $n \geq 1$, it follows from [22] that there exists a normal conditional expectation $\widetilde{\mathcal{E}}_n : p_n\mathcal{M}p_n \to \mathcal{M}_n$ satisfying

$$\widetilde{\mathcal{E}}_n(\sigma_t(x)) = \sigma_t(\widetilde{\mathcal{E}}_n(x))$$

for all $x \in p_n\mathcal{M}p_n$ and all $t \in \mathbb{R}$. We now define $\mathcal{E}_n : \mathcal{M} \to \mathcal{M}_n$ by setting, for $x \in \mathcal{M}$,

$$\mathcal{E}_n(x) = \widetilde{\mathcal{E}}_n(p_n x p_n).$$

It is clear that for every m and n in $\mathbb{N}$, $\mathcal{E}_m\mathcal{E}_n = \mathcal{E}_n\mathcal{E}_m = \mathcal{E}_{\min(n,m)}$. Moreover, for $1 \leq p < \infty$, the conditional expectation $\mathcal{E}_n$ extends in a natural way to a contractive projection from $L^p(\mathcal{M})$ onto $L^p(\mathcal{M}_n)$ (we refer to [10, Lemma 2.2] for details).

DEFINITION 2.1. For $1 \leq p \leq \infty$, a *non-commutative martingale* in $L^p(\mathcal{M})$ with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ in $L^p(\mathcal{M})$ satisfying

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all } n \geq 1.$$

For such a martingale $x \subset L^p(\mathcal{M})$ $(1 \leq p \leq \infty)$, we set

$$\|x\|_p := \sup_{n\geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, then x is called a *bounded L^p-martingale*. The *difference sequence* $dx = \{d_n(x)\}_{n\geq 1}$ of a martingale $x = (x_n)_{n\geq 1}$ is defined by

$$d_n(x) = x_n - x_{n-1}, \quad n \geq 1,$$

with the usual convention that $x_0 = 0$.

We note that for $1 < p < \infty$, every bounded L^p-martingale is of the form $(\mathcal{E}_n(x_\infty))_{n\geq 1}$ for some $x_\infty \in L^p(\mathcal{M})$. We often identify a martingale with its final value whenever the latter exists. For some concrete natural examples of non-commutative martingales, we refer to [20] and the survey paper [26].

We will now describe square functions of non-commutative martingales and non-commutative martingale Hardy spaces. Our main references for classical martingale Hardy spaces are the monographs [13, 25]. Following Pisier and Xu [20], we will consider the following row and column versions of square functions. For a martingale $x = (x_n)_{n=1}^\infty$ and $N \geq 1$, set

$$S_{\mathrm{C},N}(x) = \Big(\sum_{j=1}^N |d_j(x)|^2\Big)^{1/2} \quad \text{and} \quad S_{\mathrm{R},N}(x) = \Big(\sum_{j=1}^N |d_j(x^*)|^2\Big)^{1/2}.$$

For $1 \le p \le \infty$ and any finite sequence $a = (a_n)_{n\ge1}$ in $L^p(\mathcal{M})$, set

$$\|a\|_{L^p(\mathcal{M};l^2_{\mathrm{C}})} = \Big\| \Big(\sum_{n\ge1} |a_n|^2 \Big)^{1/2} \Big\|_{L^p(\mathcal{M},\tau)},$$
$$\|a\|_{L^p(\mathcal{M};l^2_{\mathrm{R}})} = \Big\| \Big(\sum_{n\ge1} |a_n^*|^2 \Big)^{1/2} \Big\|_{L^p(\mathcal{M},\tau)}.$$

The difference sequence dx belongs to $L^p(\mathcal{M}; l^2_{\mathrm{C}})$ (respectively, $L^p(\mathcal{M}; l^2_{\mathrm{R}})$) if and only if the sequence $(S_{\mathrm{C},n}(x))_{n=1}^\infty$ (respectively, $(S_{\mathrm{R},n}(x))_{n=1}^\infty$) is bounded in $L^p(\mathcal{M})$. In this case, the limits

$$S_{\mathrm{C}}(x) = \Big(\sum_{k=1}^\infty |d_k(x)|^2 \Big)^{1/2} \quad \text{and} \quad S_{\mathrm{R}}(x) = \Big(\sum_{k=1}^\infty |d_k(x)^*|^2 \Big)^{1/2}$$

are elements of $L^p(\mathcal{M})$. These two versions of square functions are crucial for the definition of non-commutative (martingale) Hardy spaces which we now describe for $p = 1$ (we refer to [20] for the other values of p). The space $\mathcal{H}^1_{\mathrm{C}}(\mathcal{M})$ (respectively, $\mathcal{H}^1_{\mathrm{R}}(\mathcal{M})$) is defined as the set of all L^1-martingales x with respect to the filtration $(\mathcal{M}_n)_{n\ge1}$ for which $dx \in L^1(\mathcal{M}; l^2_{\mathrm{C}})$ (respectively, $L^1(\mathcal{M}; l^2_{\mathrm{R}})$). For such x, we set

$$\|x\|_{\mathcal{H}^1_{\mathrm{C}}(\mathcal{M})} = \|S_{\mathrm{C}}(x)\|_1 \quad \text{and} \quad \|x\|_{\mathcal{H}^1_{\mathrm{R}}(\mathcal{M})} = \|S_{\mathrm{R}}(x)\|_1.$$

Equipped with the above norms, $\mathcal{H}^1_{\mathrm{C}}(\mathcal{M})$ and $\mathcal{H}^1_{\mathrm{R}}(\mathcal{M})$ are Banach spaces. The non-commutative martingale Hardy space $\mathcal{H}^1(\mathcal{M})$ of non-commutative martingales is defined as

$$\mathcal{H}^1(\mathcal{M}) = \mathcal{H}^1_{\mathrm{C}}(\mathcal{M}) + \mathcal{H}^1_{\mathrm{R}}(\mathcal{M})$$

equipped with the usual norm of sum of two Banach spaces:

$$\|x\|_{\mathcal{H}^1(\mathcal{M})} = \inf\{ \|y\|_{\mathcal{H}^1_{\mathrm{C}}(\mathcal{M})} + \|z\|_{\mathcal{H}^1_{\mathrm{R}}(\mathcal{M})} \}$$

where the infimum is taken over all y and z with $x = y + z$, $y \in \mathcal{H}^1_{\mathrm{C}}(\mathcal{M})$ and $z \in \mathcal{H}^1_{\mathrm{R}}(\mathcal{M})$.

The dual of $\mathcal{H}^1(\mathcal{M})$ can be identified (up to a constant $\sqrt{2}$) to a non-commutative analogue of martingale BMO which is the main object of this paper. BMO-spaces associated with non-commutative martingales were introduced by Pisier and Xu in [20] for the finite case and [10] for the general case. It was proved among other results that Fefferman's classical H^1-BMO duality extends to this setting. In order to define such BMO-spaces for this context, we consider first what is known as the column BMO by setting

$$\mathrm{BMO}_{\mathrm{C}}(\mathcal{M}) := \{ x = (x_n) : \sup_m \sup_{n\le m} \|\mathcal{E}_n |x_m - x_{n-1}|^2\|_\infty < \infty \}.$$

Then $\mathrm{BMO_C}(\mathcal{M})$ becomes a Banach space when equipped with the norm

$$\|x\|_{\mathrm{BMO_C}} = (\sup_{m\geq 1}\sup_{n\leq m} \|\mathcal{E}_n|x_m - x_{n-1}|^2\|_\infty)^{1/2}.$$

Similarly, the row version $\mathrm{BMO_R}(\mathcal{M})$ is defined as the space of all x for which $x^* \in \mathrm{BMO_C}(\mathcal{M})$, equipped with the natural norm $\|x\|_{\mathrm{BMO_R}(\mathcal{M})} = \|x^*\|_{\mathrm{BMO_C}(\mathcal{M})}$. The space $\mathrm{BMO}(\mathcal{M})$ associated with the filtration $(\mathcal{M}_n)$ is the intersection of the two types of BMO-spaces described above:

$$\mathrm{BMO}(\mathcal{M}) := \mathrm{BMO_C}(\mathcal{M}) \cap \mathrm{BMO_R}(\mathcal{M})$$

with the usual intersection norm of two Banach spaces:

$$\|x\|_{\mathrm{BMO}} = \max\{\|x\|_{\mathrm{BMO_C}}, \|x\|_{\mathrm{BMO_R}}\}.$$

As noted above, $(\mathcal{H}^1(\mathcal{M}))^* = \mathrm{BMO}(\mathcal{M})$ isomorphically with the isomorphism given by the fact that any $y \in \mathrm{BMO}(\mathcal{M})$ defines a linear functional ξ_y on $\mathcal{H}^1(\mathcal{M})$ defined by $\xi_y(x) = \lim_{n\to\infty} \mathrm{tr}(y_n^* x_n)$ for all $x \in \mathcal{H}^1(\mathcal{M})$.

For more information on non-commutative martingale BMO-spaces, we refer to the articles [20, 10, 15, 9]. Special attention will be given to the subspace called *vanishing mean oscillation*, denoted by $\mathrm{VMO}(\mathcal{M})$, and defined as the closure (for the BMO-norm) of the linear subspace of those $x \in \mathrm{BMO}(\mathcal{M})$ for which $\mathcal{E}_n(x) = x$ for some $n \in \mathbb{N}$.

As in the classical case, for $1 \leq p < \infty$, $\mathrm{BMO}(\mathcal{M})$ is a subset of $L^p(\mathcal{M})$ in the sense of Proposition 2.2 below. These inclusions will be used repeatedly below.

Proposition 2.2 ([9, Theorem 3.18]). *For $1 \leq p < \infty$ and $0 \leq \eta \leq 1$, let $I_p^\eta(x) = D^{(1-\eta)/p} x D^{\eta/p}$. Then the inclusion map*

$$I_p^\eta(\mathrm{BMO}) \subset L^p(\mathcal{M})$$

is bounded with norm $c(p) \leq cp$, where c is an absolute constant.

3. Main results. Our primary objective in this section is to provide a Kadets–Pełczyński type alternative for VMO-spaces. It generalizes a result of Müller and Schechtman from [14] stated in Theorem 1.1 to non-commutative setting.

For the general statement, we introduce the following norm for x in $\mathrm{BMO}(\mathcal{M})$:

$$\|\!|x\|\!|_{\mathrm{C}} := \sum_{n\geq 1} 2^{-n} \|\mathcal{E}_n(|x - \mathcal{E}_{n-1}(x)|^2)\|_\infty^{1/2}. \tag{3.1}$$

As above, the row version is defined by $\|\!|x\|\!|_{\mathrm{R}} := \|\!|x^*\|\!|_{\mathrm{C}}$ and

$$\|\!|x\|\!| := \max\{\|\!|x\|\!|_{\mathrm{C}}, \|\!|x\|\!|_{\mathrm{R}}\}. \tag{3.2}$$

One can easily verify that $\|\!|\cdot\|\!|$ is a norm in the linear space $\mathrm{BMO}(\mathcal{M})$ with $\|\!|x\|\!| \leq \|x\|_{\mathrm{BMO}}$ for all $x \in \mathrm{BMO}(\mathcal{M})$. We will see below that in general $\|\!|\cdot\|\!|$

is not equivalent to the usual BMO-norm. The following property of $\| \cdot \|$ is needed in the proof of our main result.

LEMMA 3.1. *For every $m_0 \in \mathbb{N}$ and $z \in \mathrm{VMO}(\mathcal{M})$,*

$$\|\mathcal{E}_{m_0}(z)\|_{\mathrm{VMO}} \le 2^{m_0} \|\!| z |\!\|.$$

Proof. If $m > m_0$, then

$$\|\mathcal{E}_m|\mathcal{E}_{m_0}(z) - \mathcal{E}_{m-1}(\mathcal{E}_{m_0}(z))|^2\|_\infty = \|\mathcal{E}_m|\mathcal{E}_{m_0}(z) - \mathcal{E}_{m_0}(z)|^2\|_\infty = 0.$$

We deduce from this observation that

$$\begin{aligned}\|\mathcal{E}_{m_0}(z)\|_{\mathrm{BMO_C}} &= \sup_{m\le m_0} \|\mathcal{E}_m|\mathcal{E}_{m_0}(z) - \mathcal{E}_{m-1}(\mathcal{E}_{m_0}(z))|^2\|_\infty^{1/2} \\ &\le \sup_{m\le m_0} \|\mathcal{E}_m|z - \mathcal{E}_{m-1}(z)|^2\|_\infty^{1/2} \le 2^{m_0} \|\!| z |\!\|_{\mathrm{C}}.\end{aligned}$$

The lemma follows with similar estimate on $\|\mathcal{E}_{m_0}(z)\|_{\mathrm{BMO_R}}$. ■

Our first result is a characterization of subspaces of VMO containing subspaces isomorphic to c_0 in terms of the norm $\|\!| \cdot |\!\|$.

THEOREM 3.2. *Let X be a subspace of* $\mathrm{VMO}(\mathcal{M})$. *Then either X contains a subspace isomorphic to c_0 or the norm $\|\!| \cdot |\!\|$ and the usual* $\mathrm{VMO}(\mathcal{M})$-*norm are equivalent on X.*

Theorem 3.2 will be deduced from the following result:

PROPOSITION 3.3. (a) *Let $(p_k)_{k\ge1}$ be a strictly increasing sequence of positive integers, $(\varepsilon_k)_{k\ge1} \subset \mathbb{R}_+$ satisfying $\sum_{k=1}^\infty \varepsilon_k = \delta$ and $(x_k)_{k\ge1} \subset$* $\mathrm{VMO}(\mathcal{M})$ *with*:

(i) $\mathcal{E}_{p_k}(x_k) = x_k$ *and* $\mathcal{E}_{p_{k-1}}(x_k) = 0$;
(ii) $\|x_k\|_{\mathrm{VMO}} = 1$;
(iii) $\|\mathcal{E}_{p_{k-1}}(S_{\mathrm{C}}^2(x_k))\|_\infty \le \varepsilon_k$.

Then for any finitely non-zero sequence $(\alpha_k)_{k\ge1} \subset B(l^2)$,

$$\sup_{k\ge1} \|\alpha_k\|_\infty \le \Big\|\sum_{k\ge1} \alpha_k \otimes x_k\Big\|_{\mathrm{BMO_C}(B(l^2)\overline{\otimes}\mathcal{M})} \le (1+\delta)^{1/2} \sup_{k\ge1} \|\alpha_k\|_\infty,$$

where the BMO-*norm is relative to the filtration $(B(l^2) \overline{\otimes} \mathcal{M}_n)_{n\ge1}$. In particular, $(x_k)_{k\ge1}$ is a basic sequence that is $(1+\delta)^{1/2}$-equivalent to the unit vector basis of c_0 in* $\mathrm{BMO_C}(\mathcal{M})$.

(b) *If one replaces condition* (iii) *by the weaker condition that*

$$\|\mathcal{E}_{p_{k-2}}(S_{\mathrm{C}}^2(x_k))\|_\infty \le \varepsilon_k \quad \textit{for all } k \ge 3,$$

then the same conclusion still holds with constant $(2+\delta)^{1/2}$.

Proof. Let $(\alpha_k)_k$ be a finitely non-zero sequence in $B(l^2)$ and set for convenience

$$a = \sum_k \alpha_k \otimes x_k \in B(l^2) \overline{\otimes} \mathcal{M}.$$

For $n \geq 1$, let $\mathbb{E}_n = \mathrm{Id} \otimes \mathcal{E}_n : B(l^2)\overline{\otimes}\mathcal{M} \to B(l^2)\overline{\otimes}\mathcal{M}_n$ be the corresponding conditional expectation. To prove that $\|a\|_{\mathrm{BMO_C}} \leq (1+\delta)^{1/2} \sup_k \|\alpha_k\|_\infty$, we need to estimate $\|\mathbb{E}_m|a - \mathbb{E}_{m-1}(a)|^2\|_\infty$ for all $m \geq 1$. To this end, fix $m \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ so that $p_{n-1} < m \leq p_n$.

First, $\mathbb{E}_m|a - \mathbb{E}_{m-1}(a)|^2 = \mathbb{E}_m(\sum_{s\geq m} |d_s(a)|^2)$. Moreover, we observe that for every $s \geq m$,

$$d_s(a) = \sum_{k\geq n} \alpha_k \otimes d_s(x_k). \tag{3.3}$$

Indeed, if $k < n$ then $p_k < p_{n-1} < m \leq s$ so $x_k \in \mathcal{M}_{p_k} \subset \mathcal{M}_{s-1}$ and therefore $d_s(x_k) = 0$. On the other hand, if $l, k \geq n$ with $l \neq k$, then

$$\mathcal{E}_m(d_s(x_l^*)d_s(x_k)) = 0. \tag{3.4}$$

To see this, assume that $l > k \geq n$. Then $m \leq p_k < p_{l-1} < p_l$. By assumption, x_k is in $\mathcal{M}_{p_k} \subset \mathcal{M}_{p_{l-1}}$ and so also is $d_s(x_k)$. We deduce that

$$\begin{aligned}\mathcal{E}_m(d_s(x_l^*)d_s(x_k)) &= \mathcal{E}_m\big(\mathcal{E}_{p_{l-1}}(d_s(x_l^*))d_s(x_k)\big)\\ &= \mathcal{E}_m\big([\mathcal{E}_s\mathcal{E}_{p_{l-1}}(x_l^*) - \mathcal{E}_{s-1}\mathcal{E}_{p_{l-1}}(x_l^*)]d_s(x_k)\big) = 0.\end{aligned}$$

Combining (3.3) and (3.4), we have $|d_s(a)|^2 = \sum_{k\geq n} |\alpha_k|^2 \otimes |d_s(x_k)|^2$. Thus we have the following estimates:

$$\begin{aligned}\mathbb{E}_m|a - \mathbb{E}_{m-1}(a)|^2 &= \mathbb{E}_m\Big(\sum_{s\geq m}|d_s(a)|^2\Big) = \sum_{s\geq m}\sum_{k\geq n}|\alpha_k|^2 \otimes \mathcal{E}_m|d_s(x_k)|^2\\ &\leq |\alpha_n|^2 \otimes \mathcal{E}_m\Big(\sum_{s\geq m}|d_s(x_n)|^2\Big) + \sum_{k\geq n+1}|\alpha_k|^2 \otimes \mathcal{E}_m(S_{\mathrm{C}}^2(x_k))\\ &\leq |\alpha_n|^2 \otimes \|x_n\|_{\mathrm{BMO_C}}^2 \mathbf{1} + \sum_{k\geq n+1}|\alpha_k|^2 \otimes \mathcal{E}_m(S_{\mathrm{C}}^2(x_k)).\end{aligned} \tag{3.5}$$

Observe that for $k \geq n+1$, it follows from the choice of n above that $m \leq p_{k-1}$. From condition (iii), we have for every $k \geq n+1$,

$$\mathcal{E}_m(S_{\mathrm{C}}^2(x_k)) = \mathcal{E}_m(\mathcal{E}_{p_{k-1}}(S_{\mathrm{C}}^2(x_k))) \leq \varepsilon_k \mathcal{E}_m(\mathbf{1}).$$

We conclude that $\|\mathbb{E}_m|a - \mathbb{E}_{m-1}(a)|^2\|_\infty \leq \|\alpha_n\|_\infty^2 + \sum_{k\geq n+1}\|\alpha_k\|_\infty^2 \varepsilon_k$ and taking supremum over m, we see that $\|a\|_{\mathrm{BMO_C}} \leq (1+\delta)^{1/2}\sup_k \|\alpha_k\|_\infty$. Thus the proof of (a) is complete.

For (b), we only need to adjust the last two lines of (3.5) and write

$$\begin{aligned}\mathbb{E}_m|a - \mathbb{E}_{m-1}(a)|^2 \leq |\alpha_n|^2 \otimes \|x_n\|_{\mathrm{BMO_C}}^2 \mathbf{1} + |\alpha_{n+1}|^2 \otimes \|x_{n+1}\|_{\mathrm{BMO_C}}^2 \mathbf{1}\\ + \sum_{k\geq n+2}|\alpha_k|^2 \otimes \mathcal{E}_m(S_{\mathrm{C}}^2(x_k)).\end{aligned}$$

As above, if $k \geq n+2$ then $m < p_{k-2}$ and therefore

$$\mathcal{E}_m(S_{\mathrm{C}}^2(x_k)) = \mathcal{E}_m(\mathcal{E}_{p_{k-2}}(S_{\mathrm{C}}^2(x_k)))$$

and hence $\|\mathcal{E}_m(S^2_{\mathrm{C}}(x_k))\|_\infty \le \varepsilon_k$. We conclude as above that

$$\begin{aligned}\|\mathbb{E}_m|a-\mathbb{E}_{m-1}(a)|^2\|_\infty &\le \|\alpha_n\|^2_\infty + \|\alpha_{n+1}\|^2_\infty + \sum_{k\ge n+1}\|\alpha_k\|^2_\infty\varepsilon_k\\ &\le (2+\delta)\sup_{k\ge 1}\|\alpha_k\|^2_\infty.\end{aligned}$$

The proof is complete. ∎

Proof of Theorem 3.2. Let X be a subspace of $\mathrm{VMO}(\mathcal{M})$ such that the $\mathrm{BMO}(\mathcal{M})$-norm and $|\!|\!|\cdot|\!|\!|$ are not equivalent on X. There exists a sequence (z_n) in the unit sphere of $(X, \|\cdot\|_{\mathrm{BMO}})$ with $\lim_{n\to\infty}|\!|\!|z_n|\!|\!| = 0$. We claim that there exists a subsequence (z_{n_k}) of (z_n) and a sequence (x_k) satisfying the assumptions of Proposition 3.3 and such that $\|z_{n_k} - x_k\| < 2^{-k}$ for all $k \ge 1$. This is done by induction. First observe that $\lim_{n\to\infty}|\!|\!|z_n|\!|\!| = 0$ is equivalent to

$$\lim_{n\to\infty}\max\{\|\mathcal{E}_m|z_n - \mathcal{E}_{m-1}(z_n)|^2\|_\infty, \|\mathcal{E}_m|z_n^* - \mathcal{E}_{m-1}(z_n^*)|^2\|_\infty\} = 0, \tag{3.6}$$

for all $m\in\mathbb{N}$. Furthermore, it follows from Lemma 3.1 that for every $m\in\mathbb{N}$,

$$\lim_{n\to\infty}\|\mathcal{E}_m(z_n)\|_{\mathrm{VMO}} = 0. \tag{3.7}$$

Let $n_1 = 1$ and $p_0 = 1$. Choose $p_1 > 1$ such that

$$\|z_1 - \mathcal{E}_{p_1}(z_1)\| < 4^{-1}.$$

For such p_1, we can choose $n_2 > n_1 = 1$ and $p_2 > p_1$ such that

$$\sup_{m\le p_1}\max\{\|\mathcal{E}_m|z_{n_2} - \mathcal{E}_{m-1}(z_{n_2})|^2\|_\infty, \|\mathcal{E}_m|z_{n_2}^* - \mathcal{E}_{m-1}(z_{n_2}^*)|^2\|_\infty\} \le \varepsilon_2,$$

$$\|\mathcal{E}_{p_1}(z_{n_2})\|_{\mathrm{VMO}} \le 4^{-2},$$

and

$$\|z_{n_2} - \mathcal{E}_{p_2}(z_{n_2})\|_{\mathrm{VMO}} \le 4^{-2}.$$

Assume that $n_{k-1} > \cdots > n_1$ and $p_{k-1} > \cdots > p_1$ have been chosen. One can choose $n_k > n_{k-1}$ and $p_k > p_{k-1}$ such that

$$\begin{aligned}\sup_{m\le p_{k-1}}\max\{&\|\mathcal{E}_m|z_{n_k} - \mathcal{E}_{m-1}(z_{n_k})|^2\|_\infty,\\ &\|\mathcal{E}_m|z_{n_k}^* - \mathcal{E}_{m-1}(z_{n_k}^*)|^2\|_\infty\} \le \varepsilon_k,\end{aligned} \tag{3.8}$$

$$\|\mathcal{E}_{p_{k-1}}(z_{n_k})\|_{\mathrm{VMO}} \le 4^{-k}, \tag{3.9}$$

and

$$\|z_{n_k} - \mathcal{E}_{p_k}(z_{n_k})\|_{\mathrm{VMO}} \le 4^{-k}. \tag{3.10}$$

For $k\ge 1$, set

$$x_k = \mathcal{E}_{p_k}(z_{n_k}) - \mathcal{E}_{p_{k-1}}(z_{n_k}). \tag{3.11}$$

We claim that the sequence (x_k) satisfies the following:

(i) $\mathcal{E}_{p_k}(x_k) = x_k$ and $\mathcal{E}_{p_{k-1}}(x_k) = 0$;
(ii) $\|z_{n_k} - x_k\|_{\mathrm{VMO}} \leq 2^{-k}$;
(iii) $\|\mathcal{E}_{p_{k-1}}(S^2_{\mathrm{C}}(x_k))\|_\infty \leq \varepsilon_k$,
(iv) $\|\mathcal{E}_{p_{k-1}}(S^2_{\mathrm{R}}(x_k))\|_\infty \leq \varepsilon_k$.

The first item is trivial. The second follows from (3.9) and (3.10). In fact, for every $k \geq 1$, we have

$$\|z_{n_k} - x_k\|_{\mathrm{VMO}} \leq \|z_{n_k} - \mathcal{E}_{p_k}(z_{n_k})\|_{\mathrm{VMO}} + \|\mathcal{E}_{p_{k-1}}(z_{n_k})\|_{\mathrm{VMO}} \leq 2.4^{-k} \leq 2^{-k}.$$

To verify (iii) and (iv), we observe that for every $k \in \mathbb{N}$,

$$S^2_{\mathrm{C}}(x_k) = \sum_{j=p_{k-1}+1}^{p_k} |d_j(z_{n_k})|^2.$$

This leads to the following estimates:

$$\begin{aligned}\mathcal{E}_{p_{k-1}}(S^2_{\mathrm{C}}(x_k)) &= \mathcal{E}_{p_{k-1}}\Big(\sum_{j=p_{k-1}+1}^{p_k} |d_j(z_{n_k})|^2\Big) \leq \mathcal{E}_{p_{k-1}}\Big(\sum_{j\geq p_{k-1}} |d_j(z_{n_k})|^2\Big)\\ &= \mathcal{E}_{p_{k-1}}|z_{n_k} - \mathcal{E}_{p_{k-1}-1}(z_{n_k})|^2.\end{aligned}$$

We can deduce from (3.8) that

$$\|\mathcal{E}_{p_{k-1}}(S^2_{\mathrm{C}}(x_k))\|_\infty \leq \|\mathcal{E}_{p_{k-1}}|z_{n_k} - \mathcal{E}_{p_{k-1}-1}(z_{n_k})|^2\|_\infty \leq \varepsilon_k.$$

The same argument may be used for the $S^2_{\mathrm{R}}(x_k)$'s. It follows from Proposition 3.3 that the sequence (x_k) is equivalent to the unit vector basis of c_0 in $\mathrm{VMO}(\mathcal{M})$. We conclude from (ii) that (z_{n_k}) is equivalent to the unit vector basis of c_0 in X. ∎

In general, the two possibilities in Theorem 3.2 are not necessarily mutually exclusive. This is the case when the von Neumannn algebra $\mathcal{M}$ is not hyperfinite. Indeed, assume that the filtration $(\mathcal{M}_n)$ consists of infinite-dimensional von Neumannn subalgebras. It is easy to verify that for every $m \in \mathbb{N}$, $\mathcal{M}_m$ (with its usual operator norm) is isomorphic to a subspace of $\mathrm{VMO}(\mathcal{M})$ (using the canonical inclusion). In particular, $\mathcal{M}_1$ embeds isometrically into $\mathrm{VMO}(\mathcal{M})$. It is clear that $|\!|\!|\cdot|\!|\!|$ and the usual VMO-norm are equivalent on $X = \mathcal{M}_1$ but if $\mathcal{M}_1$ is infinite-dimensional then it also contains isometric copies of c_0 and therefore satisfies both possibilities.

For the case of hyperfinite von Neumannn algebras, the dichotomy from Theorem 3.2 turns out to be mutually exclusive. Moreover, it reduces to characterization of Hilbertian subspaces as in Theorem 1.1. To this end, consider the following Hilbertian norm $|\!|\!|\cdot|\!|\!|_2$ on $\mathrm{BMO}(\mathcal{M})$. For $x \in \mathrm{BMO}(\mathcal{M})$, let

$$|\!|\!|x|\!|\!|_2 := (\|I^0_2(x)\|_2^2 + \|I^1_2(x)\|_2^2)^{1/2} = (\|xD^{1/2}\|_2^2 + \|D^{1/2}x\|_2^2)^{1/2} \tag{3.12}$$

where I^0_2 and I^1_2 are the inclusion maps described in Proposition 2.2.

THEOREM 3.4. *Assume that $\mathcal{M}$ is hyperfinite and $(\mathcal{M}_n)_{n\geq 1}$ is a filtration consisting of finite-dimensional von Neumannn subalgebras of $\mathcal{M}$. Let X be a subspace of* $\mathrm{VMO}(\mathcal{M})$. *Then either X is isomorphic to a Hilbert space and complemented in* $\mathrm{BMO}(\mathcal{M})$ *or X contains a subspace isomorphic to c_0.*

Proof. Let $T_2 = I_2^0 \oplus I_2^1 : \mathrm{BMO}(\mathcal{M}) \to L^2(\mathcal{M}) \oplus_2 L^2(\mathcal{M})$ be the linear map given by $x \mapsto (xD^{1/2}, D^{1/2}x)$ and $S = T_2|_X$.

• If S is an isomorphism then X is isomorphic to the Hilbert space $S(X)$ and if $P : L^2(\mathcal{M}) \oplus_2 L^2(\mathcal{M}) \to S(X)$ is the orthogonal projection then $\Pi := S^{-1}PT_2$ is a projection from $\mathrm{BMO}(\mathcal{M})$ onto X.

• If S is not an isomorphism then the $|\!|\!|\cdot|\!|\!|_2$-norm and the $\mathrm{VMO}(\mathcal{M})$-norm are not equivalent on X in the sense that there exists a sequence $(z_n)_{n\geq 1}$ in the unit sphere of X such that $\lim_{n\to\infty}(\|z_nD^{1/2}\|_2 + \|D^{1/2}z_n\|_2) = 0$. We claim that $\lim_{n\to\infty} |\!|\!|z_n|\!|\!| = 0$. To this end, we will verify that for every $m \in \mathbb{N}$, $\lim_{n\to\infty} \|\mathcal{E}_m|z_n - \mathcal{E}_{m-1}(z_n)|^2\|_\infty = 0$ and $\lim_{n\to\infty} \|\mathcal{E}_m|z_n^* - \mathcal{E}_{m-1}(z_n^*)|^2\|_\infty = 0$.

Let $m \geq 1$ be fixed. Since conditional expectations are bounded on $L^2(\mathcal{M})$, it is clear from the fact that $\mathcal{E}_{m-1}(z_nD^{1/2}) = \mathcal{E}_{m-1}(z_n)D^{1/2}$ (see for instance the proof of [10, Proposition 2.3]) that $\lim_{n\to\infty} \|\mathcal{E}_{m-1}(z_n)D^{1/2}\|_2 = 0$ and therefore

$$\lim_{n\to\infty} \|(z_n - \mathcal{E}_{m-1}(z_n))D^{1/2}\|_2 = 0.$$

Thus

$$\lim_{n\to\infty} \|D^{1/2}|z_n - \mathcal{E}_{m-1}(z_n)|^2 D^{1/2}\|_1 = 0.$$

Since the conditional expectation $\mathcal{E}_m$ is bounded in $L^1(\mathcal{M})$, it follows that

$$\lim_{n\to\infty} \|D^{1/2}\mathcal{E}_m|z_n - \mathcal{E}_{m-1}(z_n)|^2 D^{1/2}\|_1 = 0. \tag{3.13}$$

On the other hand, $(\mathcal{E}_m|z_n - \mathcal{E}_{m-1}(z_n)|^2)_{n\geq 1}$ is a bounded sequence in the finite-dimensional space $\mathcal{M}_m$. Any subsequence of $(\mathcal{E}_m|z_n - \mathcal{E}_{m-1}(z_n)|^2)_{n\geq 1}$ has a further subsequence that is convergent in $\mathcal{M}_m$ and since the inclusion map $I_1^{1/2} : \mathcal{M} \to L^1(\mathcal{M})$ is continuous (see Proposition 2.2), it follows from (3.13) that the limit of such convergent subsequence must be zero. This proves that

$$\lim_{n\to\infty} \|\mathcal{E}_m|z_n - \mathcal{E}_{m-1}(z_n)|^2\|_\infty = 0.$$

Since m is arbitrary, we deduce that $\lim_{n\to\infty} |\!|\!|z_n|\!|\!|_{\mathrm{C}} = 0$. A similar argument can be used for $(z_n^*)_{n\geq}$ in order to conclude that $\lim_{n\to\infty} |\!|\!|z_n|\!|\!|_{\mathrm{R}} = 0$. ∎

REMARKS 3.5. (a) Theorem 3.4 cannot be extended to the larger space $\mathrm{BMO}(\mathcal{M})$. Indeed, since $\mathrm{BMO}(\mathcal{M})$ is a dual Banach space and contains a subspace isomorphic to c_0, it necessarily contains a subspace isomorphic to l^∞ (see for instance [3, Theorem 10, p. 48]). In particular, it contains a subspace isomorphic to l^1. Such a subspace clearly fails to satisfy the conclusion of Theorem 3.2.

(b) One should note that in contrast to the case of VMO, the non-commutative generalization of the classical Kadets–Pełczyński alternatives for non-commutative L^p-spaces ($2 < p < \infty$), settled in [21], does not require the hyperfinite assumption.

We conclude this section with a quantitative form of Proposition 3.4 for the case of hyperfinite filtration. This version is much closer in spirit to the proof in the commutative case of dyadic-VMO considered in [14]. It also illustrates the involvement of Hilbert space norms.

First we will fix some notations. For $n \geq 1$, let $\gamma_n := \dim(\mathcal{M}_n) = \dim(L^2(\mathcal{M}_n))$. Inductively, we can construct a sequence $(w_j)_{j=1}^\infty$ in $\mathcal{M}$ such that:

(a) $(w_j D^{1/2})_{j=1}^\infty$ is an orthonormal basis of $L^2(\mathcal{M})$;

(b) for every $n \geq 1$, $(w_j D^{1/2})_{1\leq j\leq \gamma_n}$ is an orthonormal basis for $L^2(\mathcal{M}_n)$.

For $n \geq 1$, define

$$\Phi(n) := \sum_{j=1}^{\gamma_n} \|w_j\|_\infty^2, \qquad n \geq 1. \tag{3.14}$$

As an illustration, consider the concrete case of the type-II_1 hyperfinite factor $\mathcal{R}$ equipped with its usual trace τ and its usual filtration $(\mathcal{R}_n)_{n\geq 1}$. Since $\mathcal{R}_n$ is the matricial space $\mathbb{M}_{2^n\times 2^n}$, we have $\gamma_n = 4^n$. Moreover, the orthonormal system can be taken to be the non-commutative Walsh system $\{w_\gamma\}_{\gamma\in\widehat{D}}$ where $\widehat{D}$ denotes the dual group of the dyadic group $D = \prod_{n=1}^\infty \mathbb{Z}/2\mathbb{Z}$. For a detailed account of the non-commutative Walsh system, we refer to [1]. It was shown in [1] that $\{w_\gamma\}_{\gamma\in\widehat{D}}$ is an orthonormal basis of $L^2(\mathcal{R},\tau)$. Also, $\{w_\gamma\}_{\gamma\in D_{2n}^*}$ where $D_{2n}^* = \{\gamma \in \widehat{D} : \gamma_i = 0 \text{ when } i > n\}$ is a basis of $\mathcal{R}_n$ and $\|w_\gamma\|_\infty = 1$ for every $\gamma \in \widehat{D}$. Thus in this case $\Phi(n) = 4^n$ for every $n \geq 1$.

Proposition 3.6. *Let $(p_k)_{k\geq 1}$ be a strictly increasing sequence of integers, $(\varepsilon_k)_{k\geq 1} \subset \mathbb{R}_+$ satisfying $\sum_{k=1}^\infty \varepsilon_k = \delta$ and $(x_k)_{k\geq 1} \subset \mathrm{VMO}(\mathcal{M})$ with:*

(i) $\mathcal{E}_{p_k}(x_k) = x_k$ *and* $\mathcal{E}_{p_{k-1}}(x_k) = 0$;

(ii) $\|x_k\|_{\mathrm{VMO}} = 1$;

(iii) $\|x_k D^{1/2}\|_2^2 + \|D^{1/2}x_k\|_2^2 \leq \varepsilon_k \Phi(p_{k-2})^{-1}$ *for all* $k \geq 3$.

Then for any finitely non-zero sequence $(\alpha_k)_{k\geq 1} \subset B(l^2)$,

$$\sup_{k\geq 1} \|\alpha_k\|_\infty \leq \Big\|\sum_{k\geq 1} \alpha_k \otimes x_k\Big\|_{\mathrm{BMO}(B(l^2)\overline{\otimes}\mathcal{M})} \leq (2+\delta)^{1/2} \sup_{k\geq 1} \|\alpha_k\|_\infty.$$

Proof. It is enough to verify that condition (iii) implies the weaker condition (iii) of Proposition 3.3(b). To this end, we will show that for every $k \geq 3$, we have the estimates $\mathcal{E}_{p_{k-2}}(S_{\mathrm{C}}^2(x_k)) \leq \varepsilon_k$ and $\mathcal{E}_{p_{k-2}}(S_{\mathrm{R}}^2(x_k)) \leq \varepsilon_k$.

Let $k \geq 3$ and consider $\mathcal{E}_{p_{k-2}}(S^2_{\mathrm{C}}(x_k)D^{1/2}) \in L^2(\mathcal{M}_{p_{k-2}})$. Since the system $(w_j D^{1/2})_{j \leq \gamma_{p_{k-2}}}$ is an orthonormal basis of $L^2(\mathcal{M}_{p_{k-2}})$, we have the expansion

$$\begin{aligned}\mathcal{E}_{p_{k-2}}(S^2_{\mathrm{C}}(x_k))D^{1/2} &= \sum_{j=1}^{\gamma_{p_{k-2}}} \langle \mathcal{E}_{p_{k-2}}(S^2_{\mathrm{C}}(x_k)D^{1/2}), w_j D^{1/2}\rangle w_j D^{1/2} \\ &= \sum_{j=1}^{\gamma_{p_{k-2}}} \langle \mathcal{E}_{p_{k-2}}(D^{1/2}S^2_{\mathrm{C}}(x_k)D^{1/2}), w_j\rangle w_j D^{1/2}.\end{aligned}$$

Since $D^{1/2}$ is fully supported, it follows that

$$\mathcal{E}_{p_{k-2}}(S^2_{\mathrm{C}}(x_k)) = \sum_{j=1}^{\gamma_{p_{k-2}}} \langle \mathcal{E}_{p_{k-2}}(D^{1/2}S^2_{\mathrm{C}}(x_k))D^{1/2}, w_j\rangle w_j.$$

We deduce from the L^1-boundedness of expectations and the definition of Φ that

$$\begin{aligned}\|\mathcal{E}_{p_{k-2}}(S^2_{\mathrm{C}}(x_k))\|_\infty &\leq \sum_{j=1}^{\gamma_{p_{k-2}}} |\langle \mathcal{E}_{p_{k-1}}(D^{1/2}S^2_{\mathrm{C}}(x_k)D^{1/2}), w_j\rangle| \cdot \|w_j\|_\infty \\ &\leq \sum_{j=1}^{\gamma_{p_{k-2}}} \|\mathcal{E}_{p_{k-2}}(D^{1/2}S^2_{\mathrm{C}}(x_k)D^{1/2})\|_1 \cdot \|w_j\|^2_\infty \\ &\leq \sum_{j=1}^{\gamma_{p_{k-2}}} \|D^{1/2}S^2_{\mathrm{C}}(x_k)D^{1/2}\|_1 \cdot \|w_j\|^2_\infty \\ &= \|D^{1/2}S^2_{\mathrm{C}}(x_k)D^{1/2}\|_1 \Phi(p_{k-2}).\end{aligned}$$

Since $D^{1/2}S^2_{\mathrm{C}}(x_k)D^{1/2} = \sum_{j=p_{k-1}+1}^{p_k} |d_j(x_k D^{1/2})|^2 = S^2_{\mathrm{C}}(x_k D^{1/2})$, we get

$$\begin{aligned}\|\mathcal{E}_{p_{k-2}}(S^2_{\mathrm{C}}(x_k))\|_\infty &\leq \|S_{\mathrm{C}}(x_k D^{1/2})\|^2_2 \Phi(p_{k-2}) \\ &= \|x_k D^{1/2}\|^2_2 \Phi(p_{k-2}) \leq \varepsilon_k.\end{aligned}$$

A similar estimate can be applied to the $S^2_{\mathrm{R}}(x_k)$'s and the conclusion follows directly from Proposition 3.3(b). ∎

4. Banach space properties of VMO. Throughout this section, we assume that the von Neumannn algebra $\mathcal{M}$ is hyperfinite and the filtration consists of finite-dimensional subalgebras. We keep all notation introduced in previous sections. First we recall some Banach space concepts.

Definition 4.1. A formal series $\sum_{n=1}^\infty x_n$ in a Banach space E is called *weakly unconditionally Cauchy* (WUC) if $\sum_{n=1}^\infty |\langle x^*, x_n\rangle| < \infty$ for all $x^* \in E^*$.

It is a well known result of Bessaga and Pełczyński that a basic sequence equivalent to the unit vector basis of c_0 forms a (WUC) series, and conversely, any basic sequence for which $\inf_n \|x_n\| > 0$ and $\sum_{n=1}^{\infty} x_n$ is a (WUC) series is equivalent to the unit vector basis of c_0. We refer to [3, Theorem 6, p. 44] for equivalent characterizations of (WUC) series.

Evidently, if $\sum x_n$ is a (WUC) series and we set $e_n = \sum_{j=1}^{n} x_j$ for all n, then (e_n) is weakly Cauchy and therefore has a weak*-limit in E^{**}. We now consider a notion introduced by Pełczyński ([18, 17]).

Definition 4.2. A Banach space E has *property* (u) if for every weakly Cauchy sequence (x_n), there exists a (WUC) series $\sum a_n$ in E such that $(x_n - \sum_{j=1}^{n} a_j)$ is weakly null.

For example, any Banach space with unconditional basis has property (u) and so also do all weakly sequentially complete Banach spaces ([18]). Other classes of Banach spaces having property (u) are order continuous Banach lattices ([12, Proposition 1.c.2, p. 31]) and those which are M-ideals in their biduals ([6]).

The next result is the main focus of this section. It may be new even for the classical dyadic martingale VMO.

Theorem 4.3. *VMO($\mathcal{M}$) has property* (u).

To prove this theorem, we need two intermediate results. We assume that the next lemma is well known but we include the proof for completeness.

Lemma 4.4. *Let E be a real or complex Banach space.*

(i) *A bounded subset C of E is relatively weakly compact if and only if for a given sequence (x_n) in C, there exists a sequence (y_n) with $y_n \in \operatorname{conv}\{x_n, x_{n+1}, \dots\}$ that converges weakly.*
(ii) *A bounded sequence (x_n) in E converges weakly to an x in E if and only if for any subsequence (x_{n_k}) of (x_n) there exists a sequence (y_k) with $y_k \in \operatorname{conv}\{x_{n_k}, x_{n_{k+1}}, \dots\}$ that converges weakly to x.*

Proof. Decomposing any functional in E^* into its real and imaginary parts, we may assume that E is a real Banach space. We will only prove the non-trivial implications. For (i), we verify (according to James's theorem) that every non-trivial functional $x^* \in E^*$ attains its maximum on $\overline{\operatorname{conv}}(C)$. To this end, let $x^* \in E^* \setminus \{0\}$ and set $\alpha := \sup\{\langle x^*, x\rangle : x \in \overline{\operatorname{conv}}(C)\}$. Since $\alpha = \sup\{\langle x^*, x\rangle : x \in C\}$, there exists a sequence (x_n) in C such that $\alpha = \lim_{n\to\infty}\langle x^*, x_n\rangle$. By assumption, there exists a sequence (y_n) with $y_n \in \operatorname{conv}\{x_n, x_{n+1}, \dots\}$ that converges weakly to $x \in \overline{\operatorname{conv}}(C)$. Since $\langle x^*, y_n\rangle \in \operatorname{conv}\{\langle x^*, x_n\rangle, \langle x^*, x_{n+1}\rangle, \dots\}$, it follows that

$$\langle x^*, x\rangle = \lim_{n\to\infty}\langle x^*, y_n\rangle = \lim_{n\to\infty}\langle x^*, x_n\rangle = \alpha.$$

Thus $\overline{\operatorname{conv}}(C)$ is relatively weakly compact, and hence so is C.

To verify (ii), it is enough to observe from (i) that $C := \{x_n : n \geq 1\}$ is relatively weakly compact and since x is the only possible weak cluster point of C, (x_n) converges weakly to x. ∎

The proof of Theorem 4.3 is based on the following observation.

REMARK 4.5. First, we note that for every $x \in \mathrm{BMO}(\mathcal{M})$,

$$\|x\|_{\mathrm{BMO_C}} = \sup_{m\geq 1} \sup_{n\leq m} \sup_{n\leq k\leq m} \Big\| \mathcal{E}_n \Big(\sum_{k=n}^{m} |d_k(x)|^2 \Big) \Big\|_\infty^{1/2}.$$

A similar expression can be used for the $\mathrm{BMO_R}$-norm. It follows that martingale difference sequences are unconditional in $\mathrm{BMO}(\mathcal{M})$. In particular, for every $x \in \mathrm{BMO}(\mathcal{M})$, $\sum_{k=1}^{\infty} d_k(x)$ is a (WUC)-series in $\mathrm{VMO}(\mathcal{M})$. We thank the referee for pointing out this fact.

Proof of Theorem 4.3. Let $(x_n)_n$ be a weakly Cauchy sequence in the space $\mathrm{VMO}(\mathcal{M})$ which is not weakly convergent. Viewed as a bounded sequence in the dual Banach space $\mathrm{BMO}(\mathcal{M})$, $(x_n)_{n\geq 1}$ is a weak*-convergent sequence in $\mathrm{BMO}(\mathcal{M})$. Let $x \in \mathrm{BMO}(\mathcal{M})$ be the weak*-limit of (x_n) in $\mathrm{BMO}(\mathcal{M})$. By Remark 4.5, $\sum_{k=1}^{\infty} d_k(x)$ is a (WUC) series in $\mathrm{VMO}(\mathcal{M})$.

For $n \geq 1$, we set

$$y_n = x_n - \sum_{k=1}^{n} d_k(x) = x_n - \mathcal{E}_n(x). \tag{4.1}$$

Thus $(y_n)_n$ is a bounded sequence in $\mathrm{VMO}(\mathcal{M})$ that converges to zero for the weak* topology in $\mathrm{BMO}(\mathcal{M})$. We claim that the sequence $(y_n)_{n\geq 1}$ is weakly null in $\mathrm{VMO}(\mathcal{M})$.

To prove this claim, we first observe that $(y_n D^{1/2})_n$ and $(D^{1/2} y_n)_n$ are weakly null sequences in $L^2(\mathcal{M})$. Let $(u_n)_n$ be an arbitrary subsequence of $(y_n)_n$. One can choose a block convex combination of $(u_n)_n$ (say $(v_n)_n$) that converges to zero for the $|\!|\!| \cdot |\!|\!|_2$-norm. By Theorem 3.2 (and Proposition 3.3), one can choose a further subsequences of $(v_n)_n$ (which we still denote by $(v_n)_n$) that is equivalent to the unit vector basis of c_0. In particular, such block convex combinations are weakly null in $\mathrm{VMO}(\mathcal{M})$. We have just verified that the sequence $(y_n)_n$ is such that any of its subsequences has a block convex combination that converges to zero weakly. We can now deduce the claim from Lemma 4.4(ii).

In summary, $\sum_{k\geq 1} d_k(x)$ is a (WUC) series in $\mathrm{VMO}(\mathcal{M})$ and the sequence $(x_n - \sum_{k=1}^{n} d_k(x))_{n\geq 1}$ is weakly null, thus proving that $\mathrm{VMO}(\mathcal{M})$ has property (u). ∎

An immediate consequence of Theorem 4.3 is that every non-weakly compact operator from $\mathrm{VMO}(\mathcal{M})$ into any arbitrary Banach space must fix isomorphic copies of c_0. This property is called property (V) and is shared by C^*-algebras in general ([19]). It is known that for dual Banach spaces, property (V) implies the space being Grothendieck. In particular, it follows from [19] that von Neumannn algebras are Grothendieck spaces, i.e. if $\mathcal{N}$ is an arbitrary von Neumannn algebra then any weak*-convergent sequence of functionals in $\mathcal{N}^*$ is automatically weakly convergent. We do not know if this property is shared by the space $\mathrm{BMO}(\mathcal{M})$. In particular, the following motivating question seems to be unresolved:

PROBLEM. *Is* $\mathrm{BMO}(\mathcal{M})$ *a Grothendieck space?*

We note however that since $\mathcal{H}^1_{\mathrm{C}}(\mathcal{M})$ is a subspace of $L^1(\mathcal{M}; l^2_{\mathrm{C}})$ which is in turn a (complemented) subspace of $L^1(\mathcal{M} \overline{\otimes} B(l^2))$, it follows that $\mathrm{BMO}_{\mathrm{C}}(\mathcal{M}) = (\mathcal{H}^1_{\mathrm{C}}(\mathcal{M}))^*$ is a quotient of the von Neumannn algebra $\mathcal{M} \overline{\otimes} B(l^2)$. Thus $\mathrm{BMO}_{\mathrm{C}}(\mathcal{M})$ has property (V) and therefore is a Grothendieck space. The same observation can be applied to $\mathrm{BMO}_{\mathrm{R}}(\mathcal{M})$. We do not know if this simple argument can be applied to $\mathrm{BMO}(\mathcal{M})$.

NOTE (added May 18, 2008). After this paper was submitted, Mikhail Ostrovskii informed me that the Müller–Schechtman dichotomy stated in Theorem 1.1 was also independently obtained by M. Leĭbov for the case of BMO defined on the unit circle (see “Subspaces of the space VMO”, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 46 (1986), 51–54 (in Russian); translation in J. Soviet Math. 48 (1990), 536–538). I wish to thank M. Ostrovskii for communicating these references to me.

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